Proceedings of the Estonian Academy of Sciences,
2024, **73**, 4, 379–395
https://doi.org/10.3176/proc.2024.4.07
Available online at www.eap.ee/proceedings

MATRIX
MAPPING

Matrix transformations of double convergent sequences with powers

Maria Zeltser[a] and Şeyda Sezgek[b*]

[a] Department of Mathematics, Tallinn University, 10120 Tallinn, Estonia
[b] Department of Mathematics, Mersin University, 33343 Mersin, Turkey

Received 26 March 2024, accepted 4 July 2024, available online 14 November 2024

Abstract. In 1967, I. J. Maddox generalized the spaces c_0, c, ℓ_∞ by adding the powers p_k $(k \in \mathbb{N})$ in the definitions of the spaces to the terms of elements of sequences (x_k). Gökhan and Çolak in 2004–2006 defined the corresponding double sequence spaces for the Pringsheim and the bounded Pringsheim convergence. We will additionally define the corresponding double sequence spaces for the regular convergence. In 2009, Gökhan, Çolak and Mursaleen characterized some classes of matrix transformations involving these double sequence spaces with powers. However, many of their results appeared to be wrong. In this paper, we give corresponding counterexamples and prove the correct results. Moreover, we present the conditions for a wider class of matrices.

Keywords: matrix transformation, Maddox sequence spaces, double sequence.

1. INTRODUCTION

In [10], Maddox generalized the spaces c_0, c, ℓ_∞ by adding the powers p_k $(k \in \mathbb{N})$ to the terms of elements of sequences (x_k) in the definitions of the spaces. In [4–6], the corresponding double sequence spaces were defined for the Pringsheim and the bounded Pringsheim convergence. We will additionally define the corresponding double sequence spaces for the regular convergence.

First, let us introduce the notions and notations we need in the paper.

Let e be the double sequence with all elements 1 and e^{kl} with the (k,l)-th element equal to 1 and the others 0 $(k,l \in \mathbb{N})$.

The following variable exponent spaces were defined by Maddox [10] and Nakano [11]:

$$c_0(p) = \{(x_k) \in \omega \mid |x_k|^{p_k} \to 0\},$$
$$c(p) = \{(x_k) \in \omega \mid |x_k - l|^{p_k} \to 0 \text{ for some } l\},$$
$$\ell_\infty(p) = \left\{(x_k) \in \omega \mid \sup_k |x_k|^{p_k} < \infty\right\},$$

where ω is the space of all complex (or real) sequences. When all the terms of (p_k) are constant, we have $\ell_\infty(p) = \ell_\infty$, $c(p) = c$ and $c_0(p) = c_0$, where ℓ_∞, c, c_0 are the spaces of bounded, convergent and null sequences, respectively.

* Corresponding author, seydasezgek@mersin.edu.tr

A double sequence $x=(x_{kl})$ of real or complex numbers is said to *converge to the limit a in Pringsheim's sense* (shortly, *p-convergent to a*) if

$$\forall \varepsilon>0\ \exists N\in\mathbb{N}:\ k,l>N\ \Rightarrow |\,x_{kl}-a\,|<\varepsilon.$$

If, in addition, $\sup_{k,l}|x_{kl}|<\infty$ or the limits $\lim_k x_{kl}$ $(l\in\mathbb{N})$ and $\lim_l x_{kl}$ $(k\in\mathbb{N})$ exist, then x is said to be *boundedly convergent to a in Pringsheim's sense* (shortly, *bp-convergent to a*) and *regularly convergent to a* (shortly, *r-convergent to a*), respectively.

Let Ω denote the linear space of all double sequences. Linear subspaces of Ω are called *double sequence spaces.*

For any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathscr{C}_\nu$ and the limit of a ν-convergent double sequence x by ν-$\lim_{m,n}x_{mn}$. The sum of a double series $\sum_{k,l}x_{kl}$ is defined by ν-$\sum_{k,l}x_{kl}:=\nu$-$\lim_{m,n}\sum_{k=1}^{m}\sum_{l=1}^{n}x_{kl}$ whenever $\left(\sum_{k=1}^{m}\sum_{l=1}^{n}x_{kl}\right)_{m,n}\in\mathscr{C}_\nu$. We also use the notations

$$\begin{aligned}
\mathscr{M}_u &:= \Big\{x\in\Omega|\ \sup_{k,l}|x_{kl}|<\infty\Big\},\\
\mathscr{M}_p &:= \Big\{x\in\Omega|\ \exists B\in\mathbb{N}:\ \sup_{m,n>B}|x_{kl}|<\infty\Big\},\\
\mathscr{C}_{\nu 0} &:= \Big\{x\in\Omega|\ \nu\text{-}\lim_{m,n}x_{mn}=0\Big\},\\
\mathscr{C}\mathscr{S}_\nu &:= \Big\{x\in\Omega|\ \Big(\sum_{k,l=1}^{m,n}x_{kl}\Big)_{m,n}\in\mathscr{C}_\nu\Big\}.
\end{aligned}$$

Given a double sequence E, we define its α and $\beta(\nu)$-dual by

$$E^{\alpha}:=\left\{u\in\Omega:\sum_{k,l}|u_{kl}x_{kl}|<\infty,\ \forall x\in E\right\},$$

$$E^{\beta(\nu)}:=\left\{u\in\Omega:\nu\text{-}\sum_{k,l}u_{kl}x_{kl}\text{ exists for each }x\in E\right\},$$

where $\nu\in\{p,bp,r\}$. For other notions and notations in the area of double sequences, we refer the reader to [13].

Let $p=(p_{kl})$ be a bounded double sequence of strictly positive numbers. Then we set

$$\begin{aligned}
\mathscr{M}_u(p) &:= \Big\{x\in\Omega|\ (|x_{kl}|^{p_{kl}})\in\mathscr{M}_u\Big\},\\
\mathscr{M}_p(p) &:= \Big\{x\in\Omega|\ (|x_{kl}|^{p_{kl}})\in\mathscr{M}_p\Big\},\\
\mathscr{C}_{\nu 0}(p) &:= \Big\{x\in\Omega|\ (|x_{kl}|^{p_{kl}})\in\mathscr{C}_{\nu 0}\Big\}\quad(\nu\in\{p,bp\}),\\
\mathscr{C}_{r0}(p) &:= \Big\{x\in\mathscr{C}_{p0}(p)|\ \forall l\in\mathbb{N}:\ (x_{kl})_k\in c((p_{kl})_k)\text{ and}\\
&\qquad \forall k\in\mathbb{N}:\ (x_{kl})_l\in c((p_{kl})_l)\Big\},
\end{aligned}$$

$$\mathscr{C}_\nu(p) := \Big\{x\in\Omega|\ \exists a\in K:\ (x_{kl}-a)\in\mathscr{C}_{\nu 0}(p)\Big\}\quad(\nu\in\{p,bp,r\}).$$

Using the idea of echelon and co-echelon spaces (cf. [8]), we can represent

$$\mathscr{C}_{p0}(p) = \bigcap_N\Big\{x\in\Omega:(|x_{kl}|N^{1/p_{kl}})\in\mathscr{C}_{p0}\Big\}, \tag{1}$$

$$\mathscr{M}_u(p) = \bigcup_N \{x \in \Omega : (|x_{kl}|N^{-1/p_{kl}}) \in \mathscr{M}_u\Big\}. \tag{2}$$

Indeed, if $x \in \mathscr{C}_{p0}(p)$, then, for every $N \in \mathbb{N}$, we have $|x_{kl}|^{p_{kl}}N \to 0$; hence, $|x_{kl}|N^{1/p_{kl}} \to 0$ as $k,l \to \infty$ since (p_{kl}) is bounded. Conversely, if $\lim_{k,l} |x_{kl}|N^{1/p_{kl}} = 0$ for every $N \in \mathbb{N}$, then, for $N \in \mathbb{N}$, we have $|x_{kl}|^{p_{kl}} \leq \frac{1}{N}$ for large k,l. Thus, $x \in \mathscr{C}_{p0}(p)$. For the second equality, if $x \in \mathscr{M}_u(p)$, then there is some $N \in \mathbb{N}$ with $|x_{kl}|^{p_{kl}} \leq N$ $(k,l \in \mathbb{N})$; hence, $|x_{kl}|N^{-1/p_{kl}} \leq 1$ for $k,l \in \mathbb{N}$. Conversely, if $|x_{kl}|N^{-1/p_{kl}} \leq M$ for some $M > 0$, then, for all k,l, we have $|x_{kl}|^{p_{kl}} \leq M^{p_{kl}}N$, which is bounded since (p_{kl}) is bounded. Hence, $x \in \mathscr{M}_u(p)$.

