Finite groups whose order graph is C_4-free

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ABSTRACT

Given a finite group G, the order graph of G, denoted by $\mathcal{S}(G)$, is a graph whose vertex set is G, and two distinct vertices a and b are adjacent if $o(a) \mid o(b)$ or $o(b) \mid o(a)$, where $o(a)$ and $o(b)$ are the orders of a and b in G, respectively. In this paper, by the order of an element, we give a characterization of the finite groups whose order graph is C_4-free. As applications, we classify a few families of finite groups whose order graph is C_4-free, such as nilpotent groups, dihedral groups and symmetric groups.

1. Introduction

In the field of algebraic graph theory, a popular and interesting research topic is groups and graphs, which is the study of the graph representations of some algebraic structure, such as a group or a ring. For example, the well-known Cayley graphs are defined on a group and have a very long history. Furthermore, graphs associated with some algebraic structures have been actively investigated in the literature since they have valuable applications (see, for example, Cayley graphs in data mining [11]), and they are related to the automata theory [9].

Another well-known graph representation on a group is the power graph of a group. The *undirected power graph* of a group G, denoted by $\mathcal{P}(G)$, is a simple graph whose vertex set is G, and two distinct vertices are adjacent if one is a power of the other. Kelarev and Quinn [10] first introduced the concept of a power graph of some group, where the power graph is directed. The concept of the undirected power graph of a group was first introduced by Chakrabarty et al. in [5]. Subsequently scholars refer to the undirected power graph simply as the power graph (cf. [1]). In the past ten years, the study of power graphs has been growing. For example, the two survey papers [1,12] contain almost all of the results and open questions on power graphs.

For a group G, the *order graph* of G, denoted by $\mathcal{S}(G)$, is an undirected graph whose vertex set is G, and two distinct vertices $a, b \in G$ are adjacent if $o(a) \mid o(b)$ or $o(b) \mid o(a)$, where $o(a)$ and $o(b)$ are the orders of a and b, respectively. Notice that for any group G, $\mathcal{P}(G)$ is always a spanning subgraph of $\mathcal{S}(G)$. Hamzeh and Ashrafi [7] first introduced the order graph of a group and referred to it as the supergraph of a power graph. Also, they characterized the full automorphism group of the order graph of a finite group. In fact, in [8], Hamzeh and Ashrafi also referred to the order graph of a group as the order supergraph of a power graph. Moreover, the authors in [8] studied some properties of $\mathcal{S}(G)$ together with the relationship between $\mathcal{S}(G)$ and $\mathcal{P}(G)$. In [14], Ma and Su studied the independence number of an order graph. In [2], Asboei and Salehi studied Thompson's problem and recognized the projective special linear groups and the projective linear groups by their order graphs. In 2022, Asboei and Salehi [3] recognized many families of finite non-solvable groups by their order graphs, which is an important work for Thompson's problem.

All graphs considered in this paper are undirected without loops and multiple edges. For a graph Γ, we use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex set and edge set of Γ, respectively. We use C_4 and P_4 to denote a cycle of length 4 and a path of length 3, respectively. A graph is called Γ-*free* if this graph has no induced subgraphs isomorphic to Γ. Graphs with forbidden structures appear in many contexts such as extremal graph theory where lower and upper bounds can be obtained for various numerical invariants of the corresponding graphs. Many graphs that are encountered in the study of graph theory are characterized by a type of configuration or subgraph they possess. However, there are occasions when such graphs are more easily defined or described by the kind of subgraphs they are not permitted to contain. For example, a tree can be defined as a connected graph that contains no cycles, and Kuratowski [13] characterized planar graphs as those graphs that fail to contain subgraphs homeomorphic from the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Recently, Cameron [4] surveyed various graphs defined on groups, including power graphs, commuting graphs, enhanced power graphs, intersection graphs, Gruenberg–Kegel graphs, nongenerating graphs and more. In [4], Cameron posed the question: for which finite groups is some graph defined on a group Γ-free (cf. [4], Question 14)? Motivated by Cameron's question, we consider the following question.

Question 1.1. *For which finite groups is the order graph C_4-free?*

Hamzeh and Ashrafi [8] have shown that every order graph of a finite group is perfect, and so every order graph is C_n-free for any odd integer $n \geq 5$. Recently, in [15], the authors characterized all finite groups whose order graph is P_4-free and classified all finite groups whose order graph is $2K_2$-free. Furthermore, they classified all finite groups whose order graph is split and characterized all finite groups whose order graph is a cograph.

