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Corresponding author:
Nikolai Sovetnikov
nikolai.sovetnikov@ut.ee

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Odd and even derivations, transposed Poisson superalgebra and 3-Lie superalgebra

Viktor Abramov and Nikolai Sovetnikov

Institute of Mathematics and Statistics, University of Tartu, Narva mnt 18, 51009 Tartu, Estonia

ABSTRACT

One important example of a transposed Poisson algebra can be constructed by means of a commutative algebra and its derivation. This approach can be extended to superalgebras; that is, one can construct a transposed Poisson superalgebra given a commutative superalgebra and its even derivation. In this paper, we show that including odd derivations in the framework of this approach requires introducing a new notion. It is a super vector space with two operations that satisfy the compatibility condition of a transposed Poisson superalgebra. The first operation is determined by a left supermodule over a commutative superalgebra and the second is a Jordan bracket. Then it is proved that the super vector space generated by an odd derivation of a commutative superalgebra satisfies all the requirements of the introduced notion. We also show how to construct a 3-Lie superalgebra if we are given a transposed Poisson superalgebra and its even derivation.

1. Introduction

One of the most important structures of the mathematical methods of Hamiltonian mechanics is a Poisson bracket. If the coordinates of the $2n$-dimensional canonical phase space are denoted by (p_i, q_i), where the integer i runs from 1 to n, then the Poisson bracket of two smooth functions f and g can be written as follows:

$$\{f, g\} = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\Big). \qquad (1)$$

From (1) it follows that the Poisson bracket is skew-symmetric; that is, $\{f, g\} = -\{g, f\}$, and it satisfies the Jacobi identity

$$\big\{\{f, g\}, h\big\} + \big\{\{g, h\}, f\big\} + \big\{\{h, f\}, g\big\} = 0.$$

Thus, the vector space of smooth functions endowed with the Poisson bracket (1) is a Lie algebra [4,11]. But there is another structure on the vector space of smooth functions. If we consider the pointwise product of two smooth functions f and g, then the vector space of smooth functions becomes an associative commutative algebra, and we have

$$\{fg, h\} = f\{g, h\} + \{f, h\}g. \qquad (2)$$

The Poisson bracket of two smooth functions defined on a phase space leads to an algebraic structure called a Poisson algebra. A Poisson algebra $(P, \cdot, \{\ ,\ \})$ is a vector space P with two binary operations, where $(P, \cdot)$ is a commutative associative algebra and $(P, \{\ ,\ \})$ is a Lie algebra. Additionally, these two binary operations satisfy the relation

$$\{z, x \cdot y\} = \{z, x\} \cdot y + x \cdot \{z, y\}, \quad x, y, z \in P, \qquad (3)$$

which is usually called a compatibility condition.

The notion of a transposed Poisson algebra was introduced and studied in [6]. By definition, a transposed Poisson algebra $(\mathcal{P}, \cdot, \{\ ,\ \})$, similarly to a Poisson algebra, is a vector space $\mathcal{P}$ with two binary operations, where the first is a commutative associative multiplication $(x, y) \in \mathcal{P} \times \mathcal{P} \to x \cdot y \in \mathcal{P}$, and the second is a Lie bracket $(x, y) \in \mathcal{P} \times \mathcal{P} \to \{x, y\} \in \mathcal{P}$. However, the compatibility condition in the case of a transposed Poisson algebra has a form different from the compatibility condition (3) for a Poisson algebra. The compatibility condition in the case of a transposed Poisson algebra has the form

$$2\,z\cdot\{x,y\}=\{z\cdot x,y\}+\{x,z\cdot y\}. \tag{4}$$

The compatibility condition of a transposed Poisson algebra shows that, for any element x of a transposed Poisson algebra, the operator of multiplication from the left $L_x(y)=x\cdot y$ is similar to the derivation of a Lie bracket (if we do not take into account the factor 2 on the left). In this sense, the condition (4) is the transpose of the condition (3), and this makes it appropriate to use the term 'transposed' in the name of an algebra. An example of a transposed Poisson algebra can be constructed by means of a derivation of a commutative associative algebra [6]. If $(A,\cdot)$ is a commutative associative algebra and $D:A\to A$ is a derivation, then the bracket

$$[x,y]=x\cdot D(y)-y\cdot D(x),\quad x,y\in A, \tag{5}$$

is a Lie bracket, and it satisfies the compatibility condition (4). Hence, $(A,\cdot,[\ ,\ ])$ is a transposed Poisson algebra. An overview of the recent results in transposed Poisson algebras can be found in [8].

The notion of a transposed Poisson algebra can be extended to superalgebras if in the definition of a transposed Poisson algebra we assume that $(\mathcal{P},\cdot)$ is a commutative associative superalgebra, and $(\mathcal{P},\{\ ,\ \})$ is a Lie superalgebra [14]. Using the Koszul sign rule, we write the compatibility condition (4) in the graded form

$$2\,z\cdot\{x,y\}=\{z\cdot x,y\}+(-1)^{|x||z|}\{x,z\cdot y\}, \tag{6}$$

where $|x|,|z|$ are the parities of elements of a superalgebra. The notion of a transposed Poisson superalgebra was introduced in [3], and later the same definition was given in [16], where the author proved that the Kantor double of a transposed Poisson algebra is a Jordan superalgebra. In [3], it was shown that one can construct an important example of the transposed Poisson superalgebra using an even derivation of a commutative superalgebra in a similar way to how this is done in the case of a transposed Poisson algebra (5). In more detail, if we have a commutative superalgebra $(A,\cdot)$ and its even derivation D, then we define the bracket by the formula

$$[x,y]=x\cdot D(y)-(-1)^{|x||y|}\,y\cdot D(x), \tag{7}$$

where x,y are the elements of a commutative superalgebra, $|x|,|y|$ are their parities, and $D:A\to A$ is an even derivation. Then $(A,[\ ,\ ])$ is a Lie superalgebra called a virasorisation of A [18]. Now, it can be proved that $(A,\cdot,[\ ,\ ])$ is a transposed Poisson superalgebra [3]. It is worth noting that it is not possible to use the bracket (7) when D is an odd derivation because in this case the consistency of the parities is broken; that is, the parity of the right-hand side of the equation (7) is $|x|+|y|+1$, but it should be $|x|+|y|$. In this paper, we investigate the question of how an odd derivation of a commutative superalgebra can be used to construct (analogously to (7)) a structure similar to a transposed Poisson superalgebra. To this end, we consider a left A-supermodule $\mathcal{E}=\mathcal{E}_0\oplus\mathcal{E}_1$, where $A=A_0\oplus A_1$ is a commutative superalgebra. We assume that a left A-supermodule $\mathcal{E}$ is endowed with a bilinear bracket $(X,Y)\in\mathcal{E}\times\mathcal{E}\to\{X,Y\}\in\mathcal{E}$ that is commutative $\{X,Y\}=(-1)^{|X||Y|}\{Y,X\}$ ($|X|,|Y|$ are the parities) and satisfies $|\{X,Y\}|=|X|+|Y|$. Next, we assume that $(\mathcal{E}_0,\{\ ,\ \})$ is a Jordan algebra [13], and that the mappings $\mathcal{E}_0\times\mathcal{E}_1\to\mathcal{E}_1,\mathcal{E}_1\times\mathcal{E}_0\to\mathcal{E}_1$ are defined by $(X,Y)\to\{X,Y\}$ and $\{Y,X\}\to\{Y,X\}$, where $X\in\mathcal{E}_0,Y\in\mathcal{E}_1$; these mappings define on $\mathcal{E}_1$ the structure of a Jordan module over $\mathcal{E}_0$ [9] . Then we propose to call the left A-supermodule structure of $\mathcal{E}$ a transposed Poisson type compatible (TP-compatible, in short) with a Jordan structure of $\mathcal{E}$ if they satisfy the super transposed Poisson compatibility condition (6), written in this case in the form