The space $\mathscr{M}_u(p)$ is solid; hence, $\mathscr{M}_u(p)^\alpha = \mathscr{M}_u(p)^{\beta(\nu)}$ $(\nu \in \{p,bp,r\})$ [1]. The space $\mathscr{M}_u(p)$ is isomorphic to $\ell_\infty(q_i)$, where $(q_i) = T(p)$ (where T is an isomorphism between the spaces Ω and ω, defined in [12]). So, by Theorem 2 in [9],

$$\mathscr{M}_u(p)^{\beta(\nu)} = M^2_\infty(p) := \bigcap_{N\in\mathbb{N}} \left\{ (y_{kl}) : \sum_{k,l} |y_{kl}|N^{1/p_{kl}} < \infty \right\}.$$

Since the space $\mathscr{C}_{bp0}(p)$ is solid, then $\mathscr{C}_{bp0}(p)^\alpha = \mathscr{C}_{bp0}(p)^{\beta(\nu)}$ $(\nu \in \{p,bp,r\})$. Since, for every $l \in \mathbb{N}$ and $x = xe^l \in \mathscr{M}_u(p)$, we have $x \in \mathscr{C}_{bp0}(p)$, so, for every $y \in \mathscr{C}_{bp0}(p)^\alpha$, we have

$$(y_{kl})_k \in (\ell_\infty((p_{kl})_k))^\alpha = \bigcap_{N\in\mathbb{N}} \left\{ (z_k) : \sum_k |z_k|N^{1/p_{kl}} < \infty \right\}. \tag{3}$$

Analogously, for every $k \in \mathbb{N}$ and $y \in \mathscr{C}_{bp0}(p)^\alpha$, we have

$$(y_{kl})_l \in (\ell_\infty((p_{kl})_l))^\alpha = \bigcap_{N\in\mathbb{N}} \left\{ (z_l) : \sum_l |z_l|N^{1/p_{kl}} < \infty \right\}. \tag{4}$$

If T is the isomorphism between the spaces Ω and ω, and $\Phi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is the corresponding map of indexes, then $T^{-1}(c_0(\Phi(p))) \subset \mathscr{C}_{bp0}(p)$, hence

$$(T^{-1}(c_0(\Phi(p))))^\alpha = M^2_0(p) \supset \mathscr{C}_{bp0}(p)^\alpha,$$

where

$$M^2_0(p) := \bigcup_{N\in\mathbb{N}} \Big\{ (y_{kl}) : \sum_{k,l} |y_{kl}|N^{-1/p_{kl}} < \infty \Big\}.$$

We will verify that

$$\begin{aligned} \mathscr{C}_{bp0}(p)^\alpha &= M^{bp}_0(p) \\ &:= M^2_0(p) \bigcap \Big\{ (y_{kl}) : \forall l \in \mathbb{N} : \sum_k |y_{kl}|N^{1/p_{kl}} < \infty, \\ &\qquad \forall k \in \mathbb{N} : \sum_l |y_{kl}|N^{1/p_{kl}} < \infty \Big\}. \end{aligned} \tag{5}$$

Suppose, on the contrary, that $y \in M^{bp}_0(p)$, but $y \notin \mathscr{C}_{bp0}(p)^\alpha$. Then, for some $x \in \mathscr{C}_{bp0}(p)$, we have

$$\sum_{k,l} |x_{kl}y_{kl}| = \infty. \tag{6}$$

By (3) and (4), we have

$$\forall l \in \mathbb{N} : \sum_k |x_{kl}y_{kl}| < \infty, \quad \forall k \in \mathbb{N} : \sum_l |x_{kl}y_{kl}| < \infty.$$

So, by (6), we can find an index sequence (m_i) such that

$$\sum_{k,l=m_i+1}^{m_{i+1}} |x_{kl}y_{kl}| > i \quad (i \in \mathbb{N}). \tag{7}$$

Let us define $\tilde{x}_{kl} := x_{kl}$ for $(k,l) \in [m_i+1, m_{i+1}]^2$ $(i \in \mathbb{N})$ and $\tilde{x}_{kl} := 0$ otherwise. Then $\tilde{x} \in T^{-1}(c_0(\Phi(p)))$ while

$$\sum_{k,l} |\tilde{x}_{kl}y_{kl}| = \sum_i \sum_{k,l=m_i+1}^{m_{i+1}} |x_{kl}y_{kl}| = \infty.$$

Hence, $y \notin (T^{-1}(c_0(\Phi(p))))^\alpha = M_0^2(p)$. Thus, the contradiction follows, and therefore (5) holds.

Let $A = (a_{mnkl})$ be any 4-dimensional scalar matrix and ν be some convergence notion of double sequences. We define

$$\Omega_A^{(\nu)} := \left\{ x \in \Omega \mid \forall m,n \in \mathbb{N} \ : \ [Ax]_{mn} := \nu\text{-}\sum_{k,l} a_{mnkl}x_{kl} \text{ exists} \right\}.$$

The map

$$A : \Omega_A^{(\nu)} \to \Omega, \ \ x \mapsto Ax := ([Ax]_{mn})_{m,n}$$

is called a matrix map of type ν. We use the notation $A \in (X,Y)_\nu$ iff A is a matrix map of type ν, and $Ax \in Y$ whenever $x \in X$. If $X^{\beta(p)} = X^{\beta(bp)} = X^{\beta(r)}$ (for example if X is solid), we use the notation $A \in (X,Y)$. In addition, if X is a sequence space, Y is a double sequence space, and $B = (b_{mnk})$ is a 3-dimensional matrix, we use the notation $B \in (X,Y)$ iff $Bx := (\sum_k b_{mnk}x_k)_{m,n}$ exists, and $Bx \in Y$ whenever $x \in X$. If X,Y are sequence spaces, and $B = (b_{nk})$ is a 2-dimensional matrix, we use the notation $B \in (X,Y)$ iff $Bx := (\sum_k b_{nk}x_k)_n$ exists, and $Bx \in Y$ whenever $x = (x_k) \in X$.

In [7], the authors characterized $(\mathscr{M}_u(p), \mathscr{C}_{p0}(q)), (\mathscr{M}_u(p), \mathscr{C}_{bp0}(q)), \left(\mathscr{C}_{bp0}(p), \mathscr{C}_{p0}(q)\right)_\nu, (\mathscr{C}_{bp0}(p)),$ $(\mathscr{C}_{bp0}(q))_\nu$, $(\mathscr{C}_{bp}(p)), (\mathscr{C}_{p0}(q))_\nu$. We have discovered that many of the results in [7] are not correct. In this paper, we give corresponding counterexamples and prove the correct results. Moreover, we give the conditions for the classes of matrices (E,F), where $E \in \{\mathscr{M}_u(p), \mathscr{C}_{bp0}(p), \mathscr{C}_{bp}(p)\}$ and $F \in \left\{\mathscr{M}_u(q), \mathscr{C}_{p0}(q), \mathscr{C}_p(q), \mathscr{C}_{bp0}(q), \ \mathscr{C}_{bp}(q), \mathscr{C}_{r0}(q), \mathscr{C}_r(q)\right\}$.

2. COUNTEREXAMPLES

In this section, we give counterexamples to some results in [7].

In the following example, we verify that the conditions for $A \in (\mathscr{M}_u(p), \mathscr{C}_{bp0}(q))$ and $A \in (\mathscr{M}_u(p), \mathscr{C}_{p0}(q))$ of [7] (Theorem 2.5 and remark after Theorem 2.5 in [7]) are not correct.

Example 2.1. Let $p_{kl} = 1/(k+l)$, $q_{kl} = 1$ $(k,l \in \mathbb{N})$. We consider the matrix $A = (a_{mnkl})$ with $a_{mnkl} = 4^{-(k+l)}(mn)^{-1}$ $(m,n,k,l \in \mathbb{N})$.