In this paper, we characterize all finite groups whose order graph is C_4-free. As applications, we classify the finite groups whose order graph is C_4-free if G is a nilpotent group, a dihedral group, a generalized quaternion group, a symmetric group and an alternating group. The results answer Question 1.1.

2. Preliminaries

This section will introduce some basic definitions on groups and graphs. All groups considered in this paper are finite. As a matter of convenience, we always use G to denote a finite group, and use e to denote its identity element. Denote by $\pi(G)$ and $\pi_e(G)$ the set of all prime divisors of $|G|$ and the set of all orders of non-trivial elements in G, respectively. Given an element g in G, the *order* of g, denoted by $o(g)$, is the number of all elements of the cyclic subgroup $\langle g \rangle$ generated by g. In particular, if $o(g) = 2$, then g is called an *involution*. As usual, we denote the cyclic group with order n by $\mathbb{Z}_n$. Given a prime p, the elementary abelian p-group of order p^n is the direct product of n cyclic groups $\mathbb{Z}_p$, which is denoted by $\mathbb{Z}_p^n$. The *exponent* of a finite group G, denoted by $exp(G)$, is the smallest positive integer t so that $g^t = e$ for each $g \in G$. It is well known that for a finite group, if every Sylow p-subgroup of this group is a normal subgroup, then G is *nilpotent*. In other words, a finite group is nilpotent if and only if this group is a direct product of its Sylow subgroups.

As usual, we use C_n to denote the cycle with n vertices. If $\{x, y\} \in E(\Gamma)$, then we also denote this by $x \sim_\Gamma y$, and shortly by $x \sim y$. In a graph, for distinct vertices $x_1, x_2, \ldots, x_n \in V(\Gamma)$, we use $x_1 \sim x_2 \sim \cdots \sim x_n \sim x_1$ to denote an induced subgraph isomorphic to C_n, which is a subgraph with the vertex set $\{x_1, x_2, \ldots, x_n\}$ and the edge set $\{\{x_1, x_n\}, \{x_i, x_{i+1}\} : 1 \leq i \leq n-1\}$. The following two observations are immediate by the definition of the order graph of a group.

Observation 2.1. *Let G be a group and let H be a subgroup of G. Then $\mathcal{S}(H)$ is an induced subgraph of $\mathcal{S}(G)$.*

Observation 2.2. *Let G be a group. Then $\mathcal{S}(G)$ is C_4-free if and only if, for each subgroup H of G, $\mathcal{S}(H)$ is C_4-free.*

The above two observations are important and will be used frequently in the sequel, at times without explicit reference to it.

3. C_4-free order graphs

This section will give a characterization for the finite groups G so that $\mathcal{S}(G)$ is C_4-free (see Theorem 3.4). As an application, this section will classify all finite nilpotent groups G so that $\mathcal{S}(G)$ is C_4-free (see Corollary 3.8).

Definition 3.1. *A finite group G is called a Φ-group if the following property holds: for distinct $p, q \in \pi(G)$, if there exist $p^m q^n k_1, p^l q^t k_2 \in \pi_e(G)$, then either $p^m q^n k_1 \mid p^l q^t k_2$ or $p^l q^t k_2 \mid p^m q^n k_1$, where m, n, l, t, k_1, k_2 are positive integers and $(k_i, p) = (k_i, q) = 1$ for $i = 1, 2$.*

If every element of a finite group has a prime power order, then this group is called a *CP-group*. For example, the alternating group of degree five is a CP-group. Particularly, for any prime p, a p-group is also a CP-group. By the definition of a Φ-group, we have the following examples.

Example 3.2. (i) Every CP-group is a Φ-group; (ii) for distinct primes p, q, a group G with $\pi_e(G) = \{p, q, pq\}$ is a Φ-group. In particular, $\mathbb{Z}_p^m \times \mathbb{Z}_q^n$ is a Φ-group for all $m, n \geq 1$.

Lemma 3.3. [8], Theorem 2.3. *$\mathcal{S}(G)$ is complete if and only if G is a p-group where p is a prime.*

The main result of this section is the following theorem.

