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TOPOLOGICAL ALGEBRAS

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Constructing new Segal topological algebras from existing ones

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ABSTRACT

In this paper, we examine the properties of Segal topological algebras, looking at ways to construct new objects from existing ones. Equipped with the example of algebras (in the sense of vector spaces equipped with multiplication), we consider two approaches: one via a direct product of an arbitrary family of Segal topological algebras and another using the Dorroh extensions of the underlying topological algebras.

1. Introduction

Since the introduction of the term ‘Segal algebra’ in 1965 by Hans Reiter [4], various kinds and types of Segal algebras have found their way to different fields of mathematics, such as the study of group algebras of locally compact groups, algebras of continuous functions defined on different (quite often Banach or C^*-) algebras, quantum mechanics, mathematical logic, etc. Although all of these appearances of the term ‘Segal algebra’ have something to do with ideas that were originally written down by Irwin Ezra Segal in 1947 in a paper studying Wiener algebras, nowadays these terms in various branches of mathematics live quite different and independent lives.

Let us remember that for us, the term ‘topological algebra’ means a topological linear space (A, τ) over the field $\mathbb{K}$, where $\mathbb{K}$ stands for either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers, and where an associative and separately continuous multiplication is defined. The term ‘separately continuous multiplication’ means that for all $a, b \in A$ and each neighbourhood O of ab in (A, τ) there exist neighbourhoods U of a and V of b in (A, τ) such that $Ub = \{ub : u \in U\},\ \ aV = \{av : v \in V\} \subseteq O$.

This paper explores Segal (topological) algebras in the context that was earlier called ‘abstract Segal algebras’, where the focus was on Segal algebras associated with particular topological algebras (mainly C^*-algebras, Banach algebras, locally multiplicatively convex algebras, Fréchet algebras, etc.). In [1], Mart Abel formulated a definition for the Segal topological algebra in a way that it included all special cases of abstract Segal algebras that had been studied earlier. His definition can be stated as follows.

Definition 1.1. *We say that a topological algebra (A, τ_A) is a left (or right or two-sided) Segal topological algebra if there exists a topological algebra (B, τ_B) and an algebra homomorphism $f : A \to B$ such that*

1. *the image of A by f is dense in B, i.e. $\overline{f(A)} = B$;*
2. *f is continuous, i.e. $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$;*
3. *$f(A)$ is a left (respectively, right or two-sided) ideal of B.*

In this case we say that (A, τ_A) is a left (or right or two-sided) Segal topological algebra in (B, τ_B) via f.

2. Direct product of Segal topological algebras

Algebraists are often interested in ways to create new objects from known ones. One such way is forming a Cartesian product of several objects of the same type and adding some conditions on this Cartesian product in order to obtain another object of the same type. When one talks about algebras (with or without topology on them, in the sense of vector spaces that are equipped with multiplication, not as universal algebras), there are two common ways to turn the Cartesian product into an algebra again. One option is to consider the Cartesian product of algebras as linear spaces and define multiplication pointwise, which gives the so-called 'pointwise extension'. Another option is to use a different kind of multiplication and talk about a construction, which in the case of a nonunital algebra A over a field $\mathbb{K}$ gives us a unital algebra $A \times \mathbb{K}$. This construction is often called the 'unitization of A' or the 'Dorroh extension'.

As we are dealing with topological algebras instead of ordinary algebras here, we need to be sure that the algebraic operations are also (separately) continuous.

Lemma 2.1. *Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of algebras over the same field $\mathbb{K}$. The Cartesian product $A := \prod_{\alpha \in \mathcal{A}} A_\alpha$ is an algebra over $\mathbb{K}$ if for $x = (x_\alpha), y = (y_\alpha) \in A$ and $\lambda \in \mathbb{K}$, addition, multiplication and scalar multiplication are defined by*

$$x + y = (x_\alpha + y_\alpha), \quad xy = (x_\alpha y_\alpha), \quad \lambda x = (\lambda x_\alpha).$$

If $(A_\alpha, \tau_\alpha), \alpha \in \mathcal{A}$ are topological algebras over $\mathbb{K}$, then A is a topological algebra over $\mathbb{K}$ under the product topology $\tau = \prod_{\alpha \in \mathcal{A}} \tau_\alpha$.

