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ON OPTIMALITY CRITERIA FOR PRODUCTION SYSTEMS

In working out an optimum plan for any production system (enterprise, industry, national economy, etc.), an optimality criterion is needed. The complexity of this task is caused by the fact that a production system plan affects many politico-economic indices. We are faced with the problem of finding a criterion for a subsystem, so that the latter's optimization would ensure an optimum plan from the point of view of the criterion for the whole production system. Such a problem may arise not only in setting up criteria for subsystems, but also in decomposing the course of solving a production system's optimization problem.

Below, an attempt has been made at an abstract treatment of these problems. Explanations proceed from the theory about enterprises, industries and national economy. In order to reach quick solutions, strictness of treatment has often been sacrificed.

1. Criterion for a system

Let vector $u=(u_j)\in U^*$, $j\in K=(1, \ldots, n)$ denote the plan of a production system J, where U^* is a non-negative part of space E^n. The plan u shall satisfy conditions set by technologies and resources $F^*(u)=z^*$, where vector function $F^*(u)=(f_i^*(u))$ is called the technology function of a system, and vector $z^*=(z_i^*)$ is called a natural restriction on the system, $i\in M=(1, \ldots, m)$ where M is a set of resources. For example, the value of coordinate z_i^* can express resources of manpower at the disposal of system J. The set of plans $\bar{U}^*=(u:u\in U^*,\ F^*(u)=z^*)$ is called the set of technologically possible plans.

By its proportions, a production system may involve the whole national economy, industry or enterprise. As a rule, plans of production systems affect a number of politico-economic indices which conventionally are not expressed by one total index (see Table, column 2). Usually, a plan affects indices in opposite directions, i. e. a plan yielding better effect with respect to a certain index, gives less effect with respect to another one. To illustrate the above, increase of consumption would decrease accumulation, raising production quality reduces profit, etc.

The optimization criterion for a production system should encompass all the indices substantially affected by the respective system and relevant politico-economic effects. Let an index affected by the system's plan be $k\in R^*=\{1, \ldots, r^*\}$, i. e. the number of all relevant indices is r^*. Let $g_k(u)$ denote the effect achieved by the plan with respect to index k. Thus, effects of all indices are described by vector function $G^*(u)=(g_k(u))$.

In the further discussion of this problem we shall use the main concepts and results of an optimization problem with vector maximum.[1] Assume that plan u^1 dominates over

[1] Optimization problem with vector maximum and respective theorems were set up by Kuhn and Tucker in 1951. They are given in paper [1]. The main theorem has been published in Estonian in paper [2]. One theorem about a linear problem may be found in paper [3].

plan u^2 (possible plans), if $G^*(u^1) \geqslant G^*(u^2)$ and at least one coordinate represents a strict inequality, i. e. $g_k(u^1) > g_k(u^2)$.

We call every possible plan u^0 an effective plan if there is no other plan dominating over it.

Theorem 1. If u^0 is the effective plan of an optimization problem with vector maximum, there can be found a vector $v=(v_k)\geqslant 0$ and $\sum_{k\in R^*} v_k = 1$, so that u^0 is the solution of the problem

$$\max_{u \in U^*} \left\{ g(u) = vG(u) : F^*(u) = z^* \right\}$$

(This theorem holds for both linear and non-linear problems.)

Thus, any effective plan can be found as the solution of an optimization problem whose criterion is the weighted sum of the indices $g(u) = \sum_k v_k g_k(u)$.

Naturally, the optimization plan of the system must be one of the effective plans. Thus, for the calculation of the total optimum it is necessary to know how to calculate values of the functions $g_k(u)$ and determine the weights v_k. It depends on the development of the economic science how many effects of politico-economic indices can be quantitatively determined and their weights estimated. Conventionally only a few effects of indices (consumption, profit, stock of production funds, etc.) are calculated. The level of a number of relevant indices is still not estimated (cultural standards of the people, working conditions, leisure time, production quality, consumption at different times, defense capacity of the country, etc.). A quantitative estimation of these indices, in the future, should be based on respective theories. As to the estimation of weights v_k, it presents some difficulties [4] and should be based on a theory of its own.

Nevertheless, the mathematical optimization of economic decisions is already feasible if a properly chosen criterion is used. In determining a criterion for a production system, from among all the indices R^* affected, those should be excluded whose effects cannot be measured. This can be readily accomplished by traditional practice. Although economic leaders do not measure the effects of some indices, out of experience and traditions they can say which set-up of resources should be given preference in the interests of a certain index. Accordingly the constraints vector of the system should be complemented (for instance, by determining the investments on science, culture, labour protection, warranting quality, etc.). Thus, we obtain the new set of indices $k \in R = (1, \ldots, r)$, $r \leqslant r^*$, and the new constraints vector $z = (z_i)$, $i \in M$. The system's model should also be respectively complemented. Let us denote it by $F(u) = z$, and the set of directly bounded plans by $u \in U$. We call the set of plans $\bar{U} = (u : u \in U,\ F(u) = z)$ a set of admissible plans with respect to certain politico-economic indices (their necessary level being guaranteed by the equation of the system). If there exist weights v_k, $k \in R$ (for example, investments f and current costs c), the optimum plan among the set R of indices can be computed.

