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OPTIMAL INVESTMENT PATHS UNDER UNCERTAINTY

The aim of this paper is to derive an investment decision rule from a dynamic stochastic model of a firm in the form of an investment demand function. Also the theoretical specification for estimating an investment decision model is presented. The problem belongs to CAPM (capital asset pricing methods, see for example [1—3]). However, few dynamic treatments of this problem are available. Most investment functions rely on static neoclassical models (with free capital adjustment) to derive an optimal capital stock as a function of the current variables.

The model presented here is in general based on the model of R. Craine [4] except for the fact that we do not consider adjustment costs in our model and we use time-continuous variables. The model is constructed so that the solution for an optimal investment path is derived from a first-order non-homogeneous differential equation (which reflects the production function constraint for a restricted capital adjustment). This means that there is no unique solution to the problem. An optimal investment path depends on the type of the investment behaviour simulation function and on changes in the demand for the firm's output and supply conditions. In other words, the prices of output, wage rates and prices of investment goods are supposed to be random. Expectations of these variables determine the position of the firm at some future moment of time. According to Craine [4, p. 648] a dynamic model with random prices generates the situation where an optimal solution is uncomputable, except for a few special cases.

In Section 1 the model of the firm under certainty is presented. The purpose of this section is to derive an optimal investment demand function for maximized profits as the firm's objective. The parameters of the investment demand function are determined by the initial conditions for a differential equation, i. e. by the initial state of prices and by the given prices at some future moment of time. In Section 2 it is assumed that the described initial conditions depend on random prices in a risk-neutral case. However, the maximized profits of the firm are in this case overestimated. The reason for overestimation is that the optimal investment paths and the corresponding maximum profits are found without taking into account any higher moments of random variables than the first (the variances are neglected).

The risk-aversion case of the model including variances is presented in Section 3 to compensate for risk-neutral overoptimistic solutions. Risk analysis is based on the assumption that random variables included in the model belong to a linear class (more details in Section 3). The final result of this section is the derivation of a theoretical upper bound of random profits with the given optimal values of labour and capital, an optimal investment path and the given level of risk.

1. Model of the firm (deterministic case)

This section presents the model of the firm and derives an investment decision rule under certainty. The results presented here are based on the neoclassical theory of the firm similarly to [4, p. 649] with differences in the following aspects:

1) we are using time-continuous variables;

2) we do not assume any adjustment costs in the model;

3) as the first step we deal with the deterministic version of the model.

Exogenous variables such as the output price of the firm's product (p_t), wage rate (w_t), and the price of investment goods (q_t) are used in the model. In Section 2 these variables are considered random.

The firm tries to maximize its discounted profits by choosing the optimal paths for decision variables — labour (L_t) and investment (I_t) with restriction to the current input of capital (K_t):

$$dK_t/dt = I_t - \delta K_t \geqslant 0, \tag{1}$$

where δ denotes the depreciation (physical deterioration) rate. The capital stock is not perfectly adjustable by (1). The firm can reduce its capital stock through depreciation at rate δ; there is also a possibility to sell off capital equipment, but "made-to-order" equipment has probably poor secondary market with relatively high selling costs. Labour is considered freely adjustable.

The objective of the firm is now to maximize its profits. For this purpose we introduce the capacity (g_t) and cash flow (C_t) at time t of the firm:

$$g_t = f(K_t, L_t), \tag{2}$$

where f is production function, which is concave and nondegenerate on each input;

$$C_t = p_t f(K_t, L_t) - w_t L_t - q_t I_t. \tag{3}$$

Now, the general shape of the problem is:

$$\max_{I_t, L_t} \int_0^\infty C_t e^{-rt}\, dt \tag{4}$$

subject to (1) or:

$$\max_{I_t, L_t} \int_0^\infty e^{-rt}[p_t f(K_t, L_t) - w_t L_t - q_t I_t]\, dt \tag{5}$$

$$I_t = \dot{K}_t + \delta K_t \geqslant 0, \quad \forall t \in [0, \infty). \tag{6}$$

In (5) r is discounting rate; $\dot{K}_t = dK_t/dt$ in (6). The problem (5)—(6) is a nonlinear planning problem, and finding the optimal paths for I, L is at best very difficult. According to Craine [4, p. 650], the only available tool is dynamic programming.

