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SOME QUESTIONS AND EXAMPLES OF AN ASPECT-WISE DECOMPOSITION OF THE ECONOMIC OPTIMUM PLAN PROBLEM*

An adequate optimum plan problem of economy contains a number of questions concerning its contents and several formal characteristics. Here an attempt is made to explain heuristically the possibilities and the expediency of solving such a problem with the help of decomposition by questions as well as by formal characteristics. It appears that, in case of decomposition by these aspects, the coordination of subproblems is accomplished by means of consultations, and, in case of some problems, it is feasible to show the convergency of the method.

1. Introduction

An adequate optimum plan problem of economy comprises several aspects which we classify as aspects concerning the contents of the questions of the problem, and formal characteristics of the problem.

The questions are, for example, where, what, how and when to produce, and to whom and how to distribute. In other words, we can speak of the questions of the location, specialization, concentration, unification, etc., of the production. There also exist questions of the structure, location, level, etc., of consumption.

In addition, an adequate problem has a number of formal characteristics describing the form of the interrelations between economic indices; for example, non-linearity, randomness, discreteness, dynamics, etc.

As already said, an adequate problem comprises numerous questions as well as the characteristics described above. Naturally, it is complicated to set, compile, (quantify) and, especially, to solve such a problem directly. Hence, an interesting question arises as to whether it would be in principle possible and, at the same time, practically expedient to solve such a problem with the help of decomposition by questions as well as by characteristics, i.e. using the method of an aspect-wise decomposition. The principle of the method is, in brief, as follows. A complicated initial problem is solved approximately through the use of a system of simple subproblems. In this case, the subproblems describe separate aspects, only. The subproblems are combined into a uniform system by means of coordination.

In case of decomposition by questions, for example, the following comparatively independent problems could be considered as subproblems: the problems of the diversification of production, location of production, the choice of the production technology, etc.

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Naturally, the system being constructed in such a way, one subproblem is solved using the solutions of the other subproblems as the initial data. This yields a coordination system called consultation in the present paper.

The elements of decomposition by questions and consultation have been used by A. Aganbegyan and K. Bagrinovsky [1—3]. They have used these terms to compile concrete systems of models. The present paper, on the contrary, is an attempt to explain the general principles and elements of decomposition by questions.

In case of decomposition by characteristics, the subproblems represent problems where certain characteristics are abstracted. For example, in one problem, randomness is abstracted, in others, non-linearity, integrity, etc. The exchange of the solutions of separate problems is necessary for coordination here, as well. No treatments of decomposition by characteristics are known in literature.

At the beginning of the paper, the general problem of decomposition by questions is set. Further, a simpler problem, in case of which the convergency of the consultation method can be proved, is dealt with. Then, a brief heuristic treatment of decomposition by characteristics is presented. To achieve concrete results in this field, much work is to be done, as yet.

2. On Question-Wise Decomposition of a General Problem

A general multi-question problem of economy is formulated here as follows:

$$\max_{x} f(x) \Big\} \; x \in X \tag{1}$$

where the plan vector x belongs to a constrained closed convex set X, and objective function $f(x)$ is a concave (convex upward) differentiable scalar function.

Let us assume that, in essence, the plan x describes several problems in a complex way. For example, the plan index x_{kjl} determines the volume of production in the region k, branch j, according to technology l; thus, the plan x provides a solution to three problems: the production structure, location, and technological structure. However, all these problems are solved complexly, and therefore we cannot see the solutions to the separate problems explicitly.

Let $z, \ldots, w$ denote the vectors of the plans of the separate problems included in the complex plan x. From these vectors we form a new vector $(z, \ldots, w)$. On the grounds of the above-said we can say that $(z, \ldots, w) = S(x)$, where S is a given operator. Let us assume that the plans of the separate problems determine the complex plan unambiguously, or that $x = S^{-1}(z, \ldots, w)$.