$$2\,z\cdot\{X,Y\}=\{z\cdot X,Y\}+(-1)^{|z||X|}\{X,z\cdot Y\},\quad z\in A,\quad X,Y\in\mathcal{E}, \tag{8}$$

where the dot stands for the multiplication of elements of the left A-supermodule $\mathcal{E}$ by elements of the superalgebra A.

As an example of the TP-compatible structure, we consider a commutative superalgebra A, its odd derivation δ and the super vector space $\mathfrak{D}=\{x\cdot\delta : x\in A\}$, where $(x\cdot\delta)(y)=x\,\delta(y)$ and

$|x \cdot \delta| = |x| + 1$. Obviously, $\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}_1$ is a left A-supermodule, where $\mathfrak{D}_0 \cong A_1, \mathfrak{D}_1 \cong A_0$. We endow $\mathfrak{D}$ with the bracket [18]

$$\{X, Y\} = (x\,\delta(y) + (-1)^{(|x|+1)(|y|+1)} y\,\delta(x)) \cdot \delta, \tag{9}$$

where $X = x \cdot \delta, Y = y \cdot \delta$. We prove that $(\mathfrak{D}_0, \{\ ,\ \})$ is a Jordan algebra and $\mathfrak{D}_1$ is a Jordan module over $\mathfrak{D}_1$. Then we prove that the left A-supermodule structure of $\mathfrak{D}$ is TP-compatible with the Jordan structure of $\mathfrak{D}$, induced by the bracket (9); that is, we prove the TP-compatibility condition (8).

The second question we consider is the question of how one can construct a 3-Lie superalgebra given a transposed Poisson superalgebra and its even derivation. In the paper [6], the authors showed a very important connection between transposed Poisson algebras and 3-Lie algebras. In particular, it was shown that, given a transposed Poisson algebra and its derivation, one can construct a ternary bracket that satisfies all the requirements of a 3-Lie algebra. The concept of an n-Lie algebra was introduced by Filippov in [10]. Later, a surge of interest in these algebras [2,3,7,19] and their various generalizations was due to their applications in generalized Hamiltonian mechanics [5,15] and M-brane theory [12,17]. An n-Lie superalgebra is a $\mathbb{Z}_2$-extension of the concept of an n-Lie algebra to the super case [1,2]. Our approach to the above-mentioned question is based on the construction proposed in [6], where it was shown how one can construct a 3-Lie algebra given a transposed Poisson algebra and its derivation. To be more exact, we prove that if $\mathcal{P}$ is a transposed Poisson superalgebra and D is an even derivation of this algebra, then the ternary bracket

$$[x, y, z] = D(x) \cdot [y, z] + (-1)^{|x|(|y|+|z|)} D(y) \cdot [z, x] + (-1)^{(|x|+|y|)|z|} D(z) \cdot [x, y]$$

satisfies the super Filippov–Jacobi identity or, in other words, $(\mathcal{P}, [\ ,\ ,\ ])$ is a 3-Lie superalgebra. This theorem was proved in [21], and we present this proof in Section 4 with some minor modifications.

2. Preliminaries

In this paper, we investigate the structure of a transposed Poisson superalgebra and how it can be used to construct 3-Lie superalgebras. Throughout what follows, the field $\mathbb{K}$ means either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. A Poisson algebra $(P, \cdot, [\ ,\ ])$ is a $\mathbb{K}$-vector space P with two binary operations, where $(P, \cdot)$ is an associative commutative algebra and $(P, [\ ,\ ])$ is a Lie algebra. In addition, these two structures must be compatible; that is, the following condition must be satisfied:

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z], \qquad \forall x, y, z \in P. \tag{10}$$

Thus, the compatibility condition (10) shows that each element of P defines, by means of a Lie bracket, a derivation of an associative commutative algebra P. The well-known example of a Poisson algebra is the associative, commutative algebra of smooth functions on a phase space, equipped with a Poisson bracket. A derivation of a Poisson algebra is a linear mapping $D : P \to P$ that is a derivation of an associative commutative algebra $(P, \cdot)$ as well as a derivation of a Lie algebra $(P, [\ ,\ ])$. Thus, in the case of the derivation of a Poisson algebra, we have

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y), \quad D([x, y]) = [D(x), y] + [x, D(y)]. \tag{11}$$

The notion of a transposed Poisson algebra was first proposed and studied in [6]. This structure turned out to be very interesting from the point of view of algebra and geometry as well as from the point of view of applications in mathematical physics. A transposed Poisson algebra $(\mathcal{P}, \cdot, [\ ,\ ])$, similarly to a Poisson algebra, is a $\mathbb{K}$-vector space $\mathcal{P}$ with two binary operations, where $(\mathcal{P}, \cdot)$ is an associative commutative algebra, and $(\mathcal{P}, [\ ,\ ])$ is a Lie algebra. The difference between the notion of a Poisson algebra and that of a transposed Poisson algebra is the compatibility condition. In the case of the transposed Poisson algebra $\mathcal{P}$, the compatibility condition has the form

$$2x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z], \qquad \forall x, y, z \in \mathcal{P}. \tag{12}$$

This condition shows that each element of $\mathcal{P}$ determines, by means of the multiplication of the associative commutative algebra $(\mathcal{P}, \cdot)$, a derivation of a Lie bracket. Hence, the use of the term 'transposed' in this context is relevant. We also note one more difference between the compatibility condition (12) and the compatibility condition (10). This is the presence of the factor 2 on the left-hand side of (12). This factor plays an important role and is related to the arity of a Lie bracket. An important example of the transposed Poisson algebra proposed in [6] is an associative commutative algebra $(A, \cdot)$, equipped with the Lie bracket:

$$[x, y] = x\,D(y) - y\,D(x), \quad x, y \in A, \tag{13}$$

where D is a derivation of A.

We can extend the notion of a transposed Poisson algebra to superalgebras, assuming that $(\mathcal{P}, \cdot)$ is a commutative associative superalgebra and $(\mathcal{P}, [\ ,\ ])$ is a Lie superalgebra. As a compatibility condition for these two structures, it is natural to take a graded version of the compatibility condition (12) for a transposed Poisson algebra. The notion of a transposed Poisson superalgebra was introduced in [3]. Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ be a super $\mathbb{K}$-vector space. The parity of a homogeneous element $x \in \mathcal{P}$ will be denoted by $|x|$; that is, $|x| \in \mathbb{Z}_2$.

