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DERIVATION OF OPTIMAL FIXED INVESTMENT PATHS FROM A DYNAMIC STOCHASTIC PROFIT-MAXIMIZING MODEL

The aim of this paper is to derive a firm's optimal investment path from a dynamic optimal planning problem, considering economic risk (with given probability distributions). Investment accelerator is used to determine the firm's optimal investment behaviour under price uncertainty.

1. The problem

This paper deals with investment decisions under uncertainty. All further considerations are based on a dynamic optimal planning model, maximizing net present value as an objective for the firm. The construction of the dynamic optimal planning problem relies on the theory of the firm. In general, the topic of this paper belongs to corporate investment theory framework, a brief description of which may be quoted from [1, p. 739]: "An investment should be undertaken if and only if it increased the value of the shares. The securities market appraises the project, its expected contributions to the future earnings of the company and its risks. If the value of the project as appraised by investors exceeds the cost, then the company shares will appreciate to the benefit of existing stockholders. That is, the market will value the project more than the cash used to pay for it. If new debt or equity securities are issued to raise the cash, the prospectus leads to an increase of share prices." More particularly about corporate investment theory, capital asset pricing methods (CAPM), q-theory, etc. see for example [1—4].

The aim of this work is to find out optimal fixed investment paths from a dynamic stochastic optimal planning problem. This paper is meant to be an extension of a previous work [5]. Here a unique optimal investment strategy is chosen from the set of feasible solutions of the problem. According to the general framework of the problem, the objective for a firm is to maximize its value. The firm's maximized net present value serves as a criterion for choosing an optimal investment path. In deriving the optimal investment paths investment accelerators are used for modelling the firm's investment behaviour.

In the following sections the variables K (capital), L (labour), and I (investment) are used as decision variables in the process of maximizing the expectation of the net present value of the firm. The prices of production inputs, q (for investments) and w (for labour), are regarded as exogenous variables. For including economic risk into the model, the net present value of the firm is considered random. This may be done by regarding prices as random variables. There is a possibility to consider all prices random, but in this case the model would become too complicated mathematically. A way to reduce the complexity of the model is to make appropriate assumptions about production input and output prices: below it is assumed that the price for the firm's output is random and the production input prices are deterministic (given).

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Consequently, the expectation of random cash flow is to be maximized with random prices for the firm's product. It is worth noting that in principle there are no restrictions to build the model up in a different way, with starting from different assumptions about prices and risk.

The firm's capacity is determined through the production function (concave and nondecreasing on each input). The optimal volume of output depends on the changes in the demand for the firm's product, being thus an endogenous variable in the model. The demand for the firm's output is characterized with the help of the relationship between the volume and the price of output.

2. The model

To begin with, it seems appropriate to present the modelling of risk at a fixed moment of time (without dynamics). More precisely, a treatment of the density function of the random output price will be presented.

The random cash flow (random profits) with considering the price of output as a random variable at some arbitrarily chosen moment of time can be expressed as:

$$\tilde{C} = \tilde{p} f(K, L) - wL - qI, \quad (1)$$

with assuming that $\tilde{C}$ belongs to a "linear class of random variables", which can be described according to [6, p. 56] as a class of random variables with standardized values

$$\tilde{Z} = (\tilde{C} - E(\tilde{C}))/\sigma(\tilde{C}), \quad (2)$$

with

$$E(\tilde{Z}) = 0, \quad \sigma(\tilde{Z}) = 1, \quad (3)$$

which have the same density function; for example, normal distribution (in (2) and (3) E and σ denote expectations and standard deviations of random variables). Within a linear class all distributions can be transformed into one another merely by a shift and a proportional extension. In this case

$$P(\tilde{Z} < Z_\gamma^0) = 1(2\Pi)^{1/2} \int_{-\infty}^{z_\gamma^0} e^{-z^2/2}\, dz = 1 - \gamma \quad (4)$$

and, after merging (2) and (4):

$$P[\tilde{C} - E(\tilde{C}) < Z_\gamma^0 \sigma(\tilde{C}) = 1 - \gamma. \quad (5)$$