When no weights exist, there are two possibilities. We can solve the problem by using different weights (parametric programming) and select the best solution out of experience. To select the best solution is much easier than to determine the weights out of experience. It is also possible to retain, in the set of indices to be optimized, only one and the most suitable element (usual practice), obtaining an optimum plan with respect to that index. Which index should be chosen as a criterion, we shall discuss after having considered decomposition of the system.

Finally, it should be noted that although at present a number of politico-economic effects are not subjected to measurement, this possibility remains if the plans are represented in preference preordering on the basis of respective effects.

P. Fishburn states [5] that if the known axioms about preference preordering hold for the index k and there is a finite set of plans, then $g_k(u)$ does exist. His second important statement is that in case of quite natural assumptions $g_k(u)$ may be chosen to be additive $g_k(u) = \sum_i g_{ki}(u_i)$.

2. Criterion for a production system unit

(provided that the criterion of the system is given)

To proceed to this problem, a production system should be described as a set of certain production units. After that we can decompose the system, using the method of Lagrange multipliers. Then we shall try to interpret the results obtained so that the criteria for subsystems could be best chosen.

Model of the system

Let us consider a production system J in m-dimensional space of resources $i \in M = (1, \ldots, m)$. By resources we mean products, natural resources, manpower, energy, etc.

Let the production system consist of $n - m$ elements $(n > m)$. We shall call the element $j \in (m+1, \ldots, n)$ the production unit j, or simply unit j. Depending on the concrete choice of the system J the units may be technological lines, enterprises, industries, etc.

The position of unit $j \in N$ in the space of resources M is described by its input and output vector. Let z_{ij} be flow of resource i (amount per time unit), from the unit, if $z_{ij} > 0$, i. e. output of resource and flow into the unit, if $z_{ij} \leqslant 0$, i. e. input (dimension of z_{ij} is amount of resource i per time unit).

Assume that x_j is the intensity of the use of the unit j, and it is directly bounded by $x_j \in X_j$, where X_j is the set of admissible intensities of the use of the unit j.

Let us call the unit j's vector function $z_j(x_j) = (z_{ij}(x_j))$ the technological function determining economic structure, and vector $z_j = (z_{ij})$, i. e. values of the functions, economic structure of unit j.

According to the properties of production systems, the economic structure of the whole system is the sum of the respective structures of units:

$$z = (z_i) = \sum_{j \in N} z_j,$$

where coordinate z is the flow of resource i out of the system $(z_i > 0)$, or into the system $(z_i \leqslant 0)$.

Now assume that the economic structure of a system is bounded from below

$$z \geqslant \underline{z} = (\underline{z}_i),$$

the lower boundary being called the obligatory economic structure of the system.

Let us define vector $w = (w_i)$, $i \in M$ as follows: $w = z - \underline{z}$ and call vector w the system's selectable economic structure $w \in W = (w : w \geqslant 0)$.

We call the vector $x = (x_j) \in x = \prod_{h=m+1}^{n} x_h$ the technological structure of the system.

Now the system's equation can be written in the form of

$$\sum_{j \in N} z_j(x_j) - w = \underline{z}. \tag{1}$$

Assume that the system affects the politico-economic indices only through its economic structure. Only a part of it, w, can be varied. Consequently, w is the argument of the system's criterion. The latter can be written

$$\max g(w), \tag{2}$$

where $g(w)$ is the politico-economic effect achieved by the selectable economic structure w of the system.

In sum, we obtain the problem

$$\max_{w \in W} \left\{ g(w) : \sum z_j(x_j) - w = \underline{z}, \quad x \in X \right\}. \qquad (3)$$

Decomposition of the system

Let us set up following assumptions for the problem (3). Suppose that $g(w) = \sum_{i \in M} g_i(w_i)$,[2] where the member functions $g_i(w_i)$ are concave. Both asumptions are quite natural. The latter of them, concavity, expresses the law of decreasing effectiveness, as it exists in economy:

$$d^2 g_i / d w_i^2 \leqslant 0.$$

Now let us employ the following theorem [6].

Theorem 2. If the vectors $w^0 \in W$ and $x^0 \in X$ maximize, the function

$$L(w, x, \lambda) = \sum_{i \in M} g_i(w_i) + \sum_{i \in M} \lambda_i [\sum_{j \in N} z_{ij}(x_j) - w_i] \qquad (4)$$

containing real numbers λ, $i \in M$, $\lambda = (\lambda_i)$, then w^0 and x^0 are solution of the problem (3) for the case

$$\sum_{i \in M} z_{ij}(x_j) - w_i = \underline{z}_i, \quad i \in M.$$

To decompose the problem (4), we shall rewrite it in the shape of

$$\max_{\substack{w \in W \\ x \in X}} L\ (w, x, \lambda) = \sum_{i \in M} [g_i(w_i) - \lambda_i w_i] + \sum_{j \in M} [\sum_{i \in M} \lambda_i z_{ij}(x_j)]. \qquad (5)$$

The value of function (5) is at maximum when the values of the members enclosed in brackets reach their maximum, but do not contain common unknowns.