Definition 2.1. *A transposed Poisson superalgebra is a triple $(\mathcal{P}, \cdot, [\ ,\ ])$, where $(\mathcal{P}, \cdot)$ is a commutative associative superalgebra, $(\mathcal{P}, [\ ,\ ])$ is a Lie superalgerba and, for any three elements $x, y, z \in \mathcal{P}$, it holds*

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{|x||z|}[x, z \cdot y]. \tag{14}$$

In [3], it was shown that a transposed Poisson superalgebra can be constructed in a similar way to a transposed Poisson algebra, which is constructed on an associative commutative algebra by means of a derivation and the Lie bracket (13). However, in the case of a superalgebra, one can attribute a parity to a derivation. We remind that a derivation D of a superalgebra A is called even if it does not change the parity of a homogeneous element; that is, $|D(x)| = |x|$ for any $x \in A_0 \cup A_1$, and a derivation D of a superalgebra A is called odd if it changes the parity of a homogeneous element; that is, $|D(x)| = |x| + 1$ for the arbitrary $x \in A_0 \cup A_1$. The parity of a derivation D will be denoted by $|D|$. Hence, $|D| = 0$ if D is an even derivation and $|D| = 1$ if D is odd. The case of an even derivation was considered in [3], and it was shown that if $(A, \cdot)$ is a commutative associative superalgebra, $D : A \longrightarrow A$ is an even derivation, and the bracket is defined by

$$[x, y] = x \cdot D(y) - (-1)^{|x||y|}\, y \cdot D(x), \quad x, y \in A, \tag{15}$$

then $(A, \cdot, [\ ,\ ])$ is a transposed Poisson superalgebra. The case of an odd derivation will be considered in the next section.

One of the goals of this paper is to construct a 3-Lie superalgebra given a transposed Poisson superalgebra and its even derivation. For this we need a graded form of the identities proved in [6]. The graded form of these identities is given in the following theorem proved in [3].

Theorem 1. *Let $(\mathcal{P}, \cdot, [\ ,\ ])$ be a transposed Poisson superalgebra. Then for any $h, x, y, z, u, v \in \mathcal{P}$, we have the following identities:*

$$(-1)^{|x||z|}\, x \cdot [y, z] + (-1)^{|x||y|}\, y \cdot [z, x] + (-1)^{|y||z|}\, z \cdot [x, y] = 0, \tag{16}$$

$$(-1)^{|x||z|}\, [h \cdot [x, y], z] + (-1)^{|x||y|}\, [h \cdot [y, z], x] + (-1)^{|y||z|}\, [h \cdot [z, x], y] = 0, \tag{17}$$

$$(-1)^{|x||z|}\, [h \cdot x, [y, z]] + (-1)^{|x||y|}\, [h \cdot y, [z, x]] + (-1)^{|y||z|}\, [h \cdot z, [x, y]] = 0, \tag{18}$$

$$(-1)^{|x||z|}\, [h, x]\, [y, z] + (-1)^{|x||y|}\, [h, y]\, [z, x] + (-1)^{|y||z|}\, [h, z]\, [x, y] = 0, \tag{19}$$

$$2\, u \cdot v \cdot [x, y] = (-1)^{|x||v|}[u \cdot x, v \cdot y] + (-1)^{|u|\,(|x|+|v|)}[v \cdot x, u \cdot y], \tag{20}$$

$$(-1)^{|u||yv|}\, x \cdot [u, y \cdot v] + (-1)^{|v||xy|}\, v \cdot [x \cdot y, u] + (-1)^{|x||yv|} y \cdot [v, x] \cdot u = 0. \tag{21}$$

Another notion that will play an important role in this paper is the notion of a 3-Lie superalgebra. A 3-Lie superalgebra is an extension of the notion of a 3-Lie algebra to the case of superalgebras. The notion of a 3-Lie algebra is a particular case of the notion of an n-Lie algebra ($n \geq 2$), proposed and developed by Filippov [10]. Let L be a $\mathbb{K}$-vector space. Then $(L, [\ ,\ ,\ ])$ is said to be a 3-Lie algebra if the ternary bracket $[\ ,\ ,\ ] : L \times L \times L \longrightarrow L$ is totally skew-symmetric:

$$[x, y, z] = -[y, x, z], \quad [x, y, z] = -[x, z, y], \quad x, y, z \in L,$$

and any five elements $x, y, z, u, v \in L$ satisfy the Filippov–Jacobi identity

$$[[x, y, z], u, v] = [[x, u, v], y, z] + [[y, u, v], z, x] + [[z, u, v], x, y].$$

Definition 2.2. *Let $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ be a super $\mathbb{K}$-vector space. $(\mathcal{L}, [\ ,\ ,\ ])$ is said to be a 3-Lie superalgebra if a $\mathbb{K}$-trilinear ternary bracket $[\ ,\ ,\ ] : \mathcal{L}^3 \to \mathcal{L}$ has the following properties:*

$$|[x, y, z]| = |x| + |y| + |z|, \tag{22}$$

$$[y, x, z] = -(-1)^{|x||y|}[x, y, z], \quad [x, z, y] = -(-1)^{|y||z|}[x, y, z], \tag{23}$$

and it satisfies the super Filippov–Jacobi identity

$$\begin{aligned}[[x, y, z], u, v] = (-1)^{|yz,uv|}[[x, u, v], y, z] + (-1)^{|x,yz|+|xz,uv|}[[y, u, v], z, x]\\ + (-1)^{|xy,zuv|}[[z, u, v], x, y],\end{aligned} \tag{24}$$

where $|yz, uv| = (|y| + |z|)(|u| + |v|)$, $|x, yz| = |x|(|y| + |z|)$, $|xy, zuv| = (|x| + |y|)(|z| + |u| + |v|)$.

3. The case of odd derivation

A method for constructing a transposed Poisson algebra, if we are given a commutative associative algebra and its derivation, was proposed in [6]. In [3], it was proved that this method can also be applied to construct a transposed Poisson superalgebra; that is, if we are given a commutative associative superalgebra and its even derivation, then we can construct a transposed Poisson superalgebra by equipping the commutative superalgebra with the graded Lie bracket (15). In this section, we consider the question of how this method can be extended to odd derivations of a commutative superalgebra. This question is important because odd derivations play a significant role not only in the structure of a commutative superalgebra but also in its applications. In this case, odd derivations whose square is zero play a particularly important role; for example, the exterior differential in a graded differential algebra of differential forms and the BRST-operator in quantum field theory [20].

Thus, our goal in this section is to include odd derivations of a commutative superalgebra in the scheme of constructing algebras such as a transposed Poisson superalgebra. For this purpose, we introduce the following notion.