Theorem 3.4. *$\mathcal{S}(G)$ is C_4-free if and only if G is a Φ-group.*

Proof. We first prove the necessity. Assume that $\mathcal{S}(G)$ is C_4-free. Suppose for a contradiction that G is not a Φ-group. Then by Definition 3.1, there exist two distinct primes $p, q \in \pi(G)$ such that G has four elements a, b, c, d with

$$o(a) = p, o(c) = q, pq \mid o(b), pq \mid o(d), o(b) \nmid o(d), o(d) \nmid o(b).$$

It is easy to see that $a \sim b \sim c \sim d \sim a$ is an induced cycle isomorphic to C_4, a contradiction.

We next prove the sufficiency. Suppose, for the sake of contradiction, that $\mathcal{S}(G)$ contains an induced cycle C_4 as follows:

$$a \sim b \sim c \sim d \sim a.$$

We first claim that one of the set $\{a, b, c, d\}$ cannot have a prime power order. Suppose for a contradiction that, without loss of generality, $o(a) = p^m$, where p is a prime and m is a positive integer. Now, consider the orders of b and d. Since a is adjacent to both b and d in $\mathcal{S}(G)$, it must not be that $o(b) \mid p^m$ and $o(d) \mid p^m$, because b and d are non-adjacent in $\mathcal{S}(G)$. We conclude that one of $o(b)$ or $o(d)$ must be divided by p^m. Note that b and d are non-adjacent. It follows that $p^m \mid o(b)$ and $p^m \mid o(d)$. Since a and c are non-adjacent in $\mathcal{S}(G)$, we obtain that $o(c) \mid o(b)$ and $o(c) \mid o(d)$. Note that $o(c)$ is not a power of p. Thus, there exists a prime divisor q of $o(c)$ such that $p \neq q$. This means that $pq \mid o(b)$ and $pq \mid o(d)$. Now, since G is a Φ-group, we have that either $o(d) \mid o(b)$ or $o(b) \mid o(d)$, and so b and d are adjacent in $\mathcal{S}(G)$, which is impossible. As a result, the above claim is valid.

We conclude that, by the above claim, there exist two distinct primes p, q such that $pq \mid o(a)$. If $o(a) \mid o(b)$ and $o(a) \mid o(d)$, then, since G is a Φ-group, we have that b and d are adjacent in $\mathcal{S}(G)$, a contradiction. It follows that one of $o(b)$ or $o(d)$ must divide $o(a)$. Since b and d are non-adjacent in $\mathcal{S}(G)$, we have that $o(b) \mid o(a)$ and $o(d) \mid o(a)$. Note that a and c also are non-adjacent in $\mathcal{S}(G)$. Similarly, we have that $o(b) \mid o(c)$ and $o(d) \mid o(c)$. Now, by our claim that there exist two distinct primes r, s such that $rs \mid o(b)$, and so $rs \mid o(a)$ and $rs \mid o(c)$. This means that a and c are adjacent in $\mathcal{S}(G)$ since G is a Φ-group. This contradicts our hypothesis that $a \sim b \sim c \sim d \sim a$ is an induced cycle. The proof is now complete. □

By Theorem 3.4 and Definition 3.1, the following result holds.

Corollary 3.5. (i) *If the order of any element of G is either a prime power or a product of two distinct primes, then $\mathcal{S}(G)$ is C_4-free.* (ii) *Suppose that the order of any element of G is at most* 29. *Then $\mathcal{S}(G)$ is not C_4-free if and only if* $12, 18 \in \pi_e(G)$.

Proposition 3.6. *Suppose that $\mathcal{S}(G)$ is C_4-free. Then G has no elements of order p^2q^2, p^2qr or $pqrs$, where p, q, r, s are distinct primes.*

Proof. Suppose for a contradiction that G has an element a of order p^2q^2, where p, q are distinct primes. Then it is easy to see that

$$a^{p^2q} \sim a^p \sim a^{pq^2} \sim a^q \sim a^{p^2q}$$

is an induced cycle of $\mathcal{S}(G)$ with four vertices; this contradicts that $\mathcal{S}(G)$ is C_4-free. If G has an element b of order p^2qr, where p, q, r are distinct primes, then

$$b^{pqr} \sim b^r \sim b^{p^2r} \sim b^p \sim b^{pqr}$$

is an induced subgraph of $\mathcal{S}(G)$ isomorphic to C_4, a contradiction. Finally, if G has an element c of order $pqrs$, where p, q, r, s are distinct primes, then

$$c^{qrs} \sim c^s \sim c^{prs} \sim c^r \sim c^{qrs}$$

is an induced subgraph of $\mathcal{S}(G)$ isomorphic to C_4, a contradiction. □

By Proposition 3.6 and Theorem 3.4, the following holds.