Next, we can rewrite (5)—(6) in the following form, using the Lagrangian function, for an arbitrarily chosen moment of time $t=s$:

$$F_s = \int_s^\infty e^{-r(t-s)}[p_t f(K_t, L_t) - w_t L_t - q_t I_t + \lambda_t (I_t - \dot{K}_t - \delta K_t)] \Rightarrow \max \tag{7}$$

$$I_t \geqslant 0, \tag{8}$$

where λ_t is the shadow price of the investment. Breaking the maximization problem into two parts, one containing the maximization of the current profit, the other the maximum future profits (with the given optimal decisions), we obtain the maximization problem for F'_s (for $t=s$):

$$\max F'_s = \max \ [p_s f(K_s, \ L_s) - w_s L_s - q_s I_s + \lambda_s (I_s - \dot{K}_s - \delta K_s)] +$$

$$+ \sum_{s+1}^{\infty} [p_t f(K^*_t, \ L^*_t) - w_t L^*_t - q_t I^*_t + \lambda^*_t (I^*_t - \dot{K}^*_t - \delta K^*_t)] (1+r)^{s-t-1}, \quad (9)$$

where * denotes optimal decisions. It is obvious that $\max F_s \cong \max F'_s$. Differentiating F'_s gives the general transition equations[1]:

$$\partial F'_s / \partial I_s = -q_s + \lambda_s = 0, \quad (10)$$

$$\partial F'_s / \partial \lambda_s = I_s - \dot{K}_s - \delta K_s = 0, \quad (11)$$

$$\partial F'_s / \partial L_s = p_s \partial f(K_s, \ L_s) / \partial L_s - w_s = 0, \quad (12)$$

$$\partial F'_s / \partial K_s = p_s \partial f(K_s, \ L_s) / \partial K_s - \lambda_s \delta = 0. \quad (13)$$

Now, from (10) and (12)—(13) we obtain that

$$w_s = p_s \partial f(K_s, \ L_s) / \partial L_s, \quad q_s = p_s \partial f(K_s, \ L_s) / \partial K_s, \quad (14)$$

if $\delta = 1$. Expression (14) reflects the situation when in the optimum case the marginal products of labour and capital are equal to the prices of labour and investment goods. It is essential to notice that if $\delta = 1$, then this version of the model is equivalent to the set of static models (at different moments of time), because then $K_s = I_s$ (if $\dot{K}_s = 0$) with no restrictions to capital adjustment and with total depreciation of capital stock in one period of time. This has to be an extreme case, generally by (13) the marginal product of capital is equal to $\lambda_s \delta$ or $\lambda_s (\delta + \partial \dot{K}_s / \partial K_s)$ and the right side of (14) is a simplification.

According to (10), in the optimum case the price of the investment goods is equal to the shadow price of the investment, which is the same result as in [4, p. 651—652] where no adjustment costs are assumed.

Next, (11) gives the optimal investment path. The difficulties arising with finding the solution for I_s (investment demand function) are connected with the fact that $I_s = \dot{K}_s + \delta K_s$ is a non-homogeneous first-order differential equation. For finding the optimal capital stock it is necessary to solve (6) with respect to K. For this task it is necessary to assume some types of elementary functions (or combinations of elementary functions) to simulate investment behaviour.

It can be easily seen that the simplest available solutions are:

1) $I_s = 0$, in this case pure reducing of the capital stock at the rate δ takes place and the current capital stock can be expressed by:

$$K_s = K_{s-1} e^{-\delta s}, \quad (15)$$

where K_s is the size of the current capital stock, and K_{s-1} is the size of the capital stock from previous periods (in the beginning of the period s). Here K_{s-1} is found as the parameter for the solution of differential equation (6), based on the initial conditions which can be found in (12)—(13);

2) $I_s = I_0$ (constant investment), then:

$$K_s = K_{s-1} e^{-\delta s} + I_0 / \delta, \quad (16)$$

where I_0 denotes the size of the initial investment;

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[1] To avoid mathematical complications we assume in (13) that $\partial \dot{K}_s / \partial K_s = 0$, or $\dot{K}_s = \text{const.}$