Definition 3.1. *Let $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ be a left supermodule over a commutative superalgebra $A = A_0 \oplus A_1$; that is, $(x, X) \in A \times \mathcal{E} \mapsto x \cdot X \in \mathcal{E}$, where $|x \cdot X| = |x| + |X|$ and $|x|, |X|$ are degrees of modulo 2. Let $\{\ ,\ \} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ be a $\mathbb{K}$-bilinear mapping such that*

a) $|\{X, Y\}| = |X| + |Y|$,

b) $\{X, Y\} = (-1)^{|X||Y|}\{Y, X\}$,

c) $(\mathcal{E}_0, \{\ ,\ \})$ *is a Jordan algebra,*

d) *the mappings $\{\ ,\ \} : \mathcal{E}_0 \times \mathcal{E}_1 \to \mathcal{E}_1$ and $\{\ ,\ \} : \mathcal{E}_1 \times \mathcal{E}_0 \to \mathcal{E}_1$ define the Jordan $\mathcal{E}_1$ over $\mathcal{E}_0$.*

Then we say that a left A-supermodule $\mathcal{E}$ is of transposed Poisson type, compatible with a Jordan structure of $\mathcal{E}$ if for any $z \in A, X, Y \in \mathcal{E}$ we have the identity

$$2\, z \cdot \{X, Y\} = \{z \cdot X, Y\} + (-1)^{|z|\,|X|}\{X, z \cdot Y\}.$$

From this definition it follows that if $X_0 \in \mathcal{E}_0, X_1 \in \mathcal{E}_1$, then $\{X_0, X_1\} = \{X_1, X_0\}$. This shows that the elements of $\mathcal{E}_1$ obtained from X_1 by the left and right actions of $X_0 \in \mathcal{E}_0$ are equal, as it should be in the case of a Jordan module. Thus, the only condition that should be checked is that $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ is a Jordan algebra if we equip it with the multiplication

$$(X_0, X_1), (Y_0, Y_1) = (\{X_0, Y_0\}, \{X_0, Y_1\} + \{X_1, Y_0\}). \tag{25}$$

In what follows, we assume that A is a commutative superalgebra, and multiplication in this algebra will be denoted by means of a juxtaposition, that is, writing one element after another. Then the vector space of derivations of this algebra is a Lie superalgebra if we equip it with the bracket

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|}\, D_2 \circ D_1, \tag{26}$$

where D_1, D_2 are the derivations of A, and $|D_1|, |D_2|$ are their parities. Now, assume that D is a derivation of a superalgebra A, and D has a certain parity; that is, D is either an even or an odd derivation. We consider the super vector space $\mathfrak{D}^D = \{x \cdot D : x \in A\}$, generated by D. Here $x \cdot D$ is a derivation of a superalgebra A, defined by $(x \cdot D)(y) = xD(y)$ and $|x \cdot D| = |x| + |D|$. Using the terminology of differential geometry, we will call derivations $x \cdot D$ vector fields. Obviously, $\mathfrak{D}^D$ becomes a left A-supermodule if one defines $x \cdot (y \cdot D) = (xy) \cdot D$. Let $X = x \cdot D, Y = y \cdot D$ and, calculating the bracket (26), we find:

$$\begin{aligned} [X,Y] \;=\; & \big(x\,D(y) - (-1)^{(|x|+|D|)(|y|+|D|)}\,y\,D(x)\big) \cdot D \\ & +\big((-1)^{|y||D|} - (-1)^{|y||D|+|D|^2}\big)\,(xy) \cdot D^2. \end{aligned} \tag{27}$$

Assume D is an even derivation of A; that is, $|D| = 0$. In this case, the second term (containing D^2) on the right-hand side of the above formula vanishes, and we get

$$[X,Y] = \big(x\,D(y) - (-1)^{|x||y|}\,y\,D(x)\big) \cdot D. \tag{28}$$

Thus, the super vector space $\mathfrak{D}^D$ generated by an even degree derivation D closes with respect to the graded commutator (27), and we obtain the structure of the Lie superalgebra on $\mathfrak{D}^D$. It is easy to see that we can omit D on both sides of (28), and the consistency of the parities will not be broken. Figuratively speaking, we can descend the bracket (28) from the vector fields $\mathfrak{D}^D$ to the elements of a superalgebra A. In this case, we get

$$[x,y] = x\,D(y) - (-1)^{|x||y|}\,y\,D(x). \tag{29}$$

Then it can be proved (see [3]) that the bracket (29) satisfies the compatibility condition

$$2\,z\,[x,y] = [zx,y] + (-1)^{|z|\,|x|}[x,zy].$$

Hence, a commutative superalgebra A endowed with the bracket (29) is a transposed Poisson superalgebra.

Now, we consider the case of an odd derivation of a commutative superalgebra A. This odd derivation will be denoted by δ; that is, $|\delta| = 1$. Let us consider the super vector space $\mathfrak{D}^\delta = \{x \cdot \delta : x \in A\}$, induced by an odd derivation δ. In the case of an odd derivation, we have $|y||\delta| = |y|,\ |y||\delta| + |\delta|^2 = |y| + 1$, and the second term on the right-hand side of (27), containing δ^2, does not vanish. However, if in the right-hand side of the formula (27) we take plus instead of minus; that is, we consider the bracket

$$\begin{aligned} \{X,Y\} \;=\; & \big(x\,\delta(y) + (-1)^{(|x|+|\delta|)(|y|+|\delta|)}\,y\,\delta(x)\big) \cdot \delta \\ & +\big((-1)^{|y|} + (-1)^{|y|+1}\big)\,(xy) \cdot \delta^2, \end{aligned} \tag{30}$$

then the term containing δ^2 vanishes. For $X = x \cdot \delta$ and $Y = y \cdot \delta$, we will have

$$\{X,Y\} = \big(x\,\delta(y) + (-1)^{(|x|+1)(|y|+1)}\,y\,\delta(x)\big) \cdot \delta. \tag{31}$$

Comparing it with the case of an even derivation, we see that in the case of the odd derivation δ, we cannot omit δ in the vector fields in (31) because we will break the consistency of the parities; that is, we will get $|\{x,y\}| = |x| + |y| + 1$ instead of what it should be: $|\{x,y\}| = |x| + |y|$. Hence, we cannot, so to speak, descend the bracket (31) from the vector fields $\mathfrak{D}^\delta$ on to a superalgebra A.

Obviously, $\mathfrak{D}^\delta$ becomes a super vector space if one defines the parity of a vector field $x \cdot \delta$ as $|x| + 1$. The super vector space $\mathfrak{D}^\delta$ is a left A-supermodule if we define $y \cdot (x \cdot \delta) = (yx) \cdot \delta$.