Proof. It is known from [3], p. 19 that a similar statement holds for topological vector spaces, i.e. we know that A is a topological vector space over $\mathbb{K}$ under the product topology τ.

We will show in the following that the Cartesian product A is indeed an algebra and that the multiplication given above is separately continuous.

1. Given any two elements $x = (x_\alpha), y = (y_\alpha) \in A$, we can see that their product xy is an element of A by definition since for each $\alpha \in \mathcal{A}$, the product $x_\alpha y_\alpha \in A_\alpha$. Hence, A is an algebra over $\mathbb{K}$.

2. Let U be a neighbourhood of the product xy of two freely chosen elements $x = (x_\alpha)$, $y = (y_\alpha) \in A$. If U is a neighbourhood of xy in the product topology τ, then there exists an open set $U_\alpha \in \tau_\alpha$ for each $\alpha \in \mathcal{A}$ such that $x_\alpha y_\alpha \in U_\alpha$ and $\prod_{\alpha \in \mathcal{A}} U_\alpha \subset U$. Since the multiplication is separately continuous in every topological algebra A_α, we know that for every $\alpha \in \mathcal{A}$ there exist neighbourhoods V_α of x_α and W_α of y_α such that $x_\alpha W_\alpha \subset U_\alpha$ and $V_\alpha y_\alpha \subset U_\alpha$. Let
$$V := \prod_\alpha V_\alpha \quad \text{and} \quad W := \prod_\alpha W_\alpha.$$
Now, V is a neighbourhood of x, W is a neighbourhood of y, and it is clear that both $xW \subset U$ and $Vy \subset U$. Hence, the multiplication given above is separately continuous and (A, τ) is a topological algebra. □

The following result about the Cartesian product of topological spaces is well known.

Lemma 2.2. [2], p. 78. *For every family of topological spaces $\{(X_\alpha, \tau_\alpha) \mid \alpha \in \mathcal{A}\}$ and subsets A_α of respective spaces X_α, it holds that*

$$\overline{\prod_{\alpha \in \mathcal{A}} A_\alpha} = \prod_{\alpha \in \mathcal{A}} \overline{A_\alpha}.$$

In this article, the proofs for Theorems 2.3 and 3.1 are given only in the left-sided case, but the right- and two-sided versions of the proofs are analogous.

Theorem 2.3. *Let $\{(A_\alpha, \tau_\alpha^A) \mid \alpha \in \mathcal{A}\}$ and $\{(B_\alpha, \tau_\alpha^B) \mid \alpha \in \mathcal{A}\}$ be topological algebras, and for each $\alpha \in \mathcal{A}$ let $(A_\alpha, \tau_\alpha^A)$ be a left Segal topological algebra in $(B_\alpha, \tau_\alpha^B)$ via $f_\alpha : A_\alpha \to B_\alpha$. Let us denote $A := \prod_{\alpha\in\mathcal{A}} A_\alpha$ and $B := \prod_{\alpha\in\mathcal{A}} B_\alpha$. Then the map*

$$f : A \to B, \quad (a_\alpha) \mapsto (f_\alpha(a_\alpha))$$

is an algebra homomorphism, and $(A, \prod_{\alpha\in\mathcal{A}} \tau_\alpha^A)$ is a left Segal topological algebra in $(B, \prod_{\alpha\in\mathcal{A}} \tau_\alpha^B)$ via f.

Proof. The map f is an algebra homomorphism because for any $(x_\alpha), (y_\alpha) \in A, \lambda \in \mathbb{K}$, it holds that