Now we substitute, in the problem (1), the vectors of the separate plans, $(z, \ldots, w)$, for the complex plan vector x, and get

$$\max_{(z, \ldots, w)} \; f(S^{-1}(z, \ldots, w)) \Big\} \; S^{-1}(z, \ldots, w) \in X \tag{2}$$

In order to simplify, let us denote: $f(S^{-1}(z, \ldots, w)) = F(z, \ldots, w)$, and $U = \{(z, \ldots, w) \mid S^{-1}(z, \ldots, w) \in X\}$; thus problem (2) will obtain the form:

$$\max_{(z, \ldots, w)} \; F(z, \ldots, w) \Big\} \; (z, \ldots, u) \in U \tag{2a}$$

For a decomposed solution of the problem (2a), a component-wise optimization [4] or component-wise movement towards the gradient might be used. Component-wise optimization consists in the following. Let us have an initial approximation $(z^0, \ldots, w^0)$ and an approximation k, denoted by $(z^k, \ldots, w^k)$. Approximation $k+1$ is now found by components $z, \ldots, w$. Namely, the problem (2a) is maximized for one problem vector at a time, and the vectors of the other questions are fixed at the value corresponding to step k. Thus, the approximation $k+1$ is found by means of the following coordination algorithm;

$$\left.\begin{array}{l}\max\limits_{z} F(z, \ldots, w^k) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \max\limits_{w} F(z^{k+1}, \ldots, w)\end{array}\right\} \begin{array}{l}(z, \ldots, w^k) \in U \\ \\ (z^{k+1}, \ldots, w) \in U\end{array} \tag{3}$$

The conditions necessary for the convergence of the algorithm (3) in case of the problem (2a) will not be dealt with. However, a simplified variant of the problem (2a), for which the convergence of the algorithm (3) has been proved, will be studied below.[1]

Expression (3) does not constitute a hierarchical structure. In other words, all the subproblems belong to the same level. Coordination is carried out by way of informing the separate problems about the solutions of the other ones. There would be no sense in forming some coordinating centre, since in essence it would act as an information intermediary (consultor), only, having no functions of transforming information.

It is of interest to note that, in case of a converged solution, the values of the objective functions of all the separate problems are equal, since all the separate problems maximize the same function, though with regard to different variables.

In conclusion we may say that, in case of the convergence of algorithm (3), a question-wise decomposition of a general problem would be feasible. At the same time, for optimization by some separate problem, we need the solutions to the other problems as initial data, or mutual consultations.

3. On Question-Wise Decomposition of a Linear Problem of a General Structure

Below the contents of a concrete three-question problem of planning production will be studied.

Let plan index x_{kjl} of the initial problem be complex as regards the questions, and let it describe the volume of production in the region $k = 1, \ldots, p$, in branch $j = 1, \ldots, n$, produced by means of technology $l = 1, \ldots, s$. Thus, the plan of the initial problem, complex with regard to the questions, is vector (x_{kjl}), $k = 1, \ldots, p$, $j = 1, \ldots, n$, and $l = 1, \ldots, s$. Let the plan (x_{kjl}) have unilateral balance constraints on its input-output, and an objective function. Thus, the initial problem is:

$$\left.\max_{(x_{kjl})} \sum_{ikjl} c_{ikjl} x_{kjl}\right\} \sum_{kjl} a_{ikjl} x_{kjl} \geqslant b_i \qquad i = 1, \ldots, m \tag{4}$$

where c_{ikjl} is the efficiency index, a_{ikjl} the technological input-output coefficient, and b_i the obligation of the final product for means $i = 1, \ldots, m$. To proceed from the plan of the complex problem, (x_{kjl}), to the plans of the separate problems, we define the following new plan indices.. Let the production in the region k be $z_k = \sum\limits_{jl} x_{kjl}$, and thus vector $z = (z_k)$, $k = 1, \ldots, p$ describes the regional location of the production. Let the share of the production of branch j in region k be $v_{kj} = \sum\limits_{l} x_{kjl}/z_k$; then vector $v_k = (v_{kj})$, $j = 1, \ldots, n$ describes the branch structure of the production in region k. And, finally, let the share of technology l in branch j in region k be $w_{kjl} = x_{kjl}/\sum\limits_{l} x_{kjl}$. Thus vector $w_{kj} = (w_{kjl})$, $l = 1, \ldots, s$ desribes the technological structure of branch j in region k.