Theorem 2. $(\mathfrak{D}^\delta_0, \{\ ,\ \})$ *is a Jordan algebra. If* $\delta^2 = 0$*, then the bracket* (31) *defines on* $\mathfrak{D}^\delta_1$ *the structure of a Jordan module over* $\mathfrak{D}^\delta_0$*. The left A-supermodule* $\mathfrak{D}^\delta$ *is of transposed Poisson type compatible with the Jordan structure of* $\mathfrak{D}^\delta$*; that is, we have the identity*

$$2z \cdot \{X,Y\} = \{z \cdot X, Y\} + (-1)^{|z||X|}\{X, z \cdot Y\}. \tag{32}$$

Proof. First of all, we prove the transposed Poisson type compatibility condition (32). Applying (31) to the left-hand side of (32), we find

$$2z\cdot\{X,Y\}=2\left(z\,x\,\delta(y)+(-1)^{|X||Y|}\,z\,y\,\delta(x)\right)\cdot\delta. \tag{33}$$

Calculating the terms on the right-hand side of (32), we get

$$\begin{aligned}\{z\cdot X,Y\}&=\left(z\,x\,\delta(y)+(-1)^{(|z|+|X|)|Y|}y\,\delta(z)\,x+(-1)^{|X||Y|}z\,y\,\delta(x)\right)\cdot\delta,\\ \{X,z\cdot Y\}&=\left(x\,\delta(z)\,y+(-1)^{|z||X|}z\,x\,\delta(y)+(-1)^{|X|(|z|+|Y|)}z\,y\,\delta(x)\right)\cdot\delta.\end{aligned}$$

Now, multiplying both sides of the second equation by $(-1)^{|z||X|}$ and taking the sum with the first equation, we get the right-hand side of (33).

The proof that $(\mathfrak{D}_0^\delta,\{\ ,\ \})$ is a Jordan algebra can be found in [18]. Since the proof given in [18] is very short and, in our opinion, does not explain some details, we give a more detailed proof here. It is easy to see that the bracket (31) has the symmetry

$$\{X,Y\}=(-1)^{|X||Y|}\{Y,X\}. \tag{34}$$

Hence, if $X=x\cdot\delta, Y=y\cdot\delta\in\mathfrak{D}_0^\delta$, then $|x|=|y|=1$ and $\{X,Y\}=\{Y,X\}$; that is, the bracket (31), restricted to the subspace $\mathfrak{D}_0^\delta$, is commutative. Hence, in order to prove that $(\mathfrak{D}_0^\delta,\{\ ,\ \})$ is a Jordan algebra, we need to prove the Jordan identity

$$\big\{\{X,X\},\{Y,X\}\big\}=\big\{\{\{X,X\},Y\},X\big\}. \tag{35}$$

By a straightforward calculation, we find

$$\begin{aligned}\big\{\{X,X\},\{Y,X\}\big\}&=\big(6\,x\,(\delta(x))^2\,\delta(y)-2\,x^2\,\delta(x)\,\delta^2(y)+2\,y\,(\delta(x))^3\\ &\qquad -2\,x^2\,\delta^2(x)\,\delta(y)\big)\cdot\delta, \qquad (36)\\ \big\{\{\{X,X\},Y\},X\big\}&=\big(6\,x\,(\delta(x))^2\,\delta(y)-2\,x^2\,\delta(x)\,\delta^2(y)\\ &\qquad +2\,y\,(\delta(x))^3-4\,x^2\,\delta^2(x)\,\delta(y)+2\,x^2\,y\,\delta^3(x)\big)\cdot\delta. \qquad (37)\end{aligned}$$

Since a superalgebra A is commutative for any odd element $x\in A_1$, we have $x^2=0$. Therefore, all terms on the right-hand sides of the above equalities containing x^2 vanish, and we see that the right-hand sides are equal. From this follows the Jordan identity.

Let us now prove that if $\delta^2=0$, then $\mathfrak{D}_1^\delta$ is a Jordan module over the Jordan algebra $\mathfrak{D}_0^\delta$. First, we have to show that the left and right actions of $\mathfrak{D}_0^\delta$ on $\mathfrak{D}_1^\delta$, determined by the bracket (31), coincide; that is, for any $X\in\mathfrak{D}_0^\delta$ and $Y\in\mathfrak{D}_1^\delta$ it holds $\{X,Y\}=\{Y,X\}$. But this follows immediately from the formula (34) when $X\in\mathfrak{D}_0^\delta, Y\in\mathfrak{D}_1^\delta$. Then we must show that (35) holds when $X\in\mathfrak{D}_0^\delta$ and $Y\in\mathfrak{D}_1^\delta$. However, in this case the formulae (36), (37) do not change if we assume that X is an even vector field and Y is odd. Indeed, if X is an even vector field and Y is odd, the formula for the bracket (31) will have the same form as in the case of even vector fields; that is, $\{X,Y\}=(x\,\delta(y)+y\,\delta(x))\cdot\delta$.

The only identity that we need to prove in order to show that $\mathfrak{D}_1^\delta$ is a Jordan module over the Jordan algebra $\mathfrak{D}_0^\delta$ is the following:

$$\Big\{\big\{\{X,X\},Y\big\},Z\Big\}-\Big\{\{X,X\},\{Y,Z\}\Big\}=2\Big\{\{X,Y\},\{X,Z\}\Big\}-2\Big\{X,\big\{Y,\{X,Z\}\big\}\Big\}. \tag{38}$$

Now, we assume $\delta^2=0$. We have

$$\begin{aligned}\{X,X\}&=2\,(x\,\delta(x))\cdot\delta,\\ \Big\{\big\{\{X,X\},Y\big\},Z\Big\}&=\big(2\,x\,\delta(x)\,\delta(y)\,\delta(z)+2\,(\delta(x))^2\,y\,\delta(z)+4\,(\delta(x))^2\,\delta(y)\,z\big)\cdot\delta,\\ \Big\{\{X,X\},\{Y,Z\}\Big\}&=\big(4\,x\,\delta(x)\,\delta(y)\,\delta(z)+2\,(\delta(x))^2\,y\,\delta(z)+2\,(\delta(x))^2\,\delta(y)\,z\big)\cdot\delta.\end{aligned}$$

Hence, the left-hand side of (38) is equal to

$$-2\,x\,\delta(x)\,\delta(y)\,\delta(z)+2\,(\delta(x))^2\,\delta(y)\,z. \tag{39}$$

The terms on the right-hand side of (38) can be written as follows:

$$
\begin{aligned}
2\Big\{\{X,Y\},\{X,Z\}\Big\} &= \big(8\,x\,\delta(x)\,\delta(y)\,\delta(z)+4\,(\delta(x))^2\,y\,\delta(z)+4\,(\delta(x))^2\,\delta(y)\,z\big)\cdot\delta,\\
2\Big\{X,\{Y,\{X,Z\}\}\Big\} &= \big(10\,x\,\delta(x)\,\delta(y)\,\delta(z)+4\,(\delta(x))^2\,y\,\delta(z)+2\,(\delta(x))^2\,\delta(y)\,z\big)\cdot\delta.
\end{aligned}
$$

Hence, the right-hand side of (38) is the expression

$$-2\,x\,\delta(x)\,\delta(y)\,\delta(z)+2\,(\delta(x))^2\,\delta(y)\,z,$$

and by comparing it with the expression on the left-hand side of (39), we conclude that the Jordan module identity is satisfied and this ends the proof. □

4. 3-Lie superalgebra constructed by means of even derivation

In the article [6], the authors prove an important theorem that gives a method for constructing 3-Lie algebras by means of transposed Poisson algebras and their derivations. This theorem can be stated as follows.

