An iterative formula for finding the number of different R_0-topologies on a finite set

Mart Abel[a,b]

[a] School of Digital Technologies, Tallinn University, Narva mnt 25, Room A-416, 10120 Tallinn, Estonia

[b] Institute of Mathematics and Statistics, University of Tartu, Narva mnt 18, Room 4078, 51009 Tartu, Estonia

Proceedings of the Estonian Academy of Sciences
2025, **74**, 4, 521–524

https://doi.org/10.3176/proc.2025.4.05

www.eap.ee/proceedings
Estonian Academy Publishers

MATHEMATICS, TOPOLOGY

RESEARCH ARTICLE

Received 19 February 2025
Accepted 21 April 2025
Available online 13 October 2025

Keywords:
R_0-topology, symmetric topology, number of topologies on a finite set

Corresponding author:
Mart Abel
mabel@tlu.ee

Citation:
Abel, M. 2025. An iterative formula for finding the number of different R_0-topologies on a finite set. *Proceedings of the Estonian Academy of Sciences*, **74**(4), 521–524. https://doi.org/10.3176/proc.2025.4.05

ABSTRACT

We present an iterative formula for calculating the number of different R_0-topologies on a finite set.

1. Introduction

A topological space (X, τ) is called an R_0-*space* (see [1], p. 888, for example) or a *symmetric space* if for each $x, y \in X, x \neq y$ either

1) x has an open neighbourhood that does not contain y, and y has an open neighbourhood that does not contain x,

or

2) each open neighbourhood of x also contains y, and each open neighbourhood of y also contains x.

The topology τ making a topological space (X, τ) an R_0-space is called an R_0-*topology* on the set X.

Mathematicians have calculated the number of different R_0-topologies on an n-element set where n is a positive integer for some smaller values of n, but a general formula for finding the total number of R_0-topologies on an n-element set for an arbitrary positive integer n has not yet been found.

In 2024, Anne-Mari Vainura wrote her bachelor's thesis [3], where she used a computer programme to calculate the number of different types of topologies on a finite set, including the number of R_0-topologies for $n \in \{1, 2, 3, 4, 5, 6, 7\}$. At least for these values, the number of different R_0-topologies on an n-element set coincided with the Bell number B_n, but no general relationship between these two number sets was yet established.

It is known (see [2] or [4], for example) that the Bell numbers B_n for an arbitrary positive integer $n \geqslant 2$ can be calculated by the following iterative formula:

$$B_n = \sum_{i=0}^{n-1} C_i^{n-1} B_i,$$

where C_k^m denotes the number of different ways one can choose exactly k elements out of a set containing exactly m elements.

In the present paper, we propose a formula that is similar, yet different, for calculating the number of different R_0-topologies on a finite set, using the number of such R_0-topologies on finite sets that do not have any singleton set open.

2. Results

We start with the following proposition.

Proposition 1. *Let $n \in \mathbb{Z}^+$, X be a set with exactly n elements, τ be a topology on X, $A = \{x \in X : \{x\} \in \tau\}$, $B = X \setminus A$ and $\tau_B = \tau|_B$ be a subset topology.*

If (X, τ) is an R_0-space, then the following conditions hold:
1) (B, τ_B) *is an* R_0*-space;*
2) $\tau_B \subseteq \tau$*, i.e. each subset of* B *that is open in* τ_B *is also open in* τ*;*
3) *each nonempty subset* $C \in \tau_B$ *contains at least two elements.*
Moreover, if the condition 1) *is fulfilled, then* (X, τ) *is an* R_0*-space.*

Proof. Let (X, τ) be an R_0-space.

Take any $x, y \in B$ such that $x \neq y$. As $x, y \in X$, then there are two possibilities:

a) there exist $O_x, O_y \in \tau_X$ such that $y \notin O_x$ and $x \notin O_y$. Take $V_x = O_x \cap B$ and $V_y = O_y \cap B$. Then V_x is such an open neighbourhood of x in τ_B that does not contain y, and V_y is such an open neighbourhood of y that does not contain x;

b) each open neighbourhood $O_x \in \tau$ of x in τ also contains y, and each open neighbourhood $O_y \in \tau$ of y in τ also contains x. But then each open neighbourhood $V_x = O_x \cap B \in \tau_B$ of x in τ_B also contains y, and each open neighbourhood $V_y = O_y \cap B \in \tau_B$ of y in τ_B also contains x.

Hence, (B, τ_B) is an R_0-space, and the condition 1) holds.

To show that the condition 2) holds, it is sufficient to show that $B \in \tau$. For that, notice that for each $x \in B$ and each $a \in A$, there exists a neighbourhood $O_a = \{a\} \in \tau$ such that $x \notin O_a$. Hence, there should also exist a neighbourhood $U_a \in \tau$ of x such that $a \notin U_a$. Now,

$$O_x = \bigcap_{a \in A} U_a$$

is a neighbourhood of x in τ and, as A is a finite set, then $O_x \in \tau$. Moreover, $O_x \subseteq B$.