3) $I_s=I_0+cs$ (increasing or decreasing the capital stock linearly):

$$K_s=K_{s-1}e^{-\delta s}+(I_0\delta-c)/\delta^2+cs/\delta; \tag{17}$$

4) $I_s=I_0e^{ds}$ (increasing or decreasing the capital stock exponentially):

$$K_s=K_{s-1}e^{-\delta s}+I_0e^{ds}/(d+\delta). \tag{18}$$

From the theory of differential equations is known that the solution of the homogeneous differential equation $\dot{K}_s+\delta K_s=0$ is $y_0(s)=K_{s-1}e^{-\delta s}$ and this solution reflects the deviation from the equilibrium state (here this deviation reflects physical deterioration of the capital stock). The particular solution $y_p(s)$ of $\dot{K}_s+\delta K_s-I_s=0$ reflects the equilibrium state of the capital stock. Now the general solution describes the general state of the capital stock in the case of each investment strategy. If we assume, for example, that we have the exponential case (18), then we may say that the capital stock will stay unchanged in the period $t=s$, if

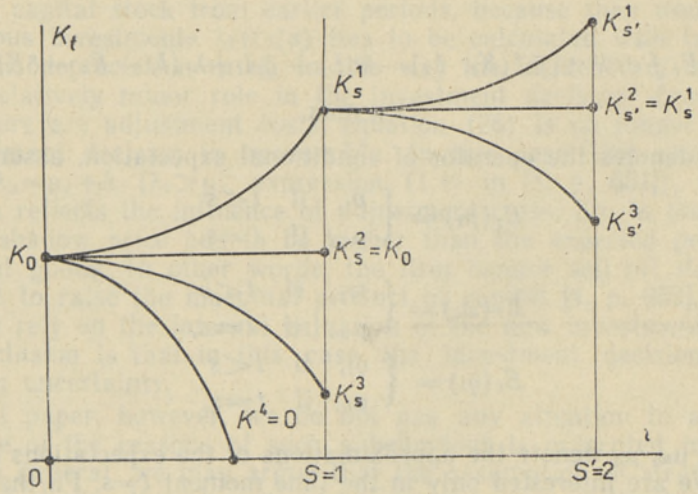
$$-d/ds(K_{s-1}e^{-\delta s})=d/ds[I_0e^{ds}/(d+\delta)]. \tag{19}$$

This is the case when the capacity of the firm, (g_t), stays unchanged because of constant technology and constant capital-labour ratio K/L. If this holds, then the changes in the capacity of the firm are proportional to the changes in the capital stock.

Generally, for the optimal investment demand function in $t=s$:

$$\text{capital stock will} \begin{Bmatrix}\text{increase}\\ \text{stay constant}\\ \text{decrease}\end{Bmatrix} \text{ if } -d/ds(y_0(s)) \begin{Bmatrix}<\\ =\\ >\end{Bmatrix} d/ds(y_p(s)). \tag{20}$$

Now it is possible to draw the conclusion that if the firm's exogenous conditions suggest expanding production activities (the capacity of the firm), then capital stock will increase, and $-d/ds(y_0(s))<d/ds(y_p(s))$, etc. will hold (see Fig.).



Optimal paths for the capital stock for time moments $s=1$, $s'=2$ in the cases of increasing capacity (K^1), constant capacity (K^2) and decreasing capacity (K^3, K^4); in the last case there will be total depreciation before $t=s$.

The parameters of the investment demand function can be found on the basis of the initial conditions which can be found in (12)—(13) for the time moments $t=0$ and $t=s$. Here we assume that K_0, L_0 reflect the initial values of labour and capital, i. e. it is assumed that K_0, $L_0 \neq 0$. Through K_0, L_0 the optimal size of the firm for the time $t=0$, in other words, the initial size of the firm is determined. From (12)—(13) we get two ordinary equations from where the values of L, K can be computed, which characterize the initial conditions of the differential equation for arbitrarily chosen $t=s$. Note that from assumption $\dot{K}_s=\text{const}$ we have that (13) will not give the initial conditions, if K_t is exponential, as then according to (18) $\dot{K}_s \neq \text{const}$. Then instead of (13) we have (13a):

$$\partial F'_s / \partial K_s = p_s \partial f(K_s, L_s) / \partial K_s - \lambda_s (\delta + \partial \dot{K}_s / \partial K_s).$$

For investment strategies (15), (16), (17) equations (12)—(13) hold.

The Figure presents two steps of solving the model for various investment strategies.