$$\begin{aligned} f((x_\alpha) + (y_\alpha)) &= f((x_\alpha + y_\alpha)) = (f_\alpha(x_\alpha + y_\alpha)) = (f_\alpha(x_\alpha) + f_\alpha(y_\alpha)) \\ &= (f_\alpha(x_\alpha)) + (f_\alpha(y_\alpha)) = f((x_\alpha)) + f((y_\alpha)), \\ f(\lambda(x_\alpha)) &= f((\lambda x_\alpha)) = (f_\alpha(\lambda x_\alpha)) = (\lambda f_\alpha(x_\alpha)) = \lambda(f_\alpha(x_\alpha)) = \lambda f((x_\alpha)), \\ f((x_\alpha)(y_\alpha)) &= f((x_\alpha y_\alpha)) = (f_\alpha(x_\alpha y_\alpha)) = (f_\alpha(x_\alpha) f_\alpha(y_\alpha)) \\ &= (f_\alpha(x_\alpha))(f_\alpha(y_\alpha)) = f((x_\alpha))f((y_\alpha)). \end{aligned}$$

Note that from the definition of f it follows that $f(A) = \prod_{\alpha\in\mathcal{A}} f_\alpha(A_\alpha)$. Next, we shall observe that the triplet (A, f, B) satisfies the three conditions from the definition of a Segal topological algebra.

1. $f(A)$ being dense in B follows directly from Lemma 2.2 and the fact that for each $\alpha \in \mathcal{A}$, $f_\alpha(A_\alpha)$ is dense in B_α:

$$\overline{f(A)} = \overline{\prod_{\alpha\in\mathcal{A}} f_\alpha(A_\alpha)} = \prod_{\alpha\in\mathcal{A}} \overline{f_\alpha(A_\alpha)} = \prod_{\alpha\in\mathcal{A}} B_\alpha = B.$$

2. Let us fix a point $(a_\alpha) \in A$ and observe a neighbourhood V of the element $f((a_\alpha)) = (f_\alpha(a_\alpha))$. From the definitions of the neighbourhood of a point and the product topology it is clear that there exist open sets $G_\alpha \in \tau_\alpha^B$ such that $\prod_{\alpha\in\mathcal{A}} G_\alpha \in \tau_B$, $\prod_{\alpha\in\mathcal{A}} G_\alpha \subset V$ is an open neighbourhood of the point $f((a_\alpha))$, and G_α is an open neighbourhood of $f_\alpha(a_\alpha)$ for each $\alpha \in \mathcal{A}$. Due to the fact that all maps f_α are continuous, we know that there exist neighbourhoods U_α of a_α for each $\alpha \in \mathcal{A}$ such that $f_\alpha(U_\alpha) \subset G_\alpha$. We notice that $\prod_{\alpha\in\mathcal{A}} U_\alpha \in A$ is a neighbourhood of the point (a_α), and applying f to that neighbourhood, we see that

$$f\Big(\prod_{\alpha\in\mathcal{A}} U_\alpha\Big) = \prod_{\alpha\in\mathcal{A}} f_\alpha(U_\alpha) \subset \prod_{\alpha\in\mathcal{A}} G_\alpha \subset V.$$

Hence, f is continuous.

3. First, let us confirm that $f(A)$ is a subalgebra of B by demonstrating that the set $f(A) = \Big(\prod_{\alpha\in\mathcal{A}} f_\alpha(A_\alpha)\Big)$ is closed with respect to the operations of algebra B. Let (x_α), $(y_\alpha) \in A$ and $\lambda \in \mathbb{K}$; then using the fact that f is a homomorphism, we know that

$$\begin{aligned} f((x_\alpha)) + f((y_\alpha)) &= (f_\alpha(x_\alpha) + f_\alpha(y_\alpha)), \\ \lambda f((x_\alpha)) &= (\lambda f_\alpha(x_\alpha)), \\ f((x_\alpha))f((y_\alpha)) &= (f_\alpha(x_\alpha) f_\alpha(y_\alpha)). \end{aligned}$$

For each $\alpha \in \mathcal{A}$ we know that $f_\alpha(A_\alpha)$ is a subalgebra of B_α. Hence, for each $\alpha \in \mathcal{A}$, it holds that $f_\alpha(x_\alpha) + f_\alpha(y_\alpha), \lambda f_\alpha(x_\alpha)$ and $f_\alpha(x_\alpha) f_\alpha(y_\alpha)$ are elements of $f_\alpha(A_\alpha)$ and the respective families (for example, $(\lambda f_\alpha(x_\alpha))_{\alpha\in\mathcal{A}}$) are all elements of $f(A)$, proving that $f(A)$ is a subalgebra of B.