Let $B = 2$. Then,

$$\sum_{k,l} |a_{mnkl}| B^{1/p_{kl}} = \frac{1}{mn} \sum_{k,l} 4^{-(k+l)} 2^{k+l} = \frac{1}{mn} \sum_{k,l} 2^{-(k+l)} < \infty \quad (m,n \in \mathbb{N})$$

and

$$\lim_{m,n} \left(\sum_{k,l} |a_{mnkl}| B^{1/p_{kl}} \right)^{q_{mn}} = \lim_{m,n} \frac{1}{mn} \sum_{k,l} 2^{-(k+l)} = 0.$$

So, the conditions (i), (ii) in Theorem 2.5 of [7] hold. In addition,

$$\sup_{m,n}\sum_{k,l}|a_{mnkl}|B^{1/p_{kl}} = \sup_{m,n}\frac{1}{mn}\sum_{k,l}2^{-(k+l)} = 1 < \infty.$$

So, the condition in the remark after Theorem 2.5 of [7] is also satisfied.

Now, for $(x_{kl}) = (8^{k+l})$, we have $|x_{kl}|^{p_{kl}} = 8$ $(k,l \in \mathbb{N})$, so, $x \in \mathscr{M}_u(p)$. On the other hand,

$$|[Ax]_{m,n}|^{q_{mn}} = \frac{1}{mn}\sum_{k,l}4^{-(k+l)}8^{k+l} = \infty \quad (m,n \in \mathbb{N}),$$

so, $Ax \notin \mathscr{C}_{p0}(q) \supset \mathscr{C}_{bp0}(q)$. Hence, $A \notin (\mathscr{M}_u(p), \mathscr{C}_{p0}(q)) \supset (\mathscr{M}_u(p), \mathscr{C}_{bp0}(q))$.

In the following example, we verify that the conditions for $A \in (\mathscr{C}_{bp}(p), \mathscr{C}_{bp})$ of [7] (Theorem 2.1 in [7]) are not correct.

Example 2.2. Let $p_{mn} = 1$ $(m,n \in \mathbb{N})$. Then, $\mathscr{C}_{bp}(p) = \mathscr{C}_{bp}$. We consider the matrix $A = (a_{mnkl})$ with $a_{mn11} = 1$ and $a_{mnkl} = 0$ for $(k,l) \neq (1,1)$ $(m,n,k,l \in \mathbb{N})$. Then,

$$\lim_{m,n}\sum_{l=1}^{\infty}a_{mn1l}B^{\frac{1}{p_{kl}}} = \lim_{m,n}\sum_{k=1}^{\infty}a_{mnk1}B^{\frac{1}{p_{kl}}} = B$$

for any $B > 1$ and

$$\lim_{m,n}a_{mn11} = 1.$$

So, the conditions (i), (iv), (v) in Theorem 2.1 of [7] are not fulfilled while $A \in (\mathscr{C}_{bp}(p), \mathscr{C}_{bp})$.

In the following example, we verify that the conditions for $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_{p0}(q))$ and $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_{bp0}(q))$ of [7] (Theorem 2.3 and Corollary 2.1 in [7]) are not correct.

Example 2.3. Let $p_{mn} = q_{mn} = 1$ $(m,n \in \mathbb{N})$. Then, $\mathscr{C}_{bp0}(p) = \mathscr{C}_{bp0}(q) = \mathscr{C}_{bp0}$ and $\mathscr{C}_{p0}(q) = \mathscr{C}_{p0}$. We consider the matrix $A = (a_{mnkl})$ with $a_{mnmn} = 1$ and $a_{mnkl} = 0$ for $(k,l) \neq (m,n)$ $(m,n,k,l \in \mathbb{N})$. Then,

$$\lim_{m,n}\left(\sum_{k,l=1}^{\infty}|a_{mnkl}|B^{-\frac{1}{p_{kl}}}\right)^{q_{mn}} = B^{-1} \neq 0$$

for any $B > 1$. So, the condition (v) in Theorem 2.3 and Corollary 2.1 of [7] are not fulfilled while $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_{bp0}(q)) \subset (\mathscr{C}_{bp0}(p), \mathscr{C}_{p0}(q))$.

In the following example, we verify that Theorem 3.1 of [7] is not correct.

Example 2.4. Let $p_{kl} = 3$, and consider the matrix $A = (a_{mnkl})$ with $a_{mn11} = 2\cdot(-1)^{m+n}$ and $a_{mnkl} = 0$ for $(k,l) \neq (1,1)$ $(m,n,k,l \in \mathbb{N})$. Now, we consider $(x_{kl}) \in \mathscr{M}_u(p)$ with $x_{kl} = 4$ $(k,l \in \mathbb{N})$. Then,

$$\limsup_{m,n}A_{mn}(x) = \limsup_{m,n}2\cdot(-1)^{m+n}\cdot 4 = 8$$

and

$$\limsup_{k,l}|x_{kl}|^{p_{kl}} = \limsup_{k,l}4^3 = 64.$$

So, all the assumptions of Theorem 3.1 of [7] are satisfied. But $A \notin (\mathscr{C}_{bp}(p), \mathscr{C}_{bp})$ and

$$\limsup_{m,n}\sum_{k,l}|a_{mnkl}| = 2 \not\leq 1.$$

3. CHARACTERIZATION OF MATRIX TRANSFORMATIONS

In what follows, we assume that $p = (p_{kl})$ and $q = (q_{kl})$ are bounded double sequences of strictly positive numbers. To characterise the matrix transformations of double convergent sequences with powers, we need the following results.

Lemma 3.1. *Let X be a sequence space. Then,*
(a) $A \in (X, \mathscr{C}_{\nu 0}(q))$ *iff* $\forall L \in \mathbb{N}: \ (a_{mnkl}L^{1/q_{mn}}) \in (X, \mathscr{C}_{\nu 0})$.
(b) $A \in (X, \mathscr{M}_u(q))$ *iff* $\exists L \in \mathbb{N}: \ (a_{mnkl}L^{-1/q_{mn}}) \in (X, \mathscr{M}_u)$.

Proposition 3.2. (cf. [3, Proposition 3]). *Let* $x = (x_{kl}) \in \Omega$. *Then,*
(a) $x \in \mathscr{C}_p(p)$ *iff* $\exists p_x \in \mathbb{K}\ \forall$ *index sequences* $(k_i), (l_i)$*:*

$$\lim_i |x_{k_i l_i} - p_x|^{p_{k_i l_i}} = 0.$$

(b) $x \in \mathscr{C}_{bp}(p)$ *iff* $x \in \mathscr{C}_p(p)$ *and* $\forall (k_i), (l_i)$ *in* $\mathbb{N}$*:* $(x_{k_i l_i}) \in \ell_\infty(p_{k_i l_i})$.
(c) $x \in \mathscr{M}_u(p)$ *iff* $\forall (k_i), (l_i)$ *in* $\mathbb{N}$*:* $(x_{k_i l_i}) \in \ell_\infty(p_{k_i l_i})$.
(d) $x \in \mathscr{M}_p(p)$ *iff* $\forall$ *index sequences* $(k_i), (l_i)$*:* $(x_{k_i l_i}) \in \ell_\infty(p_{k_i l_i})$.

Corollary 3.3. (cf. [3, Corollary 4]). *Let* $B = (b_{mnk})$ *be a 3-dimensional matrix and let X be a sequence space with* $X \subset \Omega_B^{(\nu)}$. *Then, the following statements hold:*
(a) $B \in (X, \mathscr{C}_p(p))$ *iff* $\forall$ *index sequences* $(m_i), (n_i)$:
$(b_{m_i n_i k})_{i,k} \in (X, c((p_{m_i n_i})_i))$*, and all these matrices are pairwise consistent on X.*
(b) $B \in (X, \mathscr{C}_{bp}(p))$ *iff* $B \in (X, \mathscr{C}_p(p))$ *and* $\forall (m_i), (n_i)$ *in* $\mathbb{N}$: $(b_{m_i n_i k})_{i,k} \in (X, \ell_\infty((p_{m_i n_i})_i))$.
(c) $B \in (X, \mathscr{C}_{p0}(p))$ *iff* $\forall$ *index sequences* $(m_i), (n_i)$: $(b_{m_i n_i k})_{i,k} \in (X, c_0((p_{m_i n_i})_i))$.
(d) $B \in (X, \mathscr{C}_{bp0}(p))$ *iff* $B \in (X, \mathscr{C}_{p0}(p))$ *and* $\forall (m_i), (n_i)$ *in* $\mathbb{N}$: $(b_{m_i n_i k})_{i,k} \in (X, \ell_\infty((p_{m_i n_i})_i))$.
(e) $B \in (X, \mathscr{M}_u(p))$ *iff* $\forall (m_i), (n_i)$ *in* $\mathbb{N}$: $(b_{m_i n_i k})_{i,k} \in (X, \ell_\infty((p_{m_i n_i})_i))$.