Theorem 3. *Let $(L,\cdot,[\ ,\ ])$ be a transposed Poisson algebra and D be its derivation. Define a ternary operation on L as follows:*

$$[x,y,z]:=D(x)[y,z]+D(y)[z,x]+D(z)[x,y],\quad x,y,z\in L. \tag{40}$$

Then $(L,[\ ,\ ,\ ])$ is a 3-Lie algebra.

Our aim in this section is to extend this result to the case of a transposed Poisson superalgebra. We show that, given a transposed Poisson superalgebra and an even derivation of this superalgebra, we can construct a 3-Lie superalgebra.

We start with the following lemma.

Lemma 1. *If $(\mathcal{P},\cdot,[\ ,\ ])$ is a transposed Poisson superalgebra and D is an even derivation of a Lie superalgebra $(\mathcal{P},[\ ,\ ])$, then*

$$
\begin{aligned}
&D(x)\cdot D([y,z])+(-1)^{|x,yz|}D(y)\cdot D([z,x])+(-1)^{|xy,z|}D(z)\cdot D([x,y])\\
&\quad+x\cdot[D(y),D(z)]+(-1)^{|x,yz|}y\cdot[D(z),D(x)])+(-1)^{|xy,z|}z\cdot[D(x),D(y)]=0.
\end{aligned}
$$

Proof. Since $D:\mathcal{P}\to\mathcal{P}$ is an even derivation of a Lie superalgebra $(\mathcal{P},[\ ,\ ])$, it satisfies the graded Leibniz rule. Since it is an even derivation, we also have $|D(x)|=|x|$ for any $x\in\mathcal{P}$. Making use of the graded Leibniz rule, the compatibility condition (14) and the cyclic permutations of x,y,z, we get

$$
\begin{aligned}
D(x)\cdot D([y,z]) &= \frac{1}{2}\big([D(x)\cdot D(y),z]+(-1)^{|x||y|}[D(y),D(x)\cdot z]+[D(x)\cdot y,D(z)]\\
&\qquad\qquad\qquad\qquad+(-1)^{|x||y|}[y,D(x)\cdot D(z)]\big),\\
D(y)\cdot D([z,x]) &= \frac{1}{2}\big([D(y)\cdot D(z),x]+(-1)^{|y||z|}[D(z),D(y)\cdot x]+[D(y)\cdot z,D(x)]\\
&\qquad\qquad\qquad\qquad+(-1)^{|y||z|}[z,D(y)\cdot D(x)]\big),\\
D(z)\cdot D([x,y]) &= \frac{1}{2}\big([D(z)\cdot D(x),y]+(-1)^{|x||z|}[D(x),D(z)\cdot y]+[D(z)\cdot x,D(y)]\\
&\qquad\qquad\qquad\qquad+(-1)^{|x||z|}[x,D(z)\cdot D(y)]\big).
\end{aligned}
$$

Taking the sum of these equations multiplied by $(-1)^{|x,yz|}$ (second equation) and $(-1)^{|xy,z|}$ (third equation), we get the equation whose left-hand side is

$$D(x)\cdot D([y,z])+(-1)^{|x,yz|}D(y)\cdot D([z,x])+(-1)^{|xy,z|}D(z)\cdot D([x,y]), \tag{41}$$

and the right-hand side can be written in the form

$$
\begin{aligned}
&-\frac{1}{2}\big([x\cdot D(y),D(z)]+(-1)^{|x||y|}[D(y),x\cdot D(z)]+(-1)^{|x,yz|}([y\cdot D(z),D(x)]\\
&\quad+(-1)^{|y||z|}[D(z),y\cdot D(x)]+(-1)^{|xy,z|}([z\cdot D(x),D(y)]+(-1)^{|x||z|}[D(x),z\cdot D(y)])\big).
\end{aligned}
$$

Making use of the compatibility condition for a transposed Poisson superalgebra, we can write the right-hand side in the form

$$-x\cdot[D(y),D(z)]-(-1)^{|x,yz|}y\cdot[D(z),D(x)]-(-1)^{|xy,z|}z\cdot[D(x),D(y)],$$

which ends the proof of the lemma. □

Theorem 4. *Let $(\mathcal{P},\cdot,[\ ,\ ])$ be a transposed Poisson superalgebra and D be its even derivation. Define the ternary bracket*

$$[x,y,z]:=D(x)\cdot[y,z]+(-1)^{|x,yz|}D(y)\cdot[z,x]+(-1)^{|xy,z|}D(z)\cdot[x,y], \tag{42}$$

where $x,y,z\in\mathcal{P}$. Then $(\mathcal{P},[\ ,\ ,\ ])$ is a 3-Lie superalgebra.

Proof. It is easy to verify that the ternary bracket (42) is trilinear and graded skew-symmetric. Hence, in order to prove the theorem, we have to prove the super Filippov–Jacobi identity (24). We begin with the left-hand side of (24). Applying the definition of the ternary bracket (42), we get

$$\begin{aligned}[][[x,y,z],u,v]&=D\Big(D(x)[y,z]\Big)[u,v]+(-1)^{|xyz,uv|}D(u)[v,D(x)[y,z]]\\&\quad+(-1)^{|v,xyzu|}D(v)[D(x)[y,z],u]+(-1)^{|x,yz|}D\Big(D(y)[z,x]\Big)[u,v]\\&\quad+(-1)^{|xyz,uv|+|x,yz|}D(u)[v,D(y)[z,x]]+(-1)^{|v,xyzu|+|x,yz|}D(v)[D(y)[z,x],u]\\&\quad+(-1)^{|xy,z|}D\Big(D(z)[x,y]\Big)[u,v]+(-1)^{|xyz,uv|+|xy,z|}D(u)[v,D(z)[x,y]]\\&\quad+(-1)^{|v,xyzu|+|xy,z|}D(v)[D(z)[x,y],u].\end{aligned}$$

Making use of the graded Leibniz rule for the even derivation D, we obtain

$$\begin{aligned}D\Big(D(x)[y,z]\Big)[u,v]&=D^2(x)[y,z][u,v]+\underline{D(x)D([y,z])}[u,v],\\(-1)^{|x,yz|}D\Big(D(y)[z,x]\Big)[u,v]&=(-1)^{|x,yz|}D^2(y)[z,x][u,v]+\underline{(-1)^{|x,yz|}D(y)D([z,x])}[u,v],\\(-1)^{|xy,z|}D\Big(D(z)[x,y]\Big)[u,v]&=(-1)^{|xy,z|}D^2(z)[x,y][u,v]+\underline{(-1)^{|xy,z|}D(z)D([x,y])}[u,v].\end{aligned}$$