Using the plan vectors of the separate problems, z, v_k and w_{kj}, $k = 1, \ldots, p$ and $j = 1, \ldots, n$, the problem (4) can be written in the following equivalent form:

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[1] The solution of problem (2a) by way of a component-wise movement towards the gradient will not be described. Let us only note that the structure of the algorithm is analogous to expression (3). The fact that we need not know the function $F(\ldots)$ but only its gradients, may appear as an advantage in this case.

$$\left.\max_{z_k,\, v_{kj},\, w_{kjl}} \sum_{ikjl} c_{ikjl} z_k v_{kj} w_{kjl}\right\}$$

$$\sum_{kjl} a_{ikjl} z_k v_{kj} w_{kjl} \geqslant b_i$$

$$\sum_j v_{kj} = \sum_l w_{kjl} = 1; \; z_k,\, v_{kj},\, w_{kjl} \geqslant 0, \tag{5}$$

$$k=1, \ldots, p; \; j=1, \ldots, n; \; l=1, \ldots, n$$

Analogously to algorithm (3), we obtain the following instructions for the component-wise optimization of problem (5):

$$\left.\max_{z_k} \sum c_{ikjl} \underset{kj}{v^k} \underset{kjl}{w^k} z_k \right\} \sum a_{ijk} \underset{kj}{v^k} \underset{kjl}{w^k} z_k \geqslant b_i, \; z_k \geqslant 0 \tag{5a}$$

$$\left.\max_{v_{kj}} \sum c_{ikjl} \underset{k}{z^{k+1}} \underset{kjl}{w^k} v_{kj} \right\} \sum a_{ijk} \underset{k}{z^{k+1}} \underset{kj}{w^k} v_{kj} \geqslant b_i$$

$$v_{kj} \geqslant 0, \sum_j v_{kj} = 1 \tag{5b}$$

$$\left.\max_{w_{kjl}} \sum c_{ikjl} \underset{k}{z^{k+1}} \underset{kj}{v^{k+1}} w_{kjl} \right\} \sum a_{ijk} \underset{k}{z^{k+1}} \underset{kj}{v^{k+1}} w_{kjl} \geqslant b_i$$

$$w_{kjl} \geqslant 0, \sum w_{kjl} = 1 \tag{5c}$$

As to its economic contents, problem (5a) is the problem of regional location; problem (5b) the problem of the branch structure of production in each region, and problem (5c) the problem of the technological structure of production in each region and each branch.

Coordination takes place by way of mutual consultations between the subproblems (5a—5c). As a result of the consultations, the objective functions of the subproblems (consultation by goals) as well as their technological parameters (technological consultation) are corrected.

As to the use of algorithm (5a)—(5b), we should note that it is not known to be convergent, and it does not afford any economy in the size of the problems. Indeed, problem (5c) is of the same size as the initial problem (4). However, as to the questions, problem (5c) is simpler than problem (4), including only one question — in our case, the choice of the technological structure of production. In this respect, the problem (5c) can be better "seen through" by an expert than problem (4).

Consequently, the study of the algorithm (5a)—(5c) may be useful in cases where planning is accomplished relying on the intuition of question-oriented experts, and especially when few iterations are applied. With the help of the algorithm given it is possible to learn which plan indices are relatively stable, and then begin with planning them. In this way, one iteration brings us relatively near to the optimum.

4. An Analysis of a Non-Linear Problem of a Special Structure

Here we set a non-linear three-question problem having a special structure, for which the convergency of the method of component-wise optimization has been proved [4].

Let the initial problem be:

$$\left.\max_{(x_{kjl})} f((x_{kjl}))\right\} \sum_{kjl} x_{kjl} \leqslant \bar{x}, \quad \sum_{jl} x_{kjl} \leqslant \bar{x}_k$$

$$k=1, \ldots, p \tag{6}$$

where (x_{kjl}) is the plan vector $(k=1, \ldots, p;\; j=1, \ldots, n;\; l=1, \ldots, s)$ and $\bar{x}_k$ and $\bar{x}$

are given scalars. The objective function, $f((x_{kjl}))$ is concave (convex upward) and differentiable.

Let the economic contents of the plan index x_{kjl} be the same as in the previous paragraph. Thus, $\bar{x}$ is the constraint on the total economic production, and $\bar{x}_k$ is the constraint on the production in region k. As previously, the plan indices of individual problems are denoted by vectors (z_k, v_{kj} and w_{kjl}). Now we can rewrite the problem (6) in the following equivalent form:

$$\left.\max_{(z_k,\, v_{kj},\, w_{kjl})} f((z_k v_{kj} w_{kjl}))\right\}$$

$$\sum_k z_k \leqslant \bar{x},\ z_k \leqslant \bar{x}_k \tag{7}$$

$$\sum_j v_{kj} = \sum_l w_{kjl} = 1;\ z_k,\ v_{kj},\ w_{kjl} \geqslant 0$$

$$k = 1, \ldots, p,\ j = 1, \ldots, n,\ l = 1, \ldots, s$$

Applying the method of component-wise optimization for the problem (6a) we obtain the following algorithm:

$$\left.\max_{(z_k)} f((z_k \underset{kj}{v^k}\ \underset{kjl}{w^k}))\right\} \sum z_k \leqslant \bar{x};\ 0 \leqslant z_k \leqslant \bar{x}_k \tag{7a}$$

$$\left.\max_{(v_{kj})} f((\underset{k}{z^{k+1}} v_{kj} \underset{kjl}{w^k}))\right\} \sum_j v_{kj} = 1;\ v_{kj} \geqslant 0$$

$$j = 1, \ldots, k;\ k = 1, \ldots, p \tag{7b}$$

$$\left.\max_{(w_{kjl})} f((\underset{k}{z^{k+1}} \underset{kj}{v^{k+1}} w_{kjl}))\right\} \sum_l w_{kjl} = 1;\ w_{kjl} \geqslant 0$$

$$l = 1, \ldots, s,\ j = 1, \ldots, k,\ k = 1, \ldots, p \tag{7c}$$

As already said, the algorithm (7a)—(7c) is convergent and, as it can easily be seen, decreases the size of the subproblems to be solved. Namely, the problems (7a)—(7c) can be arranged in a hierarchical structure: problem (7a), comprising p unknowns, is solved on the top level. On the following, lower level, p problems are solved (7b), each comprising n unknowns. On the lowest level, pn problems, each with s unknowns, are solved. However, such a hierarchic arrangement is only formal, since it does not correspond to the coordination scheme. From the standpoint of coordination, the subproblem of every different level must receive the solutions of all other problems. Thus, from the standpoint of coordination, all subproblems belong to the same level, and only mutual consultations are carried out. Consequently, only an informative centre could be organized here.

5. On the Decomposition of an Optimum Problem by Characteristics

A truthful composed mathematical problem of an economic optimum plan must comprise a number of formal characteristics: non-linearity, randomness, dynamics, discreteness of functions, integrity of variables, sizableness, etc. Let us describe the existence of such characteristics in the problem by the values of parameters $a, \ldots, h$, and then insert these parameters in their explicit forms in the optimum problem. Now the so-called multi-characteristic economic problem can be written as follows:

$$\max_x C(x, a, \ldots, h) \tag{8}$$

where x is the plan, and C the given operator. The equalization of the parameter of some characteristic to zero in problem (8) is to be understood as total abstraction of the relevant characteristic. A positive value of the parameter of some characteristic means that the relevant characteristic is taken into account.[2]

The decomposition of problem (8) by characteristics is extremely complicated and little studied, and, therefore, we shall confine ourselves to the discussion of two heuristic approaches.

Firstly. An investigation of the multi-characteristic problem (8) reveals that, by means of abstraction, a number of one-characteristic (or few-characteristic) special problems can be formed from it. In so doing, every special problem yields a different plan. Let us denote these plans using the mark of the corresponding unabstracted characteristic. So we obtain:

$$\begin{array}{c} \max\limits_{x_a} C(x_a,\, a,\, 0,\, \ldots,\, 0) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \max\limits_{x_h} C(x_h,\, 0,\, \ldots,\, 0,\, h) \end{array} \tag{9}$$

The application of this technique is widespread in practice. The compilation of the final compromise plan on the basis of plans $x_a, \ldots, x_h$ is carried out by the so-called analyzer [6—8]. The major difficulty in applying this technique is the fact that, in connection with abstraction, the plans $x_a, \ldots, x_h$ deviate substantially from the actual optimum plan, at the same time the analyzer cannot easily assess either the direction or the size of the deviations. For the benefit of the analyzers it would be advisable to provide fidelity boundaries for all the plans. Let us denote them by $X_a, \ldots, X_h$. Now the analyzer has to fix the compromise plan falling into the area $X_a \cap \ldots \cap X_a$; this is relatively simpler and more trustworthy.