Theorem 3.4. $A \in (\mathscr{M}_u(p), \mathscr{M}_u(q))$ *iff the following condition holds:*

$$\forall M \in \mathbb{N}: \ \sup_{m,n} \left(\sum_{k,l} |a_{mnkl}| M^{1/p_{kl}} \right)^{q_{mn}} < \infty.$$

Proof. We can identify the double sequence space $\mathscr{M}_u(p)$ with $\ell_\infty(r_i)$, where $(r_i) = T(p)$, $\mathscr{M}_u(q)$ with $\ell_\infty(s_i)$, where $(s_i) = T(q)$, and the 4-dimensional matrix A with the 2-dimensional matrix $B = (b_{ik})$. Then, $A \in (\mathscr{M}_u(p), \mathscr{M}_u(q))$ iff $B \in (\ell_\infty(r_i), \ell_\infty(s_i))$. By Theorem 5.1, 15 in [8], this is equivalent to
$\forall M \in \mathbb{N}$:

$$\sup_i \left(\sum_k |b_{ik}| M^{1/r_k} \right)^{s_i} = \sup_{m,n} \left(\sum_{k,l} |a_{mnkl}| M^{1/p_{kl}} \right)^{q_{mn}} < \infty.$$

□

Theorem 3.5. *Let* $A = (a_{mnkl})$ *be a 4-dimensional matrix. Then,*
(a) $A \in (\mathscr{M}_u(p), \mathscr{C}_p(q))$ *iff the following conditions hold:*
(i) $\forall M, m, n \in \mathbb{N}: \ \sum_{k,l} |a_{mnkl}| M^{1/p_{kl}} < \infty$;
(ii) $\forall M \in \mathbb{N}\ \exists D \in \mathbb{N}: \ \sup_{m,n>D} \sum_{k,l} |a_{mnkl}| M^{1/p_{kl}} < \infty$;
(iii) $\exists (a_{kl})\ \forall M \in \mathbb{N}: \ \lim_{m,n} (\sum_{k,l} |a_{mnkl} - a_{kl}| M^{1/p_{kl}})^{q_{mn}} = 0$.
(b) $A \in (\mathscr{M}_u(p), \mathscr{C}_{bp}(q))$ iff (i)–(iii) *and the following condition hold:*
(iv) $\forall M \in \mathbb{N}: \ \sup_{m,n} (\sum_{k,l} |a_{mnkl}| M^{1/p_{kl}})^{q_{mn}} < \infty$.
(c) $A \in (\mathscr{M}_u(p), \mathscr{C}_r(q))$ iff (i)–(iii) *and the following conditions hold:*

(v) $\forall M,n\in\mathbb{N}:\ \sup_m\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vi) $\forall M,m\in\mathbb{N}:\ \sup_n\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vii) $\exists(\beta_{mkl})\ \forall M,m\in\mathbb{N}:\lim_n(\sum_{k,l}|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{1/q_{mn}}=0$;
(viii) $\exists(\gamma_{nkl})\ \forall M,n\in\mathbb{N}:\lim_m(\sum_{k,l}|a_{mnkl}-\gamma_{nkl}|M^{1/p_{kl}})^{1/q_{mn}}=0$.

Proof. **(a)** *Necessity.* (i) follows since $\mathscr{M}_u(p)^{\beta(v)}=M^2_\infty(p)$.

To prove (ii) and (iii), we identify the double sequence space $\mathscr{M}_u(p)$ with $\ell_\infty(r)$, where $(r_i)=T(p)$ [12], and the 4-dimensional matrix A with the 3-dimensional matrix $B=(b_{mnk})$. By Corollary 3.3(a), $B\in(\ell_\infty(r),\mathscr{C}_p(p))$ iff for every index sequences $(m_i),(n_i)$: $(b_{m_in_ik})_{i,k}\in(\ell_\infty(r),c((q_{m_in_i})_i))$, and all these matrices are pairwise consistent on $\ell_\infty(r)$. By Theorem 5.1, 11 in [8], this is equivalent to the following two conditions holding:

(a) $\forall$ index sequences $(m_i),(n_i)$, $\forall M\in\mathbb{N}:\ \sup_i\sum_k|b_{m_in_ik}|M^{1/r_k}<\infty$;
(b) $\exists(b_k)$ $\forall$ index sequences $(m_i),(n_i)$ $\forall M\in\mathbb{N}$:

$$\lim_i\left(\sum_k|b_{m_in_ik}-b_k|M^{1/r_k}\right)^{q_{m_in_i}}=0.$$

By Proposition 3a in [3], the statement *(a)* is equivalent to $\forall M\in\mathbb{N}$: $\left(\sum_k|b_{mnk}|M^{1/r_k}\right)_{m,n}\in\mathscr{M}_p$, that is,

$$\forall M\in\mathbb{N}\ \exists D\in\mathbb{N}:\ \sup_{m,n>D}\sum_k|b_{mnk}|M^{1/r_k}=\sup_{m,n>D}\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}<\infty,$$

so, (ii) follows.

By Proposition 3e in [3], the statement *(b)* is equivalent to

$$\exists(b_k)\ \forall M\in\mathbb{N}:\ \left(\sum_k|b_{mnk}-b_k|M^{1/r_k}\right)^{q_{mn}}\in\mathscr{C}_{p0}.$$

Let $(a_{kl})=T^{-1}(b)$. Then, this is equivalent to (iii).

Sufficiency. By (i), it follows that Ax exists for every $x\in\mathscr{M}_u(p)$. From (ii) and (iii), it follows that the conditions (a) and (b) in the necessity proof hold. Again, as in the necessity proof by Theorem 5.1, 11 in [8] and Corollary 3.3, we get $A\in(\mathscr{M}_u(p),\mathscr{C}_p(q))$.

(b) The proof follows from (a) and Theorem 3.4.

(c) Note that $A\in(\mathscr{M}_u(p),\mathscr{C}_r(q))$ iff $A\in(\mathscr{M}_u(p),\mathscr{C}_p(q))$; for all $n\in\mathbb{N}$, the 3-dimensional matrix $(a_{mnkl})_{m,k,l}$ maps $\mathscr{M}_u(p)$ to $c((q_{mn})_m)$, and for all $m\in\mathbb{N}$, the 3-dimensional matrix $(a_{mnkl})_{n,k,l}$ maps $\mathscr{M}_u(p)$ to $c((q_{mn})_n)$. By (a), the first statement is equivalent to (i)–(iii). Since we can identify $\mathscr{M}_u(p)$ with $\ell_\infty(r)$, where $(r_i)=T(p)$, and a 3-dimensional matrix with a 2-dimensional matrix by Theorem 5.1, 11 in [8], the second and the third statements are equivalent to (v)–(viii). □

Theorem 3.6. *Let $A=(a_{mnkl})$ be a 4-dimensional matrix. Then,*
(a) *$A\in(\mathscr{M}_u(p),\mathscr{C}_{p0}(q))$ iff the following conditions hold:*
(i) $\forall M,m,n\in\mathbb{N}:\ \sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(ii) $\forall M\in\mathbb{N}:\ \lim_{m,n}\left(\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}\right)^{q_{mn}}=0$.
(b) *$A\in(\mathscr{M}_u(p),\mathscr{C}_{bp0}(q))$ iff* (i), (ii) *and the following condition hold:*
(iii) $\forall M\in\mathbb{N}:\ \sup_{m,n}\left(\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}\right)^{q_{mn}}<\infty$.
(c) *$A\in(\mathscr{M}_u(p),\mathscr{C}_{r0}(q))$ iff* (i), (ii) *and the following conditions hold:*
(iv) $\forall M,n\in\mathbb{N}:\ \sup_m\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(v) $\forall M,m\in\mathbb{N}:\ \sup_n\sum_{k,l}|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vi) $\exists(\beta_{mkl})\ \forall M,m\in\mathbb{N}:\lim_n(\sum_{k,l}|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{1/q_{mn}}=0$;
(vii) $\exists(\gamma_{nkl})\ \forall M,n\in\mathbb{N}:\lim_m(\sum_{k,l}|a_{mnkl}-\gamma_{nkl}|M^{1/p_{kl}})^{1/q_{mn}}=0$.