Corollary 3.7. *$\mathcal{S}(G)$ has an induced subgroup isomorphic to C_4 if one of the following holds:*

(a) $12, 18 \in \pi_e(G)$*;*

(b) $12, 30 \in \pi_e(G)$*;*

(c) $18, 30 \in \pi_e(G)$*;*

(d) $36 \in \pi_e(G)$.

It is well known that a finite group is a *nilpotent group* if and only if this group is the direct product of its Sylow subgroups. Particularly, in a nilpotent group, two elements of different prime orders commute. In the following, as a corollary of Theorem 3.4 and Proposition 3.6, we characterize all finite nilpotent groups whose order graph is C_4-free.

Corollary 3.8. *Let G be a nilpotent group. Then $\mathcal{S}(G)$ is C_4-free if and only if G is isomorphic to one of the following:*

(i) *a p-group, where p is a prime;*

(ii) *$P \times Q$, where P is a p-group and $exp(Q) = q$ with distinct primes p, q;*

(iii) *$P \times Q \times R$, where $exp(P) = p$, $exp(Q) = q$ and $exp(R) = r$ with distinct primes p, q, r.*

Proof. We first prove the sufficiency. By Lemma 3.3, if G is a p-group, then $\mathcal{S}(G)$ is complete and so is C_4-free. Now, let $G = P \times Q$, where P is a p-group and $exp(Q) = q$ with distinct primes p, q. Then $\pi(G)$ has precisely two distinct elements p, q. If there exist two elements $a, b \in \pi_e(G)$ such that $pq \mid a$ and $pq \mid b$, then it must be that $a = p^m q$ and $b = p^n q$, where m, n are positive integers, and so $a \mid b$ or $b \mid a$, which implies that G is a Φ-group. It follows from Theorem 3.4 that $\mathcal{S}(G)$ is C_4-free, as desired. Finally, assume that $G = P \times Q \times R$, where $exp(P) = p$, $exp(Q) = q$ and $exp(R) = r$ with distinct primes p, q, r. Then $\pi(G) = \{p, q, r\}$. For any two distinct elements in $\pi(G)$, say p, q, if there exist two distinct elements $c, d \in \pi_e(G)$ such that $pq \mid c$ and $pq \mid d$, then it must be that $c, d \in \{pq, pqr\}$, and so $c \mid d$ or $d \mid c$, thus G is a Φ-group. It follows from Theorem 3.4 again that $\mathcal{S}(G)$ is C_4-free.

For the converse, assume that $\mathcal{S}(G)$ is C_4-free. It suffices to prove that if G is not a p-group, then G is isomorphic to one group in (ii) and (iii). Now, suppose that G is not a p-group. Note that G is a nilpotent group. By Proposition 3.6, we see that G has no elements whose order is a product of four pairwise distinct primes. It follows that $2 \le |\pi(G)| \le 3$.

Suppose that $\pi(G) = \{p, q\}$, where p, q are distinct primes. Then $G = P \times Q$, where P and Q are p-group and q-group, respectively. It is impossible that G has two elements x, y with $o(x) = p^2$ and $o(y) = q^2$, otherwise, $o(xy) = p^2q^2$, a contradiction as per Proposition 3.6. It follows that one of P or Q must have a prime exponent. Without loss of generality, let $exp(Q) = q$, and so G is isomorphic to one group in (ii), as desired.

Finally, suppose that $\pi(G) = \{p, q, r\}$, where p, q, r are pairwise distinct primes. Then $G = P \times Q \times R$, where P, Q and R are p-group, q-group and r-group, respectively. By Proposition 3.6, G has no elements of order $p_1^2 p_2 p_3$, where p_1, p_2, p_3 are pairwise distinct primes. It follows that each of P, Q, R has no elements whose order is a square of a prime. As a result, $exp(P) = p$, $exp(Q) = q$ and $exp(R) = r$, as desired. □

Applying Corollary 3.8 to abelian groups, we can classify all abelian groups whose order graph is C_4-free.