Let $(a_\alpha) \in A$ and $(b_\alpha) \in B$, then

$$(b_\alpha) f((a_\alpha)) = (b_\alpha)(f_\alpha(a_\alpha)) = (b_\alpha f_\alpha(a_\alpha)).$$

For each $\alpha \in \mathcal{A}$ we know that $f_\alpha(A_\alpha)$ is a left ideal in B_α; hence $b_\alpha f_\alpha(a_\alpha) \in B_\alpha$. Then also $(b_\alpha f_\alpha(a_\alpha)) \in \prod_{\alpha\in\mathcal{A}} B_\alpha = B$, and $f(A)$ is a left ideal in B. □

3. Unitization of Segal topological algebras

In this paper, we do not assume that any of the algebras have a unit with regard to multiplication. A common move to remedy this is to unitize the algebra, i.e. to consider instead the Cartesian product of the algebra A and the field $\mathbb{K}$, where addition and scalar multiplication are defined pointwise and the multiplication $*$ of two elements $(a,\lambda),(b,\mu) \in A\times\mathbb{K}$ is defined by $(a,\lambda)*(b,\mu) = (ab+\mu a+\lambda b, \lambda\mu)$. This provides the multiplicative unit $(0_A, 1_{\mathbb{K}})$ in the new algebra $A\times\mathbb{K}$. In ring theory, the resulting product space is sometimes called the 'Dorroh extension'. Note that A can be embedded into its Dorroh extension by the mapping $\varphi : A \to A\times\mathbb{K}$, $a \mapsto (a, 0_{\mathbb{K}})$, which is an injective algebra homomorphism.

We can also employ this tactic when working with topological algebras. If (A,τ_A) is a topological algebra, we can take $A_* := A\times\mathbb{K}$, and $\tau_{A_*} := \tau_A \times \tau_{\mathbb{K}}$, where $\tau_{\mathbb{K}}$ is the natural topology of the field $\mathbb{K}$ and τ_{A_*} is a standard product topology. Then (A_*, τ_{A_*}) is a topological algebra as well.

This raises the question: can we unitize a Segal topological algebra? It turns out that unitization is almost universally possible, barring one extra condition that the map between the two topological algebras must satisfy.

Theorem 3.1. *Let (A,τ_A) and (B,τ_B) be topological algebras, and let (A,τ_A) be a left Segal topological algebra in (B,τ_B) via $f : A \to B$. Let (A_*,τ_{A_*}) and (B_*,τ_{B_*}) be the Dorroh extensions of the topological algebras A and B, respectively. Then the topological algebra (A_*,τ_{A_*}) is a left Segal topological algebra in (B_*,τ_{B_*}) via the map*

$$f' : A_* \to B_*, \quad (a,\lambda) \mapsto (f(a),\lambda)$$

if and only if f is an epimorphism.

Proof. The map f' is an algebra homomorphism because for any $a_1, a_2 \in A, \lambda, \kappa \in \mathbb{K}$, it holds that

$$\begin{aligned}
f'((a_1,\lambda)+(a_2,\kappa)) &= f'(a_1+a_2, \lambda+\kappa) = (f(a_1+a_2), \lambda+\kappa) = (f(a_1)+f(a_2), \lambda+\kappa)\\
&= (f(a_1),\lambda)+(f(a_2),\kappa) = f'((a_1,\lambda))+f'((a_2,\kappa)),\\
f'(\kappa(a,\lambda)) &= f'(\kappa a, \kappa\lambda) = (f(\kappa a),\kappa\lambda) = (\kappa f(a),\kappa\lambda) = \kappa(f(a),\lambda) = \kappa f'(a,\lambda),\\
f'((a_1,\lambda)*(a_2,\kappa)) &= f'(a_1a_2+\kappa a_1+\lambda a_2, \lambda\kappa) = (f(a_1a_2+\kappa a_1+\lambda a_2),\lambda\kappa)\\
&= (f(a_1a_2)+f(\kappa a_1)+f(\lambda a_2),\lambda\kappa) = (f(a_1)f(a_2)+\kappa f(a_1)+\lambda f(a_2),\lambda\kappa)\\
&= (f(a_1),\lambda)*(f(a_2),\kappa) = f'((a_1,\lambda))*f'((a_2,\kappa)).
\end{aligned}$$

If $f(A) = B$, then $f'(A\times\mathbb{K}) = f(A)\times\mathbb{K} = B\times\mathbb{K}$, which means that the map f' is a surjective homomorphism or epimorphism. Next, we shall observe that the triplet (A_*, f', B_*) satisfies the three conditions from the definition of a Segal topological algebra.