Proof. **(a)** *Necessity*. (i) follows since $\mathscr{M}_u(p)^{\beta(\nu)} = M^2_\infty(p)$. To prove (ii), we identify the sequence space $\ell_\infty(r)$, where $(r_i) = T(p)$, and use the 3-dimensional matrix $B = (b_{mnk})$ as in the proof of Theorem 3.5. By Corollary 3.3(c), $B \in (\ell_\infty(r), \mathscr{C}_{p0}(p))$ iff for every index sequences $(m_i),(n_i)$: $(b_{m_in_ik})_{i,k} \in (\ell_\infty(r), c_0((q_{m_in_i})_i))$. By Theorem 5.1, 7 in [8], this is equivalent to the following condition holding:

$$\forall M \; \forall \text{ index sequences } (m_i),(n_i): \; \lim_i \left(\sum_k |b_{m_in_ik}| M^{1/r_k} \right)^{q_{m_in_i}} = 0. \tag{8}$$

By Proposition 3e in [3], the statement is equivalent to

$$\forall M \in \mathbb{N}: \; \lim_{m,n} \left(\sum_k |b_{mnk}| M^{1/r_k} \right)^{q_{mn}} = \lim_{m,n} \left(\sum_{k,l} |a_{mnkl}| M^{1/p_{kl}} \right)^{q_{mn}} = 0.$$

So, (ii) follows.

Sufficiency. By (i), it follows that Ax exists for every $x \in \mathscr{M}_u(p)$. From (ii), it follows that the condition (8) in the necessity proof holds. Again, as in the necessity proof by Theorem 5.1, 7 in [8] and Corollary 3.3(c), we get $A \in (\mathscr{M}_u(p), \mathscr{C}_{p0}(q))$.

(b) The proof follows from (a) and Theorem 3.5(b).

(c) The proof follows from (a) and Theorem 3.5(c). □

Theorem 3.7. *Let $A = (a_{mnkl})$ be a 4-dimensional matrix, then $A \in (\mathscr{C}_{bp0}(p), \mathscr{M}_u(q))$ iff the following conditions hold:*

(i) $\forall M,k \in \mathbb{N} \; : \; \sup_{m,n} \left(\sum_l |a_{mnkl}| M^{1/p_{kl}} \right)^{q_{mn}} < \infty$;

(ii) $\forall M,l \in \mathbb{N} \; : \; \sup_{m,n} \left(\sum_k |a_{mnkl}| M^{1/p_{kl}} \right)^{q_{mn}} < \infty$;

(iii) $\exists M \in \mathbb{N} \; : \; \sup_{m,n} \left(\sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}} \right)^{q_{mn}} < \infty$.

Proof. Necessity. (i) Given $k \in \mathbb{N}$, the map $(a_{mnkl})_{m,n,l} : \ell_\infty((p_{kl})_l) \to \mathscr{M}_u(q)$ is defined. We can identify the double sequence space $\mathscr{M}_u(q)$ with $\ell_\infty(s)$, where $(s_i) = T(q)$, and the 3-dimensional matrix $(a_{mnkl})_{m,n,l}$ with the 2-dimensional matrix $B = (b_{il})$. So, $B: \; \ell_\infty((p_{kl})_l) \to \ell_\infty(s)$. By Theorem 5.1, 15 in [8], this is equivalent to

$$\forall M \in \mathbb{N}: \sup_i \left(\sum_l |b_{il}| M^{1/p_{kl}} \right)^{s_i} = \sup_{m,n} \left(\sum_l |a_{mnkl}| M^{1/p_{kl}} \right)^{q_{mn}} < \infty.$$

So, (i) follows. Analogously, we get (ii).

(iii) Since $T^{-1}(c_0(r)) \subset \mathscr{C}_{bp0}(p)$ $(r = T(p))$, and we can identify the double sequence space $\mathscr{M}_u(q)$ with $\ell_\infty(s)$, where $(s_i) = T(q)$, and the 4-dimensional matrix A with the 2-dimensional matrix $C = (c_{ij})$, then $C: \; c_0(r) \to \ell_\infty(s)$. So, by Theorem 5.1, 13 in [8], we get

$$\exists M \in \mathbb{N}: \; \sup_i \left(\sum_j |c_{ij}| M^{-1/r_j} \right)^{s_i} = \sup_{m,n} \left(\sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}} \right)^{q_{mn}} < \infty.$$

So, (iii) follows.

Sufficiency. First, note that from (i)–(iii), it follows that $(a_{mnkl})_{k,l} \in \mathscr{C}_{bp0}(p)^{\beta(\nu)}$ for all $m,n \in \mathbb{N}$. Hence, Ax exists for all $x \in \mathscr{C}_{bp0}(p)$.

Suppose, on the contrary, that $Ax \notin \mathscr{M}_u(q)$ for some $x \in \mathscr{C}_{bp0}(p)$. So, by (2) for all N, we have $([Ax]_{m,n} N^{-1/q_{mn}}) \notin \mathscr{M}_u$. Hence,

$$\forall N: \; \sup_{m,n} \left| \sum_{k,l} a_{mnkl} x_{kl} N^{-1/q_{mn}} \right| = \infty. \tag{9}$$

Let M be such that $|x_{kl}| \leq M^{1/p_{kl}}$ $(k,l \in \mathbb{N})$.

From (i), (ii) and (2), it follows that

$$\forall N,k:\ \alpha_{Nk} := \sup_{m,n} \sum_{l} |a_{mnkl} x_{kl} N^{-1/q_{mn}}| < \infty \tag{10}$$

and

$$\forall N,l:\ \beta_{Nl} := \sup_{m,n} \sum_{k} |a_{mnkl} x_{kl} N^{-1/q_{mn}}| < \infty. \tag{11}$$

Note that $\alpha_{Nk} \leq \alpha_{1k}$ and $\beta_{Nl} \leq \beta_{1l}$ $(k,l,N \in \mathbb{N})$.

From (iii), it follows that $A \in (T^{-1}(c_0(\Phi(p))), \mathscr{M}_u(q))$.

We will use the gliding hump method to get a contradiction. We set $k_1 := 0$. Let m_1, n_1 be such that

$$\left| \sum_{k,l} a_{m_1 n_1 kl} x_{kl} 1^{-1/q_{m_1 n_1}} \right| > 2.$$

Since the double series $\sum_{k,l} a_{m_1 n_1 kl} x_{kl}$ converges regularly (cf. (10) and (11)), we can find $k_2 > k_1$ such that

$$\left| \sum_{k,l} a_{m_1 n_1 kl} x_{kl} 1^{-1/q_{m_1 n_1}} - \sum_{k,l=1}^{k_2} a_{m_1 n_1 kl} x_{kl} 1^{-1/q_{m_1 n_1}} \right| < \frac{1}{2}$$

and

$$\left| \sum_{k,l=k_2+1}^{k_2+p} a_{m_1 n_1 kl} x_{kl} \right| < \frac{1}{2} \quad (p \in \mathbb{N}).$$

Then,

$$\left| \sum_{k,l=k_1+1}^{k_2} a_{m_1 n_1 kl} x_{kl} 1^{-1/q_{m_1 n_1}} \right| > 1.$$

Now, we find $m_2 > m_1$, $n_2 > n_1$ such that

$$\left| \sum_{k,l} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} \right| > 3 + 2\sum_{k=1}^{k_2} \alpha_{1k} + 2\sum_{l=1}^{k_2} \beta_{1l},$$

and we find $k_3 > k_2$ such that

$$\left| \sum_{k,l} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} - \sum_{k,l=1}^{k_3} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} \right| < \frac{1}{4}$$

and

$$\left| \sum_{k,l=k_3+1}^{k_3+p} a_{m_j n_j kl} x_{kl} \right| < \frac{1}{4} \quad (p \in \mathbb{N};\ j \leq 2).$$

Then,

$$\begin{aligned} &\left| \sum_{k,l=k_2+1}^{k_3} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} \right| \\ > \quad &\left| \sum_{k,l} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} \right| - \sum_{k=1}^{k_2} \alpha_{2k} - \sum_{l=1}^{k_2} \beta_{2l} \\ &- \left| \sum_{k,l} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} - \sum_{k,l=1}^{k_3} a_{m_2 n_2 kl} x_{kl} 2^{-1/q_{m_2 n_2}} \right| \\ > \quad & 2 + \sum_{k=1}^{k_2} \alpha_{1k} + \sum_{l=1}^{k_2} \beta_{1l}. \end{aligned}$$