Note that the sum of the underlined terms is equal to the sum of the first three terms in the equation in Lemma 1, multiplied by $[u,v]$. Hence, using the identity of Lemma 1, we can write this sum as follows:

$$\begin{aligned}&D(x)D([y,z])[u,v]+(-1)^{|x,yz|}D(y)D([z,x])[u,v]+(-1)^{|xy,z|}D(z)D([x,y])[u,v]\\&\quad=-x[D(y),D(z)][u,v]-(-1)^{|x,yz|}y[D(z),D(x)][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v].\end{aligned}$$

Now, the left-hand side of the super Filippov–Jacobi identity can be written in the following form:

$$\begin{aligned}&D^2(x)[y,z][u,v]+(-1)^{|x,yz|}D^2(y)[z,x][u,v]+(-1)^{|xy,z|}D^2(z)[x,y][u,v]\\&+(-1)^{|xyz,uv|}D(u)\big([v,D(x)[y,z]]+(-1)^{|x,yz|}[v,D(y)[z,x]]+(-1)^{|xy,z|}[v,D(z)[x,y]]\big)\\&+(-1)^{|xyzu,v|}D(v)\big([D(x)[y,z],u](-1)^{|x,yz|}[D(y)[z,x],u]\\&+(-1)^{|xy,z|}[D(z)[x,y],u]\big)-x[D(y),D(z)][u,v]\\&-(-1)^{|x,yz|}y[D(z),D(x)][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v].\end{aligned} \tag{43}$$

We label every term of this expression by a pair (L, n), where L stands for the left-hand side of the super Filippov–Jacobi identity and n is the positional number of the term in this expression. For example, (L, 2) is a label for $(-1)^{|x,yz|}D^2(y)[z,x][u,v]$.

Now, we calculate the terms on the right-hand side of the super Filippov–Jacobi identity (24).

We get

$$\begin{aligned}(-1)^{|yz,uv|}[[x,u,v],y,z] &= D^2(x)[y,z][u,v] - (-1)^{|yz,uv|}x[D(u),D(v)][y,z]\\ &+ (-1)^{|x,yz|}D(y)[z,D(x)[u,v]] + (-1)^{|xy,z|+|y,uv|}D(z)[D(x)[u,v],y]\\ &+ (-1)^{|xyz,uv|}D^2(u)[v,x][y,z] - (-1)^{|xyz,uv|}u[D(v),D(x)][y,z]\\ &+ (-1)^{|x,yzuv|}D(y)[z,D(u)[v,x]] + (-1)^{|xy,zuv|}D(z)[D(u)[v,x],y]\\ &+ (-1)^{|xu,v|+|yz,uv|}D^2(v)[x,u][y,z] - (-1)^{|xu,v|+|yz,uv|}v[D(x),D(u)][y,z]\\ &+ (-1)^{|x,yz|+|xu,v|}D(y)[z,D(v)[x,u]] + (-1)^{|xy,z|+|xu,v|+|y,uv|}D(z)[D(v)[x,u],y].\end{aligned} \tag{44}$$

$$\begin{aligned}(-1)^{|x,yz|+|xz,uv|}&[[y,u,v],z,x] = (-1)^{|x,yz|}D^2(y)[z,x][u,v]\\ &- (-1)^{|x,yz|+|xz,uv|}y[D(u),D(v)][z,x] + (-1)^{|xy,z|}D(z)[x,D(y)[u,v]]\\ &+ (-1)^{|z,uv|}D(x)[D(y)[u,v],z] + (-1)^{|xyz,uv|+|x,yz|}D^2(u)[v,y][z,x]\\ &- (-1)^{|xyz,uv|+|x,yz|}u[D(v),D(y)])[z,x] + (-1)^{|xy,z|+|y,uv|}D(z)[x,D(u)[v,y]]\\ &+ (-1)^{|yz,uv|}D(x)[D(u)[v,y],z] + (-1)^{|x,yz|+|u,xzv|+|xyz,v|}D^2(v)[y,u][z,x]\\ &- (-1)^{|x,yz|+|u,xzv|+|xyz,v|}v[D(y),D(u)][z,x] + (-1)^{|xy,z|+|yu,v|}D(z)[x,D(v)[y,u]]\\ &+ (-1)^{|z,uv|+|yu,v|}D(x)[D(v)[y,u],z].\end{aligned} \tag{45}$$

$$\begin{aligned}(-1)^{|xy,zuv|}[[z,u,v],x,y] &= (-1)^{|xy,z|}D^2(z)[x,y][u,v] - (-1)^{|xy,zuv|}z[D(u),D(v)][x,y]\\ &+ D(x)[y,D(z)[u,v]] + (-1)^{|x,yzuv|}D(y)[D(z)[u,v],x]\\ &+ (-1)^{|xyz,uv|+|xy,z|}D^2(u)[v,z][x,y] - (-1)^{|xyz,uv|+|xy,z|}u[D(v),D(z)][x,y]\\ &+ (-1)^{|z,uv|}D(x)[y,D(u)[v,z]] + (-1)^{|z,xuv|+|x,yuv|}D(y)[D(u)[v,z],x]\\ &+ (-1)^{|zu,xyv|+|xy,v|}D^2(v)[z,u][x,y] - (-1)^{|zu,xyv|+|xy,v|}v[D(z),D(u)][x,y]\\ &+ (-1)^{|zu,v|}D(x)[y,D(v)[z,u]] + (-1)^{|x,yzuv|+|zu,v|}D(y)[D(v)[z,u],x].\end{aligned} \tag{46}$$

Analogously, we label the terms of these expressions by pairs (m,n), where m is a Roman number I, II or III, where I stands for (44), II for (45) and III for (46), and n is the positional number of a term in the corresponding expression. For example, the pair (I,1) denotes the term $(-1)^{|yz,uv|}D^2(x)[u,v][y,z]$. We see that the terms (I,1) and (L,1), (II,1) and (L,2), (III,1) and (L,3) cancel each other. Our next goal is to show that the remaining terms containing a square of the derivation D can also be cancelled. First, collect all the terms with $D^2(u)$:

$$(-1)^{|xyz,uv|}D^2(u)\Big([v,x][y,z] + (-1)^{|x,yz|}[v,y][z,x] + (-1)^{|z,xy|}[v,z][x,y]\Big).$$

This sum is equal to zero due to the identity (19). Analogously, all the terms with $D^2(v)$ can be cancelled. The sum of the terms containing $[D(u),D(v)]$ is equal to zero due to the identity (16). Now, making use of the identity (17), we obtain:

$$\begin{aligned}(\mathrm{III},12) + (\mathrm{I},11) &= -(-1)^{|xzu,v|+|x||y|}D(y)[D(v)[x,z],u],\\ (\mathrm{II},7) + (\mathrm{I},8) &= -(-1)^{|xy,zu|}D(z)[D(u)[x,y],v],\\ (\mathrm{II},12) + (\mathrm{III},11) &= -(-1)^{|yzu,v|+|y||z|}D(x)[D(v)[z,y],u],\\ (\mathrm{I},7) + (\mathrm{III},8) &= -(-1)^{|x,yzu|+|z||u|}D(y)[D(u)[z,x],v],\\ (\mathrm{I},12) + (\mathrm{II},11) &= -(-1)^{|x,yzv|+|y,zv|+|u||v|}D(z)[D(v)[y,x],u],\\ (\mathrm{III},7) + (\mathrm{II},8) &= -(-1)^{|yz,u|}D(x)[D(u)[y,z],v].\end{aligned}$$