Secondly. In order to decrease the deviations of the solutions of one-characteristic problems, attempts might be made to supplement or conform such problems so that it would be possible to avoid, to some extent, deviations from the actual optimum resulting from abstraction. At that, the supplemented problem ought to be substantially simpler than the corresponding unabstracted one. To supplement a separate problem, it seems to be natural to use the solutions to the other problems. Denoting the supplemented problems by $G_a, \ldots, G_h$, we obtain the following coordination instructions for finding $k+1$ approximations:

$$\begin{array}{c} \max\limits_{x_a} G_a(x_a,\, \underset{b}{x^k},\, \ldots,\, \underset{h}{x^k}) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \max\limits_{x_h} G_h(x_h,\, \underset{a}{x^{k+1}},\, \ldots,\, \underset{g}{x^{k+1}}) \end{array} \tag{10}$$

where $x_a, \ldots, x_h$ are the plans of the separate supplemented problems.

For example, let the two-characteristic initial problem be stochastic and non-linear. On its basis two special problems are formed. Let one of them be stochastic and linear, and the other one non-linear but determined. Thus, both special problems are simpler than the initial one. At the same time, the former of them fixes the nomenclature of production more trustworthily and the volume of production less trustworthily, and the latter — vice versa. Now let us try to supplement these problems by each other. To this end, the nomenclature of production in the non-linear problem is fixed according to the preliminary solution of the stochastic problem, and the input-output coefficients of the stochastic problem are corrected in accordance with the solution of the non-linear problem.

As the above-described example proves, the application of the method of characteristic-wise decomposition is still an art at the present time.

[2] It is evident that the abstraction of characteristics simplifies the problem and reduces the so-called information costs. On the other hand, however, it lessens the truthfulness of the results, increases the negentropy of the results, thus decreasing also information returns [5]. Thus, there arises the problem of the choice of the characteristics to be taken into account; the study of this problem is out of the bounds of the present paper.

REFERENCES

1. К. Багриновский, О согласовании статистических моделей различных типов. Математические вопросы формирования экономических моделей. Новосибирск, 1970, 3—26.
2. А. Аганбегян, К. Багриновский, О согласовании глобальной и локальных задач планирования. Проблемы применения макроэкономических моделей в планировании. М. 1972, 53—68.
3. А. Аганбегян, К. Багриновский, О системе взаимосвязанных моделей планирования народного хозяйства. Проблемы планирования и управления экономическими целенаправленными системами. Новосибирск, 1972, 3—16.
4. J. Warga, Minimizing Certain Convex Functions. J. Soc. Industr. Appl. Math., 1963, 11, 3, 588—593.
5. Ü. Ennuste, On the Principle of the Economic Effectiveness of a Planning System. Eesti NSV TA Toimetised — Ühiskonnateadused, 1970, XIV, 4, 415—426.
6. J. Kornai, Theoretische Probleme bei Modellsystemen. Wirtschaftswissenschaft, 1968, 4, 543—552.
7. W. Dück, L. Wunderlich, Gedanken zum Aufbau von Modellensystemen der Operationsforschung. Mathematik und Wirtschaft. Berlin, 1972, 8, 194—208.
8. R. Swinth, Organizational Joint Problem-Solving. Management Science, 1971, 18, 2, 68—79.

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MAJANDUSE OPTIMUMPLAANI ÜLESANDE ASPEKTILISE DEKOMPOSITSIOONI PROBLEEME JA NÄITEID

Resümee

Majanduse optimumplaani adekvaatne ülesanne sisaldab rea aspekte — sisulisi probleeme ning formaalseid matemaatilisi omadusi. Käesolevas uurimuses on heuristiliselt püütud selgitada sellise ülesande lahendamise võimalusi ja otstarbekust aspektilise dekompositsiooni abil. Selgub, et siin on loomulik alamülesandeid koordineerida vastastikuste konsultatsioonide abil. Mõnede ülesannete puhul on võimalik näidata, et see meetod viib koonduvusele.

Eesti NSV Teaduste Akadeemia
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Toimetusse saabunud
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Ю. ЭННУСТЕ

О ПРИНЦИПЕ И ПРИМЕРАХ АСПЕКТНОЙ ДЕКОМПОЗИЦИИ ЗАДАЧИ ОПТИМАЛЬНОГО ПЛАНИРОВАНИЯ ЭКОНОМИКИ

Резюме

Адекватная задача оптимального планирования экономики содержит ряд аспектов (экономических проблем и формальных математических свойств). В статье сделаны некоторые попытки внести ясность в проблему возможности и целесообразности решения такой задачи с помощью аспектной декомпозиции. Оказывается, что здесь естественно координировать подзадачи при помощи взаимных консультаций и для некоторых классов такой метод сходится.

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