Continuing inductively, we find $m_i > m_{i-1}$, $n_i > n_{i-1}$ such that

$$\left|\sum_{k,l} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}}\right| > i + 1 + 2\sum_{k=1}^{k_i} \alpha_{1k} + 2\sum_{l=1}^{k_i} \beta_{1l},$$

and we find $k_{i+1} > k_i$ such that

$$\left|\sum_{k,l} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}} - \sum_{k,l=1}^{k_{i+1}} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}}\right| < \frac{1}{2^i}$$

and

$$\left|\sum_{k,l=k_{i+1}+1}^{k_{i+1}+p} a_{m_j n_j kl} x_{kl}\right| < \frac{1}{2^i} \quad (p \in \mathbb{N};\ j \le i).$$

Then,

$$\begin{aligned}
&\left|\sum_{k,l=k_i+1}^{k_{i+1}} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}}\right| \\
\ge\ & \left|\sum_{k,l} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}}\right| - \sum_{k=1}^{k_i} \alpha_{ik} - \sum_{l=1}^{k_i} \beta_{il} \\
& - \left|\sum_{k,l} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}} - \sum_{k,l=1}^{k_{i+1}} a_{m_i n_i kl} x_{kl} i^{-1/q_{m_i n_i}}\right| \\
>\ & i + \sum_{k=1}^{k_i} \alpha_{1k} + \sum_{l=1}^{k_i} \beta_{1l}.
\end{aligned}$$

Let us define $\tilde{x}_{kl} := x_{kl}$ for $(k,l) \in [k_i+1, k_{i+1}]^2$ $(i \in \mathbb{N})$ and $\tilde{x}_{kl} := 0$ otherwise. Then, we get $x \in (T^{-1}(c_0(\Phi(p))))$. On the other hand, for $j \le i$, we get

$$\begin{aligned}
&\left|\sum_{k,l} a_{m_i n_i kl} \tilde{x}_{kl} j^{-1/q_{m_i n_i}}\right| \\
\ge\ & \left|\sum_{k,l=k_i+1}^{k_{i+1}} a_{m_i n_i kl} x_{kl} i^{-1/q_{mn}}\right| - \sum_{k=1}^{k_i} \alpha_{jk} - \sum_{l=1}^{k_i} \beta_{jl} - \sum_{s=i+1}^{\infty} \left|\sum_{k,l=k_s+1}^{k_{s+1}} a_{m_i n_i kl} x_{kl}\right| \\
\ge\ & i - \sum_{s=i+1}^{\infty} \frac{1}{2^s} \ge i - 1.
\end{aligned}$$

Hence,

$$\forall N:\ \sup_i \left|\sum_{k,l} a_{m_i n_i kl} \tilde{x}_{kl} N^{-1/q_{m_i n_i}}\right| \ge \sup_i (i-1) = \infty;$$

hence, $A\tilde{x} \notin \mathscr{M}_u(q)$, which contradicts $A \in (T^{-1}(c_0(\Phi(p)), \mathscr{M}_u(q))$. □

Theorem 3.8. *Let $A = (a_{mnkl})$ be a 4-dimensional matrix. Then,*

(a) $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_p(q))$ *iff the following conditions hold:*

(i) $\forall k,l \in \mathbb{N}\ \exists (a_{kl})\ :\ \lim_{m,n} |a_{mnkl} - a_{kl}|^{q_{mn}} = 0$;

(ii) $\forall M,m,n,l \in \mathbb{N}\ :\ \sum_k |a_{mnkl}| M^{1/p_{kl}} < \infty$;

(iii) $\forall M,l \in \mathbb{N}\ \exists D \in \mathbb{N}\ :\ \sup_{m,n>D} \sum_k |a_{mnkl}| M^{1/p_{kl}} < \infty$;

(iv) $\forall M,l\in\mathbb{N}\ :\ \lim_{m,n}(\sum_k|a_{mnkl}-a_{kl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(v) $\forall M,m,n,k\in\mathbb{N}\ :\ \sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vi) $\forall M,k\in\mathbb{N}\ \exists D\in\mathbb{N}\ :\ \sup_{m,n>D}\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vii) $\forall M,k\in\mathbb{N}\ :\ \lim_{m,n}(\sum_l|a_{mnkl}-a_{kl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(viii) $\forall m,n\in\mathbb{N}\ \exists M\in\mathbb{N}\ :\ \sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(ix) $\exists M,D\in\mathbb{N}\ :\ \sup_{m,n>D}\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(x) $\forall L\ \exists M,D\in\mathbb{N}\ :\ \sup_{m,n>D}L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-a_{kl}|M^{-1/p_{kl}}<\infty$.
(b) $A\in(\mathscr{C}_{bp0}(p),\mathscr{C}_{bp}(q))$ iff (i)–(x) *and the following conditions hold:*
(xi) $\forall M,l\in\mathbb{N}\ :\ \sup_{m,n}(\sum_k|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}<\infty$;
(xii) $\forall M,k\in\mathbb{N}\ :\ \sup_{m,n}(\sum_l|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}<\infty$;
(xiii) $\exists M\in\mathbb{N}\ :\ \sup_{m,n}(\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}})^{q_{mn}}<\infty$.
(c) $A\in(\mathscr{C}_{bp0}(p),\mathscr{C}_{r}(q))$ iff (i)–(x) *and the following conditions hold:*
(xiv) $\exists(\beta_{mkl})\ \forall k,l\in\mathbb{N}\ :\ \lim_n|a_{mnkl}-\beta_{mkl}|^{q_{mn}}=0$;
(xv) $\forall M,l,m\in\mathbb{N}\ :\ \sup_n\sum_k|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xvi) $\forall M,l,m\in\mathbb{N}\ :\ \lim_n(\sum_k|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xvii) $\forall M,k,m\in\mathbb{N}\ :\ \sup_n\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xviii) $\forall M,k,m\in\mathbb{N}\ :\ \lim_n(\sum_l|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xix) $\exists(\gamma_{nkl})\ \forall k,l\in\mathbb{N}\ :\ \lim_m|a_{mnkl}-\gamma_{nkl}|^{q_{mn}}=0$;
(xx) $\forall M,l,n\in\mathbb{N}\ :\ \sup_m\sum_k|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xxi) $\forall M,l,n\in\mathbb{N}\ :\ \lim_m(\sum_k|a_{mnkl}-\gamma_{nkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xxii) $\forall M,k,n\in\mathbb{N}\ :\ \sup_m\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xxiii) $\forall M,k,n\in\mathbb{N}\ :\ \lim_m(\sum_l|a_{mnkl}-\gamma_{nkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xxiv) $\forall m\ \exists M\in\mathbb{N}\ :\ \sup_n\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(xxv) $\forall L,m\ \exists M\in\mathbb{N}\ :\ \sup_n L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-\beta_{mkl}|M^{-1/p_{kl}}<\infty$;
(xxvi) $\forall n\ \exists M\in\mathbb{N}\ :\ \sup_m\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(xxvii) $\forall L,n\ \exists M\in\mathbb{N}\ :\ \sup_m L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-\gamma_{nkl}|M^{-1/p_{kl}}<\infty$.

Proof. **(a)** *Necessity.* (i) follows since $e^{kl}\in\mathscr{C}_{bp0}(p)$ $(k,l\in\mathbb{N})$ and (viii) follows since $(a_{mnkl})_{k,l}\in\mathscr{C}_{bp0}(p)^{\beta(\nu)}$ $=M_0^{bp}(p)$ $(m,n\in\mathbb{N})$.

Given $l\in\mathbb{N}$, the map $(a_{mnkl})_{m,n,k}:\ell_\infty((p_{kl})_k)\to\mathscr{C}_p(q)$ is defined. We can identify the sequence space $\ell_\infty((p_{kl})_k)$ with the double sequence space $\mathscr{M}_u(T^{-1}((p_{kl})_k))$ and the 3-dimensional matrix $(a_{mnkl})_{m,n,k}$ with the 4-dimensional matrix $B=(b_{mnij})$. So, $B:\ \mathscr{M}_u(T^{-1}((p_{kl})_k))\to\mathscr{C}_p(q)$. Therefore, by Theorem 3.5, we get (ii)–(iv).