Consequently, to optimize the production system, the production units j should maximize their economic structures in λ-prices. To put it otherwise, unit j should maximize the difference of its incomes and outcomes (profit) in λ-price. If the system is considered to be dynamic, unit j should maximize profit throughout the whole period. Thus, when taking into account only one interval, one can get the erroneous impression that maximization of profit does not represent the criterion of the unit.

As to prices λ_i, it should be noted that they balance demand and supply in the system I. Conditions necessary for their existence are given paper [7].

Now it is clear that if a subsystem or unit does not know λ-prices (balancing demand and supply), then optimization of subsystems with the aim of maximum profit does not lead to an optimum for the whole system.

It follows from the expression (4) that λ_i depends on all the remaining indices. An analytical expression of λ-prices is difficult, but we can examine the direction in which λ_i-s depend on indices. But the numerical values can be found, and they are known as the solution of the dual problem of the production optimization problem.

3. On the choice of a suitable criterion

We have seen that the choice of a system's criterion will simultaneously determine the system of internal prices λ; by means of the latter a system can be decomposed. At present, economic practice lacks λ-prices (chronic deficit of some resources proves it), and we have no grounds to assume that maximization of the enterprise's profits during a plan period would ensure an optimum plan for the whole system.

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[2] $g(w)$ can also be transformed into an additive function. For example, $g(w) = A w_1^{a_1} \ldots w_m^{a_m}$ as a target function is equivalent to the function $g(w) = a_1 lg w_1 + , \ldots, + a_m lg w_m$.

To secure a material balance of the national economy, it seems to be practical to fix, in production units, the output and input of the most important resources (in form

Class of production system	Politico-economic indices with relevant effects	Suitable criteria	Notes
1	2	3	4
National economy			
	Population's material consumption	Consumption	Normal situation
	Balance of foreign trade		
Economic region	Leisure time of population	Balance of foreign trade	Big foreign trade
Union republic	Defence capacity	Need in manpower	Tense balance of manpower
	Natural resources		
	Cultural level of people		
	Volume of production funds		
	Material balance of national economy		
Complex of industries	Costs of resources	Costs of resources	Fixed production
	Capital investments	Capital investments	„
Industry	Volume of production funds	Reduced costs	„
Enterprise	Profit	Profit	Partly fixed production
Farm	Working conditions	Profit	Proper prices
Shop	Balance of products	Capital investments	Fixed production
	Capital investments	Reduced costs	„

of a restriction). In production and consumption of the remaining kinds of resources, profit can be maximized [8, 9]. This criterion stimulates an economical utilization of resources and an increased output of production useful for the unit.

In optimizing systems of national economy, maximization of material consumption should be considered as a general case, provided that the level of other costs in national economy is fixed. In special cases (unemployment, shortage of manpower, big share of foreign trade) it may be reasonable to maximize other indices.

In our view, the most suitable criteria are those given in column 3 of the Table with respective notes in column 4.

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TOOTMISSÜSTEEMI OPTIMISEERIMISKRITEERIUMIDEST

Resümee

Tootmissüsteemi (ettevõte, tööstusharu, rahvamajandus jt.) optimumplaani koostamisel ükskõik mistahes tasemel on möödapääsmatu süsteemi optimiseerimiskriteeriumi määramine. Ülesande keerukus seisneb asjaolus, et üldiselt mõjutab tootmissüsteemi plaan mitut majanduspoliitilist näitajat. Peale selle on süsteemi kriteeriumi määramisega tihedalt seotud probleem, milline peab olema mingi tootmissüsteemi alamsüsteemi kriteerium, et alamsüsteemi optimiseerimine viimase järgi tagaks optimaalse plaani kogu süsteemile kehtestatud kriteeriumi seisukohalt. Sama küsimus võib kerkida ka tootmissüsteemi optimumülesande lahendamiskäigu dekomponeerimisel. Neid küsimusi püütaksegi selgitada — üldiselt abstraktselt, mõningad selgitused aga esitatakse nii ettevõtte-, haru- kui ka rahvamajandusteooria kohta.

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О КРИТЕРИЯХ ОПТИМИЗАЦИИ ПЛАНА ПРОИЗВОДСТВЕННЫХ СИСТЕМ

Резюме

В статье рассматриваются две проблемы: во-первых, план производственной системы влияет на многие политико-экономические показатели и на основе их надо определить единый критерий; во-вторых, имеется критерий для всей системы и определяются критерии для подсистем.

Проблемы исследуются на основе абстрактных моделей, но некоторые интерпретации даны на базе теории предприятия, отрасли, экономического района и народного хозяйства.

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