Notice that

$$B = \bigcup_{x \in B} O_x \in \tau$$

is a union of open sets. Hence, every element of $\tau_B = \{V \cap B : V \in \tau\}$ is open in τ as an intersection of two sets that are open in τ. Thus, the condition 2) holds.

Let $C \in \tau_B \setminus \{\emptyset\}$. Then $C \in \tau$, by the condition 2), and $C \subseteq B$. If C were a singleton, then there would exist an element $x \in B$ such that $C = \{x\}$ and $\{x\} \in \tau$. But all such elements were already included in A. Thus, each nonempty element of τ_B should contain at least two elements.

Hence, all three conditions hold for an R_0-space (X, τ).

Suppose that the condition 1) holds for the subset topology τ_B. Notice that

$$\tau = \{C \cup D : C \in \tau_B, D \subseteq A\}$$

because all elements of A, as singleton sets, have to be open in τ.

Take any $x, y \in X$ with $x \neq y$. Now, we have three possibilities:

a) $x, y \in A$. Then $O_x = \{x\}, O_y = \{y\} \in \tau$, $y \notin O_x, x \notin O_y$, and the case 1) of the definition of an R_0-space realizes.

b) $x, y \in B$. Then, by the condition 1), there are two possibilities.

b1) There exist neighbourhoods $V_x, V_y \in \tau_B$ such that $x \in V_x, y \in V_y, y \notin V_x, x \notin V_y$. By the definition of a subspace topology, there exist neighbourhoods $O_x, O_y \in \tau$ of x and y, respectively, such that $V_x = O_x \cap B$ and $V_y = O_y \cap B$. But then O_x and O_y satisfy the condition 1) of the definition of an R_0-space.

b2) Every neighbourhood V_x of x in τ_B also contains y, and every neighbourhood V_y of y in τ_B also contains x. Take arbitrary neighbourhoods $O_x \in \tau$ of x and $O_y \in \tau$ of y. Then $O_x \cap B$ is an open neighbourhood of x in τ_B, and $O_y \cap B$ is an open neighbourhood of y in τ_B. Hence, O_x must contain y, and O_y must contain x. It means that the case 2) of the definition of an R_0-space realizes.

c) One of the elements x, y is in A and another in B. Without loss of generality, we may assume that $x \in A, y \in B$. Then there exist $O_x = \{x\} \in \tau$ and $O_y = B \in \tau$ such that the condition 1) of the definition of an R_0-space is fulfilled.

Therefore, if the condition 1) is fulfilled, (X, τ) is an R_0-space. □

Remark 1. With that, we have demonstrated that (X, τ) is an R_0-space if and only if the conditions 1), 2) and 3) are fulfilled. Indeed, under the assumptions of Proposition 1, from the condition 1) it also follows that the conditions 2) and 3) must hold because the condition 1) guarantees that (X, τ) is an R_0-space, which means that all of the conditions 1), 2) and 3) must hold. Hence, there could not be any R_0-topology on X without a subset R_0-topology on B where the conditions 2) and 3) are also fulfilled.

Corollary 1. *Let X be a finite set, $B = \{x \in X : \{x\} \notin \tau\}$ and $\tau_B = \tau|_B$ be a subset topology on B. Then there is a bijection between the R_0-topologies on X and such R_0-topologies on B where no singleton is open, i.e. the following conditions hold:*

1) *to each R_0-topology τ on X there corresponds a unique R_0-topology τ_B on B where each element of τ_B is either an empty set or contains at least two elements;*

2) *to each R_0-topology τ_B on B where each element of τ_B is either an empty set or contains at least two elements, there corresponds exactly one R_0-topology τ in X such that $\tau_B = \tau|_B$.*

Proof. If $X = \emptyset$, then $B = \emptyset$, and there is exactly one topology on both X and B, which makes both of them R_0-spaces.

If $X \neq \emptyset$, then the condition 1) follows from Proposition 1 by taking $\tau_B = \tau|_B$.

Suppose that τ_B is such R_0-topology on B where each element of τ_B is either an empty set or contains at least two elements, and take any topology τ on X such that $\tau_B = \tau|_B = \{O \cap B : O \in \tau\}$. Then the conditions 1) and 3) of Proposition 1 hold, whence $\tau = \{C \cup D : C \in \tau_B, D \subseteq X \setminus B\}$ is the unique topology on X, making it an R_0-space where $\tau_B = \tau|_B$, and for each $a \in X \setminus B$, we have $\{a\} \in \tau$. Thus, the condition 2) holds. □

Denote the number of all topologies making an n-element set an R_0-space by $R_0(n)$, and the number of all topologies making an n-element set an R_0-space where no singleton is open by $R_0(n, 2)$. Then, according to the results proven above, we obtain the formula

$$R_0(n) = \sum_{i=0}^{n} C_i^n R_0(n-i, 2),$$

where C_k^m again denotes the number of different ways one can choose exactly k elements out of a set containing exactly m elements.