2. Model of the firm (uncertainty case)

In the stochastic model of the firm the exogenous variables of the model (5)—(6) are considered as random variables from some moment of time $t=s>0$ up to the horizon. This means that the prices are observed for $t=0$ and uncertainty starts later. From the time $t \geqslant s$ the prices, i. e. the output prices $\tilde{p}_t$, wage rates $\tilde{w}_t$ and prices of investment goods $\tilde{q}_t$, are random. The distributions of these random variables are supposed in this section not to be known: we assume that only the expectations are known, and higher moments of random variables are ignored. Also, we consider risk-neutrality. Under such assumptions the random cash flow is:

$$\tilde{C}_t = \tilde{p}_t f(K_t, L_t) - \tilde{w}_t L_t - \tilde{q}_t I_t, \tag{21}$$

the expectational cash flow is expressed as:

$$E(\tilde{C}_t) = E[\tilde{p}_t f(K_t, L_t) - \tilde{w}_t L_t - \tilde{q}_t I_t]. \tag{22}$$

The maximization of the expectational discounted cash flow according to (7) is:

$$E(\tilde{F}_s) = E_t \int_s^\infty e^{-r(t-s)} [\tilde{p}_t f(K_t, L_t) - \tilde{w}_t L_t - \tilde{q}_t I_t + \lambda_t (I_t - \dot{K}_t - \delta K_t)] dt \Rightarrow \Rightarrow \max, \tag{23}$$

where E_t denotes the operator of conditional expectation, assuming that:

$$E_t(\tilde{p}_t) = \begin{cases} p_t, & \text{if} \quad t<s \\ \mu_p, & \text{if} \quad t=s, \end{cases} \tag{24.1}$$

$$E_t(\tilde{w}_t) = \begin{cases} w_t, & \text{if} \quad t<s \\ \mu_w, & \text{if} \quad t=s, \end{cases} \tag{24.2}$$

$$E_t(\tilde{q}_t) = \begin{cases} q_t, & \text{if} \quad t<s \\ \mu_q, & \text{if} \quad t=s, \end{cases} \tag{24.3}$$

where μ_p, μ_w, μ_q denote the approximations of the expectations of $\tilde{p}$, $\tilde{w}$, $\tilde{q}$, if $t=s$. We are interested only in the time moment $t=s$. Further development of the dynamics of prices which is necessary for the next step of solving a dynamic planning problem demands more information Ω $(=\mu_{tj}$, $t=s$, $s+\Delta t, \ldots)$. The next step of dynamic planning involves an assumption that the initial conditions for $t=s$ are considered as "zero-state", and then the process for $s'=s+\Delta t$ is repeated again.

In further considerations we substitute random variables with their expectations, if $t=s$. Then in (23) $\cong$ will be used instead of $=$, according to Jensen's inequality[2]:

$$E(F_s) \cong \mu_{F_s} = \int_s^\infty e^{-r(t-s)} [\mu_p f(K_t, L_t) - \mu_w L_t - \mu_q I_t + \lambda_t (I_t - \dot{K}_t - \delta K_t)] dt, \tag{25}$$

where μ_F denotes approximated expectational profits. In this case the results obtained in Section 1 are valid here, too.

The exact timing of the moment s is left open here, about it see, for example, Nickell [5, p. 250—252]. Now we can interprete the problem in such a way that the optimal investment path is determined by the initial conditions for differential equation (6). The initial conditions are derived as the values of K, L for the time moment $t=s$ by (12) and (13) (from $\mu_p \partial f(K_s, L_s)/\partial L_s = \mu_w$, $\mu_p \partial f(K_s, L_s)/\partial K_s = \mu_q \delta$). Now the initial conditions for the time $t=s$ reflect changed conditions for the firm. More precisely, the expectations of the prices μ_p, μ_w, μ_q reflect the expected changes in the demand and supply conditions. The response of all the firms interacting with either the demand for their product, or the supply of factor inputs generates the prices. So the expectations of prices determine the optimal size of the capital stock K_s and the employment L_s at the time $t=s$. In this way the optimal investment strategy has to have such properties that the capital stock would reach size K_s for the time moment s (with taking into account restriction (6)). Labour was considered freely adjustable.