In (4) and (5) γ denotes the subjective rate of risk aversion. For risk aversive decision makers $Z_\gamma^0 < 0$. Now it can be stated that with probability $1 - \gamma$:

$$\tilde{C} - E(\tilde{C}) < Z_\gamma^0 \sigma(\tilde{C}) \quad (6)$$

or

$$\tilde{C} < E(\tilde{C}) + Z_\gamma^0 \sigma(\tilde{C}). \quad (7)$$

The right side of expression (7) denotes the upper bound of random cash flow for the given risk rate γ. In other words, the right side of (7) is the fractile of normally distributed $\tilde{C}$ (it is obvious that if $\tilde{p}$ is normally distributed, then $\tilde{C}$ is normally distributed as well). Denoting the expectation and standard deviation of random cash flow as μ_C and σ_C, we obtain the following result:

$$\tilde{C} < E(\tilde{C}) + Z^0_\gamma \sigma(\tilde{C}) = B(\mu_C, \sigma_C), \tag{8}$$

where $B(\mu_C, \sigma_C)$ is the fractile of normally distributed $\tilde{C}$, whose maximizing means the maximizing of the expectation of cash flow (because for a given distribution of $\tilde{C}$ and a given rate of risk aversion the expression $Z^0_\gamma \sigma(\tilde{C})$ is determined). Thus, the aim is to maximize the expectation of cash flow. It is essential to notice that all considerations have so far taken place at a discrete moment of time.

Next, the dynamic model of maximizing discounted cash flow (net present value) will be presented. The dynamic model can be divided into two parts, one containing the maximizing of current profits, the other the maximum future profits with the assumed optimal values of the decision variables. The previous considerations about economic risk hold in the case of maximizing the current profits in the time-discrete case. The solution of the dynamic planning problem reduces to the solutions of the set of time-discrete optimization problems at different moments of time; therefore it is possible to use previous time-discrete considerations about risk. The optimizing of decision variables K, L, and I is carried out with the help of the dynamic planning problem:

$$E\left[\int_0^\infty e^{-rt}(\tilde{p}_t f(K_t, L_t) - w_t L_t - q_t I_t)\, dt\right] \to \max, \tag{9}$$

$$f(K_t, L_t) \geqslant Q_t \geqslant 0,$$
$$I_t \geqslant dK_t + \delta K_t \geqslant 0,$$

where δ is depreciation rate, Q is the volume of output, and elasticity $\tilde{\eta}$ relates the volume of output and the price of output in the following way:

$$\tilde{p}_t = b Q_t^{\tilde{\eta}} \tag{10}$$

or

$$\tilde{p}_t f(K_t, L_t) = \bar{p}_t Q_t = b Q_t^{\tilde{\eta}+1}. \tag{11}$$

Now $1/\tilde{\eta}$ characterizes the (random) elasticity of demand. Next some extreme cases will be considered (for deterministic η, this may help to clarify some consequences of using (10)). At first, if $\eta = 0$, then output price is stable and equals b. Secondly, there is a special case when $\eta = -1$, then profits do not depend on the output price and volume; in this case profit-maximizing problem (9) reduces to a cost-minimizing problem. Generally $-1 < \eta < 0$.

To solve problem (9), it is necessary to write it into Lagrangian form:

$$F_s = \max \int_0^\infty e^{-rt}\{\mu_p(Q_t) - w_t L_t - q_t I_t + \lambda_{1t}[f(K_t, L_t) - Q_t] + \lambda_{2t}[I_t - dK_t - \delta K_t]\}\, dt, \tag{12}$$

where $\mu(Q_t) = E(bQ_t^{\tilde{\eta}+1})$ denotes the expected value of output.