Corollary 3.9. *Let G be an abelian group. Then $\mathcal{S}(G)$ is C_4-free if and only if G is isomorphic to*

$$\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_t}},\ \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_t}} \times \mathbb{Z}_q^n,\ \mathbb{Z}_p^m \times \mathbb{Z}_q^n \times \mathbb{Z}_r^t,$$

where p, q, r are pairwise distinct primes, and for all $1 \le i \le t$, m, n, t, α_i are positive integers.

Applying Corollary 3.9 to cyclic groups, we can classify all cyclic groups whose order graph is C_4-free.

Corollary 3.10. *Let G be a cyclic group. Then $\mathcal{S}(G)$ is C_4-free if and only if G is isomorphic to*

$$\mathbb{Z}_{p^m},\ \mathbb{Z}_{p^m q},\ \mathbb{Z}_{pqr},$$

where p, q, r are pairwise distinct primes, and m is a positive integer.

4. Non-nilpotent groups

Corollary 3.8 classifies all nilpotent groups whose order graph is C_4-free. This section will classify a few families of non-nilpotent groups whose order graph is C_4-free, including dihedral groups, generalized quaternion groups, symmetric groups, alternating groups and sporadic simple groups.

For a positive integer $n \ge 3$, the *dihedral group* of order $2n$, denoted by D_{2n}, is the group of symmetries of a regular polygon with n vertices, including both rotations and reflections. D_{2n} plays an important role in group theory, geometry and chemistry. D_{2n} has a presentation as follows:

$$D_{2n} = \langle a, b : b^2 = a^n = e, ab = ba^{-1} \rangle. \tag{1}$$

It is easy to check that for all $1 \le i \le n$, we have that $o(a^i b) = 2$. Also, D_{2n} has a partition

$$\{\langle a \rangle, \{b, ab, a^2 b, \ldots, a^{n-1} b\}\}. \tag{2}$$

Combining Theorem 3.4, Corollary 3.10 and (2), we can easily classify all dihedral groups whose order graph is C_4-free. Note that D_{2n} is nilpotent if and only if n is a power of 2.

Theorem 4.1. *Let D_{2n} be the dihedral group as presented in* (1). *Then $\mathcal{S}(D_{2n})$ is C_4-free if and only if $n = p^m$, $p^m q$ or pqr, where p, q, r are pairwise distinct primes and $m \ge 1$.*

The *generalized quaternion group* Q_{4m} of order $4m$ is

$$Q_{4m} = \langle x, y : y^2 = x^m, y^4 = x^{2m} = e, xy = yx^{-1} \rangle. \tag{3}$$

Observe that if $m = 1$, then $Q_4 \cong \mathbb{Z}_4$; if $m \ge 2$, then Q_{4m} is non-abelian. Also, it is easy to see that Q_{4m} has a unique involution $x^m = y^2$ and $o(x^i y) = 4$ for all $1 \le i \le 2m$. Note that

$$\{\langle x \rangle, \{x^i y : 1 \le i \le 2m\}\} \tag{4}$$

is a partition of Q_{4m}, and Q_{4m} is nilpotent if and only if m is a power of 2.

Theorem 4.2. *Let Q_{4m} be the generalized quaternion group as presented in* (3). *Then $\mathcal{S}(Q_{4m})$ is C_4-free if and only if n is one of the following:*

$$2^m,\ p^m,\ pq,\ 2^m p,$$

where p, q are two distinct odd primes and $m \ge 1$ is a positive integer.

Proof. The required result follows from Theorem 3.4, Corollary 3.10 and (4). □

The *symmetric group* of order $n!$, denoted by $\mathbf{S}_n$, is the group consisting of all permutations on n letters. As we know, the symmetric group is important in many different areas of mathematics, including combinatorics and group theory, since every finite group is a subgroup of some symmetric group.