This treatment of the problem ignores the internal adjustment costs of the firm. According to [4, p. 652] this problem is reflected by the following equation (based on expression (10)):

$$\lambda_s = \mu_q. \tag{26}$$

According to [4] this means that the firm can reverse freely its investment decision as more information becomes available. This is possible because if the shadow price of the investment and the expected price of the investment goods are equal, the firm can sell off its recently obtained equipment in the case of changed conditions with no loss (that does not affect the capital stock from earlier periods, because then user value of the previous investments $\lambda_t (t<s)$ has to be calculated with taking into account the depreciation rate). In this way the influence of uncertainty plays a relatively minor role in the investment decision. Craine shows that if there are adjustment costs, equation (26) is no longer true, and the investment decision is irreversible. In this case (26) changes into equality $\lambda_s = \mu_q + h$ ($\lambda_s > \mu_s$, expression (1.4) in [4, p. 651]), where the member h reflects the influence of adjustment costs. So, in the optimum case the shadow price has to be higher than the expected price of the investment goods. In other words, the firm cannot sell off its obtained equipment to raise the marginal product of capital [4, p. 652]. Then the firm must rely on the internal valuation of the new investment. Craine's final conclusion is that in this case the investment decision depends mainly on uncertainty.

In this paper, however, we do not pay any attention to adjustment costs. One of the reasons of such a behaviour is presented in Nickell's work: "In general, we may argue that the assumption of strictly convex adjustment costs, though extremely convenient analytically, is really a simple mechanistic device for obtaining reasonable investment paths

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[2] Jensen's inequality says that the expected profits are always smaller than or equal to the profits computed using the approximations.

from the profit-maximizing activities of firms rather than a realistic attempt to analyze the investment decision" [5, p. 249]. Another reason for avoiding adjustment costs is connected with difficulties of finding an adequate analytical form for such costs. For example, Craine [4, p. 649] uses expression $aI_t^2/2$ for adjustment costs, Abel [6, p. 228] uses γI_t^β (for investment and adjustment costs together), Hartman [7, p. 259] leaves the question open and simply assumes that $C(I, q)$ denotes the investment cost function with no further specification, etc.

In general, adjustment costs can be expressed as part of the stochastic investment cost function $C(I_t, \tilde{q}_t)$. Then in our model expression $\tilde{q}_t I_t$ in (22) has to be replaced by $C(I_t, \tilde{q}_t)$. After the substitution of the expectations of prices with their approximations and differentiating with respect to I on the basis of (10):

$$\lambda_s = \partial C(I_s, \mu_q)/\partial I_s. \tag{27}$$

Now the firm's shadow price (user value) of the investment λ_s is greater than the expected price of the investment in the case of reselling the investment goods iff $\partial C(I_s, \mu_q)/\partial I_s > \mu_q$. In this case the previous considerations about irreversibility of investment decisions hold. Obviously, this means that the increment of investment cost function per unit of investment is greater than the investment's expected price. The stochastic adjustment process including an investment cost function with a random price of investment goods is complicated and has to be treated as a separate problem.

To conclude, it seems that so far the best solution has been presented in Nickell's work [5] where no adjustment costs but the combined influence of uncertainty and distributed lags is used. Nickell shows that this combination yields a solution similar to solutions that arise from adjustment cost models, but closer to reality. However, just because Nickell's results are hard to connect with this paper's material [3], we use the assumption about the possibility of reversing investment decisions (the obtained capital equipment can be sold without adjustment costs) as more information is available. To compensate for some underestimation of uncertainty in this section, we try to connect our results with risk aversion.

3. Maximum profits in the case of risk aversion with optimal values of decision variables

In Section 2 the problem of optimizing investment paths in a profit maximizing problem was given with a consideration of risk neutrality. In (25) we used Jensen's inequality, according to which the expectational profits are always less than or equal to the approximated expectational profits. In a risk aversion case the expectational profits have to be much less than that.

In Section 2 no higher moments of random variables than the first were taken into account; this means that variances were not involved. In this Section we consider also variances. To do so it is necessary to specify more precisely random variables. So, we assume that random variables, used in Section 2, belong to a linear class. In [8, p. 56] it is stated that, "... random variables $\tilde{V}_1$, $\tilde{V}_2$, ... form a linear class if their standardized values

$$\tilde{Z} = (\tilde{V}_i - E(\tilde{V}_i))/\sigma(\tilde{V}_i) \ \forall i$$
$$\text{with} \quad E(\tilde{Z}) = 0, \ \sigma(\tilde{Z}) = 1 \tag{28}$$

[3] In Nickell's work uncertainty is specified differently: the main problem there is the timing of predictable demand jump with including distributions of random variables in time.

have the same density function. Within a linear class all distributions can be transformed into one another merely by a shift and a proportional extension. Since $E(\tilde{V})$ acts as a measure for the shift and $\sigma(\tilde{V})$ as a measure for extension around the mean, these two parameters are sufficient to characterize the whole distribution, given the shape of the standardized distribution".