Analogously, since given $k\in\mathbb{N}$, the map $(a_{mnkl})_{m,n,l}:\ell_\infty((p_{kl})_l)\to\mathscr{C}_p(q)$ is defined. So, (v)–(vii) follow.

Since $T^{-1}(c_0(r))\subset\mathscr{C}_{bp0}(p)$ $(r=T(p))$, and we can identify the 4-dimensional matrix A with the 3-dimensional matrix $B=(b_{mni})$, then $B:\ c_0(r)\to\mathscr{C}_p(q)$.

By Corollary 3.3(a), $B\in(c_0(r),\mathscr{C}_p(q))$ iff for every index sequences $(m_i),(n_i)$: $(b_{m_in_ik})_{i,k}\in(c_0(r),c((q_{m_in_i})_i))$, and all these matrices are pairwise consistent on $c_0(r)$. By Theorem 5.1, 9 in [8], it follows that the following three conditions hold:

(a) $\exists(b_k)$ $\forall$ index sequences $(m_i),(n_i)$: $\lim_i|b_{m_in_ik}-b_k|^{q_{m_in_i}}=0$;

(b) $\forall$ index sequences $(m_i),(n_i)$, $\exists M\in\mathbb{N}$:

$$\sup_i\sum_k|b_{m_in_ik}|M^{-1/r_k}<\infty;$$

(c) $\forall$ index sequences $(m_i),(n_i)$ $\forall L\in\mathbb{N}$ $\exists M\in\mathbb{N}$:

$$\sup_i\left(\sum_k L^{1/q_{m_in_i}}|b_{m_in_ik}-b_k|M^{-1/r_k}\right)<\infty.$$

(a) is equivalent to (i).

We will show that *(b)* implies (ix). Suppose, on the contrary, that (ix) does not hold. Then,

$$\forall M, D \in \mathbb{N}: \sup_{m,n>D} \sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}} = \infty.$$

Hence, we can find index sequences $(m_i), (n_i)$ such that

$$\sum_{k,l} |a_{m_i n_i kl}| i^{-1/p_{kl}} > i.$$

Hence,

$$\sum_{k} |b_{m_i n_i k}| j^{-1/r_k} = \sum_{k,l} |a_{m_i n_i kl}| j^{-1/p_{kl}} > i \quad (j \leq i).$$

So,

$$\forall M \in \mathbb{N}: \sup_{i} \sum_{k} |b_{m_i n_i k}| M^{-1/r_k} = \infty$$

that contradicts *(b)*. Hence, (ix) follows.

Analogously, we get (x) from *(c)*.

Sufficiency. First, we verify that $a = (a_{kl}) \in \mathscr{C}_{bp0}(p)^{\beta(\nu)}$.

By (x), we can choose $M_1, D_1 \in \mathbb{N}$ such that

$$\sup_{m,n>D_1} 1^{1/q_{mn}} \sum_{k,l} |a_{mnkl} - a_{kl}| M_1^{-1/p_{kl}} < \infty.$$

By (ix), we can choose $M_2, D_2 \in \mathbb{N}$ such that

$$\sup_{m,n>D_2} \sum_{k,l} |a_{mnkl}| M_2^{-1/p_{kl}} < \infty.$$

If we replace M_1, M_2 with $M := \max\{M_1, M_2\}$ and D_1, D_2 with $D := \max\{D_1, D_2\}$, both inequalities still hold.

Now, let $m, n > D$. Then,

$$\sum_{k,l} |a_{kl}| M^{-1/p_{kl}} \leq \sum_{k,l} |a_{mnkl} - a_{kl}| M^{-1/p_{kl}} + \sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}} < \infty.$$

Hence, $a \in M_0^2(p)$.

Now, let $M, l \in \mathbb{N}$ be fixed. By (1) and (iv),

$$\lim_{m,n} \sum_{k} |a_{mnkl} - a_{kl}| M^{1/p_{kl}} = 0.$$

We choose $m, n \in \mathbb{N}$ such that $\sum_k |a_{mnkl} - a_{kl}| M^{1/p_{kl}} < 1$. Then, by (ii),

$$\sum_{k} |a_{kl}| M^{1/p_{kl}} \leq \sum_{k} |a_{mnkl} - a_{kl}| M^{1/p_{kl}} + \sum_{k} |a_{mnkl}| M^{1/p_{kl}} < \infty.$$

Analogously, we get that $\sum_l |a_{kl}| M^{1/p_{kl}} < \infty$ for all $M, k \in \mathbb{N}$. Altogether, we get $a \in M_0^{bp}(p) = \mathscr{C}_{bp0}(p)^{\beta(\nu)}$.

Now, let $x \in \mathscr{C}_{bp0}(p)$ be fixed. We will prove that $\mathscr{C}_p(q) - \lim Ax = \sum_{k,l} a_{kl} x_{kl}$. For that, we will show that for all $L \in \mathbb{N}$, we get

$$\lim_{m,n}\left(\sum_{k,l}a_{mnkl}x_{kl}-\sum_{k,l}a_{kl}x_{kl}\right)L^{q_{mn}}=0. \tag{12}$$

Let $L\in\mathbb{N}$ and $\varepsilon>0$ be fixed. By (x), we can choose $M,D\in\mathbb{N}$ such that

$$A_L:=\sup_{m,n>D}L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-a_{kl}|M^{-1/p_{kl}}<\infty.$$

Let $k_0\in\mathbb{N}$ be such that $|x_{kl}|M^{1/p_{kl}}\le\varepsilon/(3A_L)$ for $k,l>k_0$. So,

$$\begin{aligned}L^{1/q_{mn}}\sum_{k,l>k_0}|a_{mnkl}-a_{kl}||x_{kl}| &= L^{1/q_{mn}}\sum_{k,l>k_0}|a_{mnkl}-a_{kl}||x_{kl}|M^{1/p_{kl}}M^{-1/p_{kl}}\\ &\le \frac{\varepsilon}{3}\ (m,n>D).\end{aligned}$$

Since $x\in\mathscr{M}_u(p)$, we can find M_0 such that $|x_{kl}|\le M_0^{1/p_{kl}}$ for $k,l\in\mathbb{N}$. By (iv) and (vii), we can find $D_1>D$ such that

$$\begin{aligned}&L^{1/q_{mn}}\sum_{l=1}^{k_0}\sum_{k}|a_{mnkl}-a_{kl}|M_0^{1/p_{kl}}\le\frac{\varepsilon}{3},\\ &L^{1/q_{mn}}\sum_{k=1}^{k_0}\sum_{l}|a_{mnkl}-a_{kl}|M_0^{1/p_{kl}}\le\frac{\varepsilon}{3}\quad(m,n>D_1).\end{aligned}$$

Also, we have

$$\begin{aligned}L^{1/q_{mn}}\sum_{k,l}(a_{mnkl}-a_{kl})x_{kl} &= L^{1/q_{mn}}\sum_{k,l>k_0}(a_{mnkl}-a_{kl})x_{kl}\\ &+ L^{1/q_{mn}}\sum_{l=1}^{k_0}\sum_{k}(a_{mnkl}-a_{kl})x_{kl}\\ &+ L^{1/q_{mn}}\sum_{k=1}^{k_0}\sum_{l=k_0+1}^{\infty}(a_{mnkl}-a_{kl})x_{kl}.\end{aligned}$$

Hence, (12) follows. So, $Ax\in\mathscr{C}_p(q)$.

(b) The proof follows from (a) and Theorem 3.7.

(c) *Necessity.* (i)–(x) follow from (a). Let $m\in\mathbb{N}$ be fixed. Then, $(a_{mnkl})_{n,k,l}:(\mathscr{C}_{bp0}(p),c((q_{mn})_n))$. This implies

$$(a_{mnkl})_{n,k}:(\ell_\infty((p_{kl})_k),c((q_{mn})_n))\ (l\in\mathbb{N}),$$

$$(a_{mnkl})_{n,l}:(\ell_\infty((p_{kl})_l),c((q_{mn})_n))\ (k\in\mathbb{N})$$

and

$$(a_{mnkl})_{n,k,l}:\ (T^{-1}(c_0(r)),c((q_{mn})_n)).$$

By Theorem 5.1, 9 in [8], $(a_{mnkl})_{n,k,l}:\ (T^{-1}(c_0(r)),c((q_{mn})_n))$ is equivalent to (xiv), (xxiv) and (xxv).