1. $f'(A_*)$ being dense in B_* follows directly from Lemma 2.2 and the fact that $\overline{f(A)} = B$.

2. Let us fix a point $(a,\lambda) \in A_*$ and observe a neighbourhood V of the mapping $f'(a,\lambda) = (f(a),\lambda)$. From the definitions of the neighbourhood of a point and the product topology it is clear that there exist open sets $G_1 \in \tau_B$ and $G_2 \in \tau_{\mathbb{K}}$ such that $G_1\times G_2 \in \tau_{B_*}$, $G_1\times G_2 \subset V$ is an open neighbourhood of the point $f'(a,\lambda)$, G_1 is an open neighbourhood of $f(a)$ in τ_A, and G_2 is an open neighbourhood of λ in $\tau_{\mathbb{K}}$. Due to the continuity of f, we know that there exists a neighbourhood U_1 of a in τ_A such that $f(U_1) \subset G_1$. We notice that $U_1\times G_2 \in A_*$ is a neighbourhood of the point (a,λ), and, applying f' to that neighbourhood, we see that
$$f'(U_1\times G_2) = f(U_1)\times G_2 \subset G_1\times G_2 \subset V.$$
Hence, f' is continuous.

3. It follows from the equality $f'(A_*) = B_*$ that $f'(A_*)$ is the left ideal of B_*.

We have demonstrated that f' is a homomorphism that satisfies all three conditions shown above; therefore, the triplet (A_*, f', B_*) forms a Segal topological algebra.

Let the triplet (A_*, f', B_*) form a Segal topological algebra, and let $b_0 \in B$ be any element. Since $f'(A_*)$ is a left ideal in the algebra B_*, then for any $a \in A$, we have

$$(b_0, 0_{\mathbb{K}})*(f(a), 1_{\mathbb{K}}) = (b_0f(a)+b_0, 0_{\mathbb{K}}) \in f'(A_*) = f(A)\times\mathbb{K}.$$

In other words, $b_0 f(a) + b_0 \in f(A)$ for any $a \in A$. Due to the fact that $f(A)$ is a left ideal in B, we know that $-b_0 f(a) \in f(A)$. Hence,

$$b_0 = b_0 f(a) + b_0 - b_0 f(a) \in f(A).$$

We have shown that each element of B belongs to $f(A)$; therefore, f must be an epimorphism. □

4. Conclusion

In the present paper, we proposed two new ways to construct Segal topological algebras from existing ones. First, we proved that one can naturally define a direct product over any family of Segal topological algebras. We then also proved that unitization is possible for almost any Segal topological algebra, assuming an extra (sufficient and necessary) condition of surjectivity placed on its algebra homomorphism.

Data availability statement

All data are available in the article.

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Uute Segali topoloogiliste algebrate konstrueerimine olemasolevatest

René Piik ja Mart Abel

Artiklis esitatakse kaks viisi olemasolevatest Segali topoloogilistest algebratest uute konstrueerimiseks. Esmalt näidatakse, et Segali topoloogiliste algebrate peal töötab ning on kooskõlas neil antud topoloogiatega tavaline otsekorrutise konstruktsioon. Seejärel tõestatakse, et pea kõiki Segali topoloogilisi algebraid on võimalik ühikustada, lähtudes topoloogilisse algebrasse Dorroh' laiendi abil ühikelemendi juurdetoomisest. Ainsaks (tarvilikuks ja piisavaks) tingimuseks osutub, et Segali topoloogilist algebrat määrav homomorfism peab olema sürjektiivne.