It is easy to see that $R(0,2) = 1$ and $R(1,2) = 0$, but how to calculate $R(n,2)$ for $n \geqslant 2$ in general, we do not know yet. By computer experiments run by Vainura, we know that $R_0(2,2) = 1 = R_0(3,2), R_0(4,2) = 4$, $R_0(5,2) = 11, R_0(6,2) = 41$ and $R_0(7,2) = 162$. Hence, the values of $R_0(n,2)$ for n changing from 0 to 7 correspond to the first eight members of the sequence $A000296$ in the On-Line Encyclopedia of Integer Sequences (OEIS).

Notice that the obtained formula is different from the formula

$$B_n = \sum_{i=0}^{n-1} C_i^{n-1} B_i$$

for calculating the Bell numbers because here the summation goes until n, while in the case of the Bell numbers it finishes with $n-1$. Moreover, at least for $n \in \{5,6,7,8\}$, $R_0(n-i,2)$ is, in general, not equal to any of the Bell numbers. So, the obtained formula is certainly diffrent from the iterative formula for calculating the Bell numbers.

We finish the paper by including some additional results on the number of such topologies on an n-element set where all open sets contain at least m elements, hoping that these will be useful for someone who wants to use the idea of looking at such type of topologies.

For $m, n \in \mathbb{Z}^+$ with $m \leqslant n$, denote by $T(n,m)$ the collection of all such topologies on an n-element set where each open nonempty set contains at least m elements. For a subset A of an n-element set X, denote by $|A|$ the cardinality of A, i.e. the number of elements in A. So, for example, if $X = \{x_1, x_2, \ldots, x_7\}$ and $A = \{x_2, x_4, x_5\}$, then $n = 7$ and $|A| = 3$.

Lemma 1. *Let $m, n \in \mathbb{Z}^+$ with $m \leqslant n$ and (X, τ) be a topological space for which X is an n-element set and $\tau \in T(n,m)$. Then, for each $A, B \in \tau$ we have that either $A \cap B = \emptyset$ or $|A \cap B| \geqslant m$. In particular, for $|A| = m = |B|$, one has $A \cap B = \begin{cases} A, if\, A = B \\ \emptyset, otherwise \end{cases}$.*

Proof. Let $C = A \cap B$. Then $C \subseteq A$ and $C \subseteq B$. Since $A, B \in \tau$, then $C = A \cap B \in \tau$. As $\tau \in T(n,m)$, then either $C = \emptyset$ or $|C| \geqslant m$. If $|A| = m = |B|$, then, if $C \neq \emptyset$, we must have both $C = A$ and $C = B$. Hence, either $C = \emptyset$ or $C = A = B$, and the claim holds. □

Lemma 2. *Let $m, n \in \mathbb{Z}^+$ with $m \leqslant n$ and (X, τ) be a topological space for which X is an n-element set and $\tau \in T(n, m)$. Then in τ there could be at most $\lfloor \frac{n}{m} \rfloor$ different open sets with exactly m elements in each.*

Proof. By Lemma 1, all different open sets with exactly m elements should be disjoint. Hence, the total number of such different sets could be at most $\lfloor \frac{n}{m} \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the floor function). □

3. Conclusion

In this paper, we found an iterative formula for calculating the number of distinct R_0-topologies on a finite set using the numbers of such distinct R_0-topologies on finite sets that do not have any open singletons.

Data availability statement

All data are available in the article.

Acknowledgement

The publication costs of this article were covered by the Estonian Academy of Sciences.

References

1. Davis, A. S. Indexed systems of neighborhoods for general topological spaces. *Am. Math. Mon.*, 1961, **68**(9), 886–893.
2. Rota, G.-C. The number of partitions of a set. *Am. Math. Mon.*, 1964, **71**(5), 498–504.
3. Vainura, A.-M. *On the number of topologies on finite sets*. Bachelor's thesis. Tallinn University, Estonia, 2024.
4. *Bell number*. Wikipedia. https://en.wikipedia.org/wiki/Bell_number

Iteratiivne valem erinevate R_0-topoloogiate arvu leidmiseks lõplikel hulkadel

Mart Abel

Artiklis tuletatakse iteratiivne valem, mille abil saab välja arvutada lõplikul hulgal defineeritavate R_0-topoloogiate arvu, kasutades selliste R_0-topoloogiate arve lõplikel hulkadel, kus ei leidu ühepunktilisi lahtisi hulki.