In (12) λ_{1t} denotes Lagrangian coefficient which reflects the increment of profits per an additional unit of production, λ_{2t} is the shadow price of investment. Next, to solve (12), it is necessary to present the transition equations. After dividing (12) into two parts, with one comprising the maximization of current profits at an arbitrarily chosen time moment $t=s$ and the other reflecting maximum future profits with assumed optimal values of the decision variables for the time interval $[s+1, \infty)$:

$$\max F'_s = \max \{[\mu(Q_s) - w_s L_s - q_s I_s + \lambda_{1s}[f(K_s, L_s) - Q_s] + \\ + \lambda_{2s}[I_s - dK_s - \delta K_s]\} + \\ + \sum_{s+1}^{\infty} \{[\mu(Q^*_s) - w_s L^*_s - q_s I^*_s + \lambda_{1s}[f(K^*_s, L^*_s) - Q^*_s] + \\ + \lambda_{2s}[I^*_s - dK^*_s - \delta K^*_s]\}, \tag{13}$$

where $\max F_s \cong \max F'_s$. Next, the transition equations, based on (13), will take the following form ($f_s = f(K_s, L_s)$):

$$\partial F'_s / \partial K_s = \lambda_{1s} \partial f_s / \partial K_s - \lambda_{2s}[\partial(dK_s)/\partial K_s + \delta] = 0, \tag{14}$$

$$\partial F'_s / \partial L_s = -w_s + \lambda_{1s} \partial f_s / \partial L_s = 0, \tag{15}$$

$$\partial F'_s / \partial I_s = -q_s + \lambda_{2s} = 0, \tag{16}$$

$$\partial F'_s / \partial \lambda_{2s} = I_s - dK_s - \delta K_s = 0, \tag{17}$$

$$\partial F'_s / \partial Q_s = \partial \mu(Q_s) / \partial Q_s - \lambda_{1s} = 0, \tag{18}$$

$$\partial F'_s / \partial \lambda_{1s} = f_s - Q_s = 0. \tag{19}$$

Formula (16) shows that adjustment costs are not considered in this work (if adjustment costs are considered, then in optimum case shadow price of investment has to be higher than the price of the investment goods). This does not lead to the conclusion that the marginal efficiency of capital and marginal efficiency of investments are undistinguishable. This is so because of the depreciation δ and the member $\partial(dK_s)/\partial K_s$ in (14). There is no distinction between the marginal efficiency of capital and that of investments only when $\delta=1$ (and $\partial(dK_s)/\partial K_s=0$, $\lambda_{1s}=1$). In this case the dynamic model (12) reduces to a set of one-period static models with a constant capital stock and total depreciation of capital stock in the given period of time.

It follows from (12) and (18) that for $\eta=0$ the Lagrangian coefficient λ_{1s} is constant. That is, in this case the increment of profits corresponding to an additional unit of production is reflected through the given price b. Therefore maximum profits would be obtained in the case of infinite expanding of production activities as then increasing output would result in increasing profits at the speed b (for a linearly homogeneous production function).

In (17) optimal investment demand function is determined:

$$I_s = dK_s + \delta K_s. \tag{20}$$

In (20) investment behaviour can be modelled with elementary functions such as linear function, exponential function, etc. For linear investment demand function

$$I_s = I_0 + cs. \tag{21}$$

where I_0 denotes initial investments. According to (21) the dynamics of the capital stock can be expressed:

$$K_s = K_{s-1} e^{-\delta s} + (I_0 - c)/\delta^2 + cs/\delta, \quad (22)$$

where K_{s-1} is the size of capital stock from previous periods. The first member of the right side of equation (22) is the solution of homogeneous differential equation $dK_s + \delta K_s = 0$ and the second member of the right side is the particular solution of the differential equation $dK_s + \delta K_s - I_s = 0$. It is obvious that the solution of the homogeneous differential equation reflects the deviation of capital stock from equilibrium (depreciation) while the particular solution reflects equilibrium state. Now the general solution (22) describes the general state of the capital stock for each investment strategy (presently it was linear). This approach has, of course, some disadvantages:

1) the general solution (22) depends on *a priori* determined modelling function of investment behaviour;

2) the general solution (22) depends on the value of the capital stock of the previous period K_{s-1}, so (22) can be solved only step-by-step for each moment of time $t = 1, 2, \ldots$;

3) no unique solution to the problem exists.