After all cancellations on the right-hand side of the super Filippov–Jacobi identity, we have the following terms:

$$
\begin{aligned}
&-(-1)^{|yz,u|}D(x)[D(u)[y,z],v]-(-1)^{|x,yzu|+|z||u|}D(y)[D(u)[z,x],v]\\
&\qquad-(-1)^{|xy,zu|}D(z)[D(u)[x,y],v]\\
&-(-1)^{|yzu,v|+|y||z|}D(x)[D(v)[z,y],u]-(-1)^{|xzu,v|+|x||y|}D(y)[D(v)[x,z],u]\\
&\qquad-(-1)^{|x,yzv|+|y,zv|+|u||v|}D(z)[D(v)[y,x],u]\\
&+D(x)[y,D(z)[u,v]]+(-1)^{|x,yz|}D(y)[z,D(x)[u,v]]\\
&\qquad+(-1)^{|xy,z|}D(z)[x,D(y)[u,v]]\\
&+(-1)^{|z,uv|}D(x)[D(y)[u,v],z]+(-1)^{|x,yzuv|}D(y)[D(z)[u,v],x]\\
&\qquad+(-1)^{|xy,z|+|y,uv|}D(z)[D(x)[u,v],y]\\
&-(-1)^{|xyz,uv|}u\big([D(v),D(x)][y,z]+(-1)^{|x,yz|}[D(v),D(y)][z,x]\\
&\qquad+(-1)^{|xy,z|}[D(v),D(z)][x,y]\big)\\
&+(-1)^{|xyz,uv|+|u||v|}v\big([D(u),D(x)][y,z]+(-1)^{|x,yz|}[D(u),D(y)][z,x]\\
&\qquad+(-1)^{|xy,z|}[D(u),D(z)][x,y]\big).
\end{aligned}
\tag{47}
$$

The left-hand side has the form

$$
\begin{aligned}
&(-1)^{|xyz,uv|}D(u)\big([v,D(x)[y,z]]+(-1)^{|x,yz|}[v,D(y)[z,x]]+(-1)^{|xy,z|}[v,D(z)[x,y]]\big)\\
&+(-1)^{|xyzu,v|}D(v)\big([D(x)[y,z],u]+(-1)^{|x,yz|}[D(y)[z,x],u]+(-1)^{|xy,z|}[D(z)[x,y],u]\big)\\
&-x[D(y),D(z)][u,v]-(-1)^{|x,yz|}y[D(z),D(x)][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v].
\end{aligned}
\tag{48}
$$

Making the following substitution $x\to D(u)$, $u\to v$, $y\to[y,z]$, $v\to D(x)$ in the identity (21), we get

$$
\begin{aligned}
(-1)^{|xyz,uv|}D(u)[v,D(x)[y,z]]&=-(-1)^{|yz,u|}D(x)[D(u)[y,z],v]\\
&\quad+(-1)^{|xyz,uv|+|u||v|}v\,[D(u),D(x)][y,z].
\end{aligned}
\tag{49}
$$

In this equation, the left-hand side is equal to the first term on the left-hand side of the super Filippov–Jacobi identity (48), and the right-hand side can be found on the right-hand side of the super Filippov–Jacobi identity (47) (terms underlined by solid black line). Hence, these terms can be cancelled. Analogously, one can see that all the underlined terms in (48) are equal to the sum in the same way as the underlined terms in (47).

After all cancellations, the left-hand side of the super Filippov–Jacobi identity has the form

$$
-x[D(y),D(z)][u,v]-(-1)^{|x,yz|}y[D(z),D(x)][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v],\tag{50}
$$

and the right-hand side of the same identity has the form

$$
\begin{aligned}
&D(x)[y,D(z)[u,v]]+(-1)^{|x,yz|}D(y)[z,D(x)[u,v]]+(-1)^{|xy,z|}D(z)[x,D(y)[u,v]]\\
&\qquad+(-1)^{|z,uv|}D(x)[D(y)[u,v],z]+(-1)^{|x,yzuv|}D(y)[D(z)[u,v],x]\\
&\qquad+(-1)^{|xy,z|+|y,uv|}D(z)[D(x)[u,v],y].
\end{aligned}
\tag{51}
$$

The terms that are underlined in the same way can be cancelled due to the identity (21). This completes the proof of the super Filippov–Jacobi identity and the proof of the theorem. □

5. Conclusion

Given a commutative superalgebra and its even derivation, one can construct the transposed Poisson superalgebra using the bracket (7). In this paper, we considered the question of how this approach can be extended to odd derivations of a commutative superalgebra. To solve this question, we need an analogue of the bracket (7) for the case of an odd derivation. This analogue is the bracket (9). However, this bracket is defined not on the elements of a commutative superalgebra but on the left supermodule over this algebra generated by an odd derivation, and we cannot omit an odd derivation (as in the case of an even derivation) without breaking the consistency of the parities. This leads us to consider a supermodule over a commutative superalgebra. We equip this supermodule with the bracket (9). This bracket induces a Jordan algebra structure on the subspace of even elements and a Jordan module structure on the subspace of odd elements. Thus, we come to the conclusion that if our goal is to treat odd derivations within a structure similar to a transposed Poisson superalgebra, we must, first, pass from a commutative superalgebra to a supermodule over this superalgebra and, second, not use a Lie superalgebra but a Jordan algebra and a Jordan module over this algebra. The latter structure is close in its properties to the notion of a Jordan superalgebra, and such a conjecture was made in [18]. However, our calculations showed that the super Jordan identity does not hold in this case.

Data availability statement

All data are available in the article.

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Paaritud ja paarisderivatsioonid, transponeeritud Poissoni superalgebra ja 3-Lie superalgebra

Viktor Abramov ja Nikolai Sovetnikov

Ühe olulise näite transponeeritud Poissoni algebrast saab konstrueerida kommutatiivse algebra ja selle derivatsiooni abil. Seda lähenemist saab laiendada superalgebratele, st konstrueerida transponeeritud Poissoni superalgebra, kui on antud kommutatiivne superalgebra ja selle paarisderivatsioon. Artiklis näitame, et paaritute derivatsioonide kaasamine selle lähenemisviisi raamistikku eeldab uue struktuuri defineerimist. See on supervektorruum kahe tehtega, mis rahuldavad transponeeritud Poissoni superalgebra kooskõlatingimust. Esimese tehte määrab vasakpoolne supermoodul üle kommutatiivse superalgebra ja teine on Jordani sulg. Seejärel tõestatakse, et kommutatiivse superalgebra paaritu derivatsiooni tekitatud supervektorruumi korral on kõik nimetatud struktuuri tingimused täidetud. Samuti näidatakse, kuidas konstrueerida 3-Lie-superalgebrat, kui on antud transponeeritud Poissoni superalgebra ja selle paarisderivatsioon.