Theorem 4.3. *$\mathcal{S}(\mathbf{S}_n)$ is C_4-free if and only if $n \leq 9$.*

Proof. Note that $\pi_e(\mathbf{S}_9) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 20\}$. Thus, by Definition 3.1, it is easy to see that $\mathbf{S}_9$ is a Φ-group. It follows from Theorem 3.4 that $\mathcal{S}(\mathbf{S}_9)$ is C_4-free. This forces that for all $n \leq 9$, $\mathcal{S}(\mathbf{S}_n)$ is C_4-free by Observation 2.2. Now, note that for any $n \geq 10$, we have that $\mathcal{S}(\mathbf{S}_{10})$ is an induced subgraph of $\mathcal{S}(\mathbf{S}_n)$. Now, it suffices to prove that $\mathcal{S}(\mathbf{S}_{10})$ is not C_4-free. In fact, $\mathbf{S}_{10}$ contains this element $(1, 2)(3, 4, 5)(6, 7, 8, 9, 10)$ having order 30, and another element $(1, 2, 3, 4)(5, 6, 7)$ having order 12. This implies that $\mathbf{S}_{10}$ is not a Φ-group, and so $\mathcal{S}(\mathbf{S}_{10})$ is not C_4-free by Theorem 3.4. □

The set of all even permutations of $\mathbf{S}_n$ constitutes precisely a group, which is called the *alternating group* on n letters and is denoted by $\mathbf{A}_n$. Remark that for any $n \geq 5$, A_n is simple.

Theorem 4.4. *$\mathcal{S}(\mathbf{A}_n)$ is C_4-free if and only if $n \leq 11$.*

Proof. Note that $\pi_e(\mathbf{A}_{11}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 20, 21\}$. By Definition 3.1, it is easy to check that $\mathbf{A}_{11}$ is a Φ-group, and so $\mathcal{S}(\mathbf{A}_{11})$ is C_4-free from Theorem 3.4. This means that, from Observation 2.2, $\mathcal{S}(\mathbf{A}_n)$ is C_4-free for all $n \leq 11$. Now, note that for any $n \geq 12$, $\mathcal{S}(\mathbf{A}_{12})$ is an induced subgraph of $\mathcal{S}(\mathbf{A}_n)$. It suffices to prove that $\mathcal{S}(\mathbf{A}_{12})$ is not C_4-free. In fact, we have that $\mathbf{A}_{12}$ contains the element $(1, 2)(3, 4)(5, 6, 7)(8, 9, 10, 11, 12)$ having order 30 and the element $(1, 2, 3, 4)(5, 6, 7)(8, 9)$ having order 12. It is easy to see that $\mathbf{A}_{12}$ is not a Φ-group, and so $\mathcal{S}(\mathbf{A}_{12})$ is not C_4-free by Theorem 3.4. □

We conclude this paper by the following result, which classifies all sporadic simple groups whose order graph is C_4-free. In fact, by Corollaries 3.5 and 3.7, Theorem 3.4 and the $\mathbb{ATLAS}$ of finite groups [6], one can easily verify this result, so we omit its proof.

Theorem 4.5. *Let G be a sporadic simple group. Then $\Gamma(G)$ is C_4-free if and only if G is isomorphic to one of the following groups:*

$$M_{11},\ M_{12},\ M_{22},\ M_{23},\ M_{24},\ HS,\ J_1,\ J_2,\ J_3,\ He,\ O'N,\ Ru.$$

5. Conclusions

In the field of algebraic graph theory, a popular and interesting research topic is groups and graphs, which is the study of the graph representations of some algebraic structure, such as a group or a ring. In this paper, we studied the order graph $\mathcal{S}(G)$ of a finite group G. More specifically, we gave a complete characterization of the finite groups whose order graph is C_4-free by orders of elements. As some applications, we classified a few families of finite groups whose order graph is C_4-free, such as nilpotent groups, dihedral groups and symmetric groups.

Data availability statement

All data are available in the article.

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C_4-vaba järgugraafiga lõplikud rühmad

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Olgu G lõplik rühm ja $\mathcal{S}(G)$ selle järgugraaf. Graafi $\mathcal{S}(G)$ tipud on rühma G elemendid ning kaks erinevat tippu a ja b on kaassed parajasti siis, kui $o(a)|o(b)$ või $o(b)|o(a)$, kus $o(a)$ ja $o(b)$ on elementide a ja b järgud rühmas G. Graafi nimetatakse C_4-vabaks, kui see ei sisalda tsüklit pikkusega 4. Artiklis kirjeldatakse lõplikke rühmi, mille järgugraaf on C_4-vaba. Rakendusena kirjeldatakse mõningaid C_4-vaba järgugraafiga lõplike rühmade klasse.