A way for avoiding these disadvantages will be described in the next section. Particularly, if now the solution depends on the type of the investment modelling elementary function, then in the next section the solution depends on the qualities of the production function.

3. An alternative form of the initial problem

As stated at the end of Section 1, it is possible to formulate the problem differently. We shall use the same dynamic optimization problem with one difference: the form of investment demand function in (20) will be changed. For this purpose an investment acceleration mechanism will be used in the following way:

$$I_s = \beta_s \, dQ_s + \delta K_s, \quad (23)$$

or

$$dK_s = \beta_s \, dQ_s. \quad (24)$$

In (23) and (24) the traditional approach to a simple accelerator is presented, where the increment of the capital stock is described with the help of the increment of the volume of output and the accelerator $\beta_s = K_s/Q_s$. Although (24) is an extremely simple form of the acceleration mechanism, it is useful for describing the principle of using accelerators in optimal investment analysis.

Now (23) can be rewritten in the following form, using production functions:

$$\beta_s = K_s/f_s, \quad (25)$$

$$dQ_s = df_s = \partial f_s/\partial K_s * dK_s + \partial f_s/\partial L_s * dL_s, \quad (26)$$

and

$$I_s = \beta_s \, df_s + \delta K_s, \quad (27)$$

where $f_s = f(K_s, L_s)$. With the help of (27) the increment of the capital stock is estimated (approximated). To ensure that this approach will give computable results, it is necessary to assume that the ratio of capital and labour is constant ($K/L = \text{const}$). This quite widespread assumption makes expression (26) reliable.

Now it is possible to rearrange the original problem (9) into the following form:

$$E\{\int_0^\infty e^{-rt}[\tilde{p}_t f(K_t, L_t) - w_t L_t - q_t I_t]\,dt\} \Rightarrow \max \tag{28}$$

$$f(K_t, L_t) \geqslant Q_t \geqslant 0,$$

$$I_t \geqslant \beta_s\, df_s + \delta K_s.$$

It is possible to rewrite (28) into Lagrangian form to divide it into two parts and reach the transition equations (14)—(19), with the only difference that instead of (17) stands (27). Now it is possible to obtain the unique solution, solving directly the transition equations:

$$\partial F_s/\partial K_s = \lambda_{1s}\,\partial f_s/\partial K_s - \lambda_{2s}[\partial(dK_s)/\partial K_s + \delta] = 0, \tag{14}$$

$$\partial F_s/\partial L_s = -w_s + \lambda_{1s}\,\partial f_s/\partial L_s = 0, \tag{15}$$

$$\partial F_s/\partial I_s = -q_s + \lambda_{2s} = 0, \tag{16}$$

$$\partial F_s/\partial \lambda_{2s} = I_s - \beta_s\, df_s - \delta K_s = 0, \tag{27}$$

$$\partial F_s/\partial Q_s = \partial\mu(Q_s)/\partial Q_s - \lambda_{1s} = 0, \tag{18}$$

$$\partial F_s/\partial \lambda_{1s} = f_s - Q_s = 0. \tag{19}$$

There is no longer any need to model investment behaviour with the help of some elementary function, because the solution depends now on the qualities of the production function f (through $df_s, K/L = \text{const}$) in (27)).

The approach of finding optimal paths for investments with the help of accelerators has its disadvantages as well.

First of all, this approach depends on the type of the accelerator. In this work a simple accelerator was used; this causes several problems, e.g.:

1) the firm does not have to work at full capacity, this means that if demand increases, the firm can expand production without investments;

2) the mechanism of acceleration is nonsymmetric; in other words, a decline of demand cannot cause a decrease of capital stock;

3) accelerators can be used for modelling only the strategic changes in the capacity of the firm, there is no chance to react to temporary changes in demand.

To conclude this section, it should be noted that in the framework of this work the investment strategy is determined by means of:

1) the function modelling investment behaviour;

or

2) estimating (approximating) the increment of capital stock with the help of accelerators; in this case the investment demand function is determined.