In the case of normal distribution (which is one of these classes and can often be used as a good approximation for the class of distributions occurring in real-life problems [8, p. 57]), the density function of a random variable, standardized according to (28) is:

$$\varphi_z(z;\ 0,\ 1)=1/(2\pi)^{0.5}e^{-0.5z^2}. \tag{29}$$

By integration this yields:

$$P(z<z^0)=1/(2\pi)^{0.5}\int_{-\infty}^{z^0} e^{-0.5z^2}\,dz. \tag{30}$$

Now, assuming like in Section 2 that the cash flow at the time t is random ($\tilde{C}_t$), and denoting it for the time moment of $t=s$ as $\tilde{C}_s=\tilde{C}$, we obtain (with the assumption that $\tilde{C}$ is normally distributed)[4]:

$$Z=(\tilde{C}-E(\tilde{C}))/\sigma(\tilde{C}),\quad \text{and} \tag{31}$$

$$P[(\tilde{C}-E(\tilde{C}))<Z^0_\alpha\,\sigma(\tilde{C})]=1-\alpha. \tag{32}$$

In (32) α is subjective permissible level of risk. In other words, according to (32) bigger than or equal to $Z^0_\alpha\sigma(\tilde{C})$ deviations from mean are supposed to occur with the probability α. Now we may say that with the probability $1-\alpha$:

$$\tilde{C}<E(\tilde{C})+Z^0_\alpha\,\sigma(\tilde{C}). \tag{33}$$

Generally it is known about Z^0_α that

$$Z_\alpha\begin{cases}<0, & \text{if } \alpha<0.5 \text{ (risk aversion)}\\ =0, & \text{if } \alpha=0.5 \text{ (risk neutrality)}\\ >0, & \text{if } \alpha>0.5 \text{ (risk loving).}\end{cases} \tag{34}$$

Consequently, in a risk-neutral case:

$$\tilde{C}<E(\tilde{C}) \tag{35}$$

with the probability 0.5. With the probability $1-\alpha$ for risk aversion:

$$\tilde{C}<E(\tilde{C})-Z^0_\alpha\,\sigma(\tilde{C}), \tag{36}$$

where Z^0_α is determined through (30) and α, $\sigma(\tilde{C})$ is the standard deviation of a normally distributed $\tilde{C}$. As to $E(\tilde{C})$, it is known by Jensen's inequality that $E(\tilde{C})\leqslant\mu_C$, where μ_c stands for approximated expectational profits:

$$\mu_C=\mu_p f(K^*,\ L^*)-\mu_w L^*-\mu_q I^*, \tag{37}$$

where * denotes optimal values of decision variables. Rewriting (36) as:

$$\tilde{C}<\mu_C-Z^0_\alpha\,\sigma(\tilde{C}) \tag{38}$$

we obtain a theoretical upper bound of the random profits $\tilde{C}$ for the given level α.

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[4] All considerations now and later in Section 3 are made at the time $t=s$ without discounting the results which can be simply added.

Now it is possible to calculate the value of the right side of (38). For calculating $\sigma(\tilde{C})$ it is essential to notice that the distribution of $\tilde{C}$ is a joint distribution. With assuming independent random variables $\tilde{p}, \tilde{w}, \tilde{q}$, the density function of $\tilde{C}$ is given by

$$\varphi_c = \varphi_p \varphi_w \varphi_q, \tag{39}$$

where $\varphi_p, \varphi_w, \varphi_q$ must have such properties (or some of them must be degenerated) that the joint distribution of $\tilde{C}$ is normal. In this case:

$$\sigma_c^2 = \sigma_p^2 (f(K^*, L^*))^2 + \sigma_w^2 (L^*)^2 + \sigma_q^2 (I^*)^2. \tag{40}$$

In (38) $Z_\alpha^0 \delta(\tilde{C})$ stands for the part of expectational revenues which are not expected by a risk-aversive decision-maker. For that reason in [9, p. 432] the value $|Z_\alpha^0 \delta(\tilde{C})|$ is called a "safety buffer" for the decision-maker for the case anything goes wrong. Through α it depends on the decision-maker's personal attitude towards risk.