The use of accelerators is based on the fact that the necessary conditions for maximum profits in the dynamic model leave enough space for various assumptions about the dynamics of investments and capital stock. This means that maximum profits can be obtained by means of various investment paths (with considering various complexes of parameters of the production function). Probably, better results would be obtained if more flexible accelerators were used; here only the principle of using accelerators was presented.

4. Optimal investment demand and risk

Throughout this paper it is assumed that the problems involving risk are solved with assuming known distributions of the random prices $\tilde{p}$. Prices q (for capital) and w (for labour) are expected to be known, too. First of all, there must be a set of information about the dynamics of demand; this is reflected through:

$$\tilde{p}_t = bQ_t^{\tilde{\eta}}. \tag{29}$$

If $\eta=0$, then $p_t=b$, or differently, the output price is given. So there is no room left for risk. Or (with the help of (12)):

$$E(\tilde{p}_t) = E(bQ_t^{\tilde{\eta}}) = \mu(Q_t)/Q_t. \tag{30}$$

It is important to notice here that the results of Section 1 can be used if and only if the random variable $\tilde{p}_t$ belongs to a linear class. This sets an additional restriction to the distribution of $\tilde{\eta}$, however, for the special case $-1<\tilde{\eta}<0$ this problem can be solved. Hence, it is assumed that

$$E(\tilde{p}_t) = \mu_p = \mu(Q_t)/Q_t,$$
$$\sigma(\tilde{p}_t) = \sigma_p = [E(\tilde{p}_t - \mu_p)^2]^{1/2}. \tag{31}$$

An investment strategy for the firm is presented through (27) as an investment demand function. Consequently, the investment strategy depends directly on the type of accelerator, depreciation rate, and total differential of the production function. Indirectly (through the dynamic optimal planning problem) prices of output, labour, and capital are involved as well. The whole problem of optimizing investment paths under uncertainty takes the following form (for the time moment $t=s$ the expressions are presented without the subscript t for simplicity):

$$\max B(\mu_C, \sigma_C) = \max \mu_C + Z_\gamma^0 \sigma_C, \tag{32}$$

$$\mu_C = \mu_p f(K, L) - wL - qI, \tag{33}$$

$$\gamma = \text{const}, \quad \sigma_C = \sigma_p [f(K, L)^2]^{1/2}, \tag{34}$$

$$I = \beta df(K, L) + \delta K, \tag{27}$$

with μ_p, σ_p determined in (31).

In general, the maximizing of fractile $B(\mu_C, \sigma_C)$ in (8) and (32)—(33) is reduced to the maximizing of the expectation of the net present value in (9). The optimal values of K and L are found from the dynamic planning problem and optimal investments from (27).

Now one more question concerning the uncertainty horizon is left open. If the time moment $t=s$ is freely chosen, then the bigger is s, the bigger must be risk. Although γ is the subjective rate of risk aversion, it is possible to estimate it by means of the so-called "survival probability". For this purpose it is necessary, according to [7], to introduce the following expression:

$$\varphi(s) = \int_s^\infty h(t)\,dt, \quad \varphi(0)=1, \quad \varphi(\infty)=0, \tag{35}$$

where $\varphi(s)$ is the probability of a rapid change in demand (demand shock, disaster) for the firm's output expected to happen at some time moment t, larger than s; $h(t)\,dt$ is the probability of the disaster striking in the time interval $[t, t+dt]$. For example, $\varphi(s)$ can be specified as follows:

$$\varphi(s)=\exp\left[-\int_0^s u(t)\,dt\right], \tag{36}$$

where $u(t)\,dt$ is the conditional probability that the disaster occurs in the time interval $s<t<s+ds$, conditional upon the survival of the project to the time s inclusive. Expression $u(t)$ is also called the hazard rate. If in (36) the hazard rate $u(t)$ equals a, then $\varphi(s)=e^{-as}$ and the probability $\varphi(s)$ declines exponentially at the speed a (for more details see [7]). Next, there is a possibility to merge (35) and (5) in the following way:

$$P[\tilde{C}-E(\tilde{C})<Z_\gamma\sigma_C]=1-\gamma=\varphi(s) \tag{37}$$

or

$$\gamma=1-\varphi(s). \tag{38}$$

Expressions (35)—(38) unite the time-discrete treatment of risk described in Section 1 with its dynamic treatment. Now the subjective rate of risk depends on time, more exactly, on the length of the time interval $[0, s]$. It is obvious that $\varphi(s)$ declines with considering the value of s growing. In this way the time-discrete rate of risk aversion γ grows as the survival probability $\varphi(s)$ declines. The discrete risk aversion rate γ is now "dynamized" (also in a subjective way, as the hazard rate a is subjective as well).

5. Conclusion

To conclude, I would like to stress two aspects of this work.

First, the use of accelerators for estimating the dynamics of capital stock. This approach makes it possible to find out the unique solution for the optimal investment path. According to the investment demand function derived in this work an investment strategy depends directly on the depreciation rate, the investment accelerator, and the total differential of the production function. In this way the characteristics of the production function (returns to scale, technical growth factor, etc.) become involved in the process of making investment decisions. Consequently, the process of determining the investment strategy is sufficiently flexible. An investment decision depends indirectly on the prices of production input factors and on predictable changes in the output prices (in demand). The role of the discounting rate is here not so remarkable as, for example, in q-theory. In other words, the discounting rate is not a decision variable in this framework.

Secondly, risk analysis with linear class variables, etc. as made in this work, is a time-discrete approach. The use of such an approach in a dynamic model is appropriate only because of the possibility of reducing the dynamic model into a set of time-discrete models. Even so the question about the risk (uncertainty) horizon is left open. In this work a suggestion is made for connecting time-discrete (subjective) rate of risk aversion with the planning horizon with the help of so-called survival probability. In connection with the survival probability the question about the dynamics of prices arises. In the framework of the suggested model discounted cash flow is maximized as a concave function, which rules out the possibility of rapid changes (shocks) in the firm's profits. Consequently, rapid changes in prices are not considered possible. In this way the survival probability is estimated as corresponding to the period of the firm's "normal life" and the shocks in demand and supply are an issue for future researches.

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OPTIMAALSETE INVESTEERINGUTE TULETAMINE DÜNAAMILISEST KASUMI MAKSIMEERIMISE MUDELIST, ARVESTADES MAJANDUSLIKKU RISKI

On vaadeldud dünaamilist firma mudelit ning tuletatud optimaalne nõudlus investeeringute järele. Ühese lahendi leidmiseks on kasutatud investitsiooniprotsessi modelleerimisel aktseleratsioonimehhanismi. Sellisel juhul sõltub firma investitsioonistrateegia kasutatavast aktseleraatorist, amortisatsiooninormatiivist ning firma tootmisvõimsust peegeldava tootmisfunktsiooni parameetritest. Majandusliku riski arvestamine toimub juhuslike hindade kaudu, mis eksogeensete teguritena mõjutavad kogu mudelit, sealhulgas investitsioonistrateegiat.

Ильмар ЛЕПИК

ОПТИМАЛЬНАЯ ИНВЕСТИЦИОННАЯ СТРАТЕГИЯ ФИРМЫ НА БАЗЕ ДИНАМИЧЕСКОЙ МОДЕЛИ МАКСИМИЗАЦИИ ПРИБЫЛИ В УСЛОВИЯХ ЭКОНОМИЧЕСКОГО РИСКА

Рассматривается определение оптимальной инвестиционной стратегии фирмы на базе динамической модели максимизации прибыли. В целях получения однозначного решения используется механизм акселерации при моделировании инвестиционного процесса. В итоге инвестиционная стратегия фирмы зависит от акселератора, от норматива амортизации и от параметров производственной функции, при помощи которой определяется производственная мощность фирмы. Учет экономического риска связан с использованием случайных цен (как экзогенных переменных), которые оказывают существенное влияние на все результаты модели, в том числе и на инвестиционную стратегию фирмы.