Conclusion

The model presented above allows us to give the optimal investment problem a rather precise specification (initial conditions of labour and capital stock, marginal flows in optimum case, etc.). Our treatment which uses time-continuous variables reflects the expected changes in capital stock (for the chosen investment strategy) for uncertain future prices. Changes in prices reflect changes in demand and supply conditions. The main result of this paper is a model for deriving the optimal investment path(s) depending on the expected values of prices. There is not, and there probably cannot exist any unique solution for the optimal investment path.

The optimal solution is derived without using variances of prices for risk neutrality. In such case we may say that the maximized profits for the optimal investment path(s) are obviously overoptimistic. So the risk-aversive case in Section 3 is meant to compensate for the neglect of variances in Section 2. The dynamics of prices is described rather poorly, but in this model the expectations of prices can be taken as information for the first step in solving the dynamic planning problem. Next steps demand more information. However, few special ways for describing the dynamics of prices without oversimplification or serious mathematical complications are available (for example, the Wiener process from [6, p. 228]).

The model presented here makes it possible to find the values of the optimal capital stock and labour with optimal investment path(s) for some moment of time in uncertain future. The model does not involve adjustment costs and exact timing of uncertainty, but it gives perfectly described initial conditions for the initial state and for some later state of the firm in uncertain future, conditional on prices.

REFERENCES

1. *Fama, E. F.* Multiperiod consumption-investment decisions // Am. Econ. Rev. 1970, **LX**, N 1, 163—174.
2. *Merton, R. C.* An intertemporal capital asset pricing model // Econometrica, 1973, **41**, N 5, 867—888.
3. *Chen Son-Nan, Moore, W. T.* Investment decisions under uncertainty: application of estimation risk in the Hillier approach // J. Fin. and Quant. Anal., 1982, **XVII**, N 3, 425—440.

4. *Craine, R.* Investment, adjustment costs and uncertainty // Intern. Econ. Rev., 1975, **16**, N 3, 648—661.
5. *Nickell, S.* Uncertainty and lags in the investment decisions of firms // Rev. Econ. Studies, 1977, **XLIV (2)**, N 137, 249—263.
6. *Abel, A. B.* Optimal investment under uncertainty // Am. Econ. Rev., 1983, N 73, 228—233.
7. *Hartman, R.* The effects of price and cost uncertainty on investment // J. of Econ. Theory, 1972, N 5, 258—266.
8. *Sinn, H.-W.* Economic decisions under uncertainty. Amsterdam, 1983.
9. *Bretschneider, S., Schroeder, L.* Revenue forecasting, budget setting and risk // Socio-Econ. Plan. Sci., 1985, **19**, N 6, 431—439.

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OPTIMAALSED INVESTEERINGUD MÄÄRAMATUSE TINGIMUSTES

On vaadeldud investitsiooniprotsessi stohhastilist optimeerimist optimaalse investitsioonifunktsiooni leidmise kaudu. Investitsioonifunktsioon on tuletatud dünaamilisest stohhastilisest kasumi maksimeerimismudelist diferentsiaalvõrrandi lahendina. Algtingimused diferentsiaalvõrrandi jaoks on määratud firma toodangu, tööjõu ja investitsioonikaupade eeldatavate hindade kaudu. Oletades hindade juhuslikkust on investitsioonifunktsioon määratud erinevatel ajamomentidel hindade keskväärtuste aproksimatsioonide kaudu kasumi maksimeerimisprotsessis.

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Ильмар ЛЕПИК

ОПТИМАЛЬНЫЕ ТРАЕКТОРИИ ИНВЕСТИЦИЙ В УСЛОВИЯХ НЕОПРЕДЕЛЕННОСТИ

Рассматривается стохастическая оптимизация инвестиционного процесса путем определения оптимальной инвестиционной функции. Эта функция выводится из стохастической динамической модели максимизации прибыли как решение дифференциального уравнения, начальные условия которого зависят от ожидаемых затрат и цен на продукцию фирмы. В предположении случайности цен на выпускаемую продукцию, на рабочую силу и на инвестиционные товары инвестиционная функция определяется через аппроксимацию математических ожиданий цен (в разные моменты времени) в процессе максимизации прибыли.

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