By Theorem 5.1, 11 in [8], $(a_{mnkl})_{n,k}:(\ell_\infty((p_{kl})_k),c((q_{mn})_n))\ (l\in\mathbb{N})$ is equivalent to (xv) and (xvi); $(a_{mnkl})_{n,l}:(\ell_\infty((p_{kl})_l),c((q_{mn})_n))\ (k\in\mathbb{N})$ is equivalent to (xvii) and (xviii).

Analogously, since for any $n\in\mathbb{N}$, the map $(a_{mnkl})_{m,k,l}:(\mathscr{C}_{bp0}(p),c((q_{mn})_m))$, (xix)–(xxiii), (xxvi) and (xxvii) follow.

Sufficiency. From (a) it follows that $A(x) \in \mathscr{C}_p(q)$ for each $x \in \mathscr{C}_{bp0}(p)$. We need to verify that given $x \in \mathscr{C}_{bp0}(p)$, we have $((Ax)_{m,n})_m \in c((q_{mn})_m)$ $(n \in \mathbb{N})$ and $((Ax)_{m,n})_n \in c((q_{mn})_n)$ $(m \in \mathbb{N})$. First, we verify that $((Ax)_{m,n})_n \in c((q_{mn})_n)$ $(m \in \mathbb{N})$.

In the same way as in the proof of (a), we get from (xxiv) and (xxv) that $(\beta_{mkl})_{k,l} \in \mathscr{C}_{bp0}(p)^{\beta(\nu)}$ $(m \in \mathbb{N})$. Further, in the same way as in (a) from (xvi), (xviii) and (xxv), it follows that

$$c((q_{mn})_n)) - \lim((Ax)_{m,n})_n = \sum_{k,l} \beta_{mkl} x_{kl} \quad (m \in \mathbb{N}).$$

Hence, $((Ax)_{m,n})_n \in c((q_{mn})_n)$ $(m \in \mathbb{N})$. Analogously, we get $((Ax)_{m,n})_m \in c((q_{mn})_m)$ $(n \in \mathbb{N})$. □

Theorem 3.9. *Let $A = (a_{mnkl})$ be a 4-dimensional matrix. Then,*

(a) $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_{p0}(q))$ *iff the following conditions hold:*
(i) $\forall k,l \in \mathbb{N} \,:\, \lim_{m,n} |a_{mnkl}|^{q_{mn}} = 0$;
(ii) $\forall M,m,n,l \in \mathbb{N} \,:\, \sum_k |a_{mnkl}| M^{1/p_{kl}} < \infty$;
(iii) $\forall M,l \in \mathbb{N} \,:\, \lim_{m,n} (\sum_k |a_{mnkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(iv) $\forall M,m,n,k \in \mathbb{N} \,:\, \sum_l |a_{mnkl}| M^{1/p_{kl}} < \infty$;
(v) $\forall M,k \in \mathbb{N} \,:\, \lim_{m,n} (\sum_l |a_{mnkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(vi) $\forall L \, \exists M,D \in \mathbb{N} \,:\, \sup_{m,n>D} L^{1/q_{mn}} \sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}} < \infty$;
(vii) $\forall m,n \in \mathbb{N} \, \exists M \in \mathbb{N} \,:\, \sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}} < \infty$.

(b) $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_{bp0}(q))$ *iff* (i)–(vii) *and the following conditions hold:*
(viii) $\forall M,l \in \mathbb{N} \,:\, \sup_{m,n} (\sum_k |a_{mnkl}| M^{1/p_{kl}})^{q_{mn}} < \infty$;
(ix) $\forall M,k \in \mathbb{N} \,:\, \sup_{m,n} (\sum_l |a_{mnkl}| M^{1/p_{kl}})^{q_{mn}} < \infty$;
(x) $\exists M \in \mathbb{N} \,:\, \sup_{m,n} (\sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}})^{q_{mn}} < \infty$.

(c) $A \in (\mathscr{C}_{bp0}(p), \mathscr{C}_{r0}(q))$ *iff* (i)–(vii) *and the following conditions hold:*
(xi) $\exists(\beta_{mkl}) \, \forall k,l \in \mathbb{N} \,:\, \lim_n |a_{mnkl} - \beta_{mkl}|^{q_{mn}} = 0$;
(xii) $\forall M,l,m \in \mathbb{N} \,:\, \lim_n (\sum_k |a_{mnkl} - \beta_{mkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(xiii) $\forall M,k,m \in \mathbb{N} \,:\, \lim_n (\sum_l |a_{mnkl} - \beta_{mkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(xiv) $\exists(\gamma_{nkl}) \, \forall k,l \in \mathbb{N} \,:\, \lim_m |a_{mnkl} - \gamma_{nkl}|^{q_{mn}} = 0$;
(xv) $\forall M,l,n \in \mathbb{N} \,:\, \lim_m (\sum_k |a_{mnkl} - \gamma_{nkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(xvi) $\forall M,k,n \in \mathbb{N} \,:\, \lim_m (\sum_l |a_{mnkl} - \gamma_{nkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(xvii) $\forall L,m \, \exists M \in \mathbb{N} \,:\, \sup_n L^{1/q_{mn}} \sum_{k,l} |a_{mnkl} - \beta_{mkl}| M^{-1/p_{kl}} < \infty$;
(xviii) $\forall L,n \, \exists M \in \mathbb{N} \,:\, \sup_m L^{1/q_{mn}} \sum_{k,l} |a_{mnkl} - \gamma_{nkl}| M^{-1/p_{kl}} < \infty$.

Proof. Since $e^{kl} \in \mathscr{C}_{p0}(p)$ $(k,l \in \mathbb{N})$, the proof follows from Theorem 3.8 by taking $a_{kl} = 0$. □

Applying Theorems 3.7–3.9 and the fact that $\mathscr{C}_{bp}(p) = \mathscr{C}_{bp0}(p) \oplus < e >$, we get the following theorems:

Theorem 3.10. *Let $A = (a_{mnkl})$ be a 4-dimensional matrix. Then, $A \in (\mathscr{C}_{bp}(p), \mathscr{M}_u(q))$ iff the following conditions hold:*
(i) $\forall M,k \in \mathbb{N} \,:\, \sup_{m,n} \left(\sum_l |a_{mnkl}| M^{1/p_{kl}}\right)^{q_{mn}} < \infty$;
(ii) $\forall M,l \in \mathbb{N} \,:\, \sup_{m,n} \left(\sum_k |a_{mnkl}| M^{1/p_{kl}}\right)^{q_{mn}} < \infty$;
(iii) $\exists M \in \mathbb{N} \,:\, \sup_{m,n} \left(\sum_{k,l} |a_{mnkl}| M^{-1/p_{kl}}\right)^{q_{mn}} < \infty$;
(iv) $\sup_{m,n} |\sum_{k,l} a_{mnkl}|^{q_{mn}} < \infty$.

Theorem 3.11. *Let $A = (a_{mnkl})$ be a 4-dimensional matrix. Then,*

(a) $A \in (\mathscr{C}_{bp}(p), \mathscr{C}_p(q))$ *iff the following conditions hold:*
(i) $\forall k,l \in \mathbb{N} \, \exists(a_{kl}) \,:\, \lim_{m,n} |a_{mnkl} - a_{kl}|^{q_{mn}} = 0$;
(ii) $\forall M,m,n,l \in \mathbb{N} \,:\, \sum_k |a_{mnkl}| M^{1/p_{kl}} < \infty$;
(iii) $\forall M,l \in \mathbb{N} \, \exists D \in \mathbb{N} \,:\, \sup_{m,n>D} \sum_k |a_{mnkl}| M^{1/p_{kl}} < \infty$;

(iv) $\forall M,l\in\mathbb{N}\;:\;\lim_{m,n}(\sum_k|a_{mnkl}-a_{kl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(v) $\forall M,m,n,k\in\mathbb{N}\;:\;\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vi) $\forall M,k\in\mathbb{N}\;\exists D\in\mathbb{N}\;:\;\sup_{m,n>D}\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(vii) $\forall M,k\in\mathbb{N}\;:\;\lim_{m,n}(\sum_l|a_{mnkl}-a_{kl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(viii) $\forall m,n\in\mathbb{N}\;\exists M\in\mathbb{N}\;:\;\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(ix) $\exists M,D\in\mathbb{N}\;:\;\sup_{m,n>D}\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(x) $\forall L\;\exists M,D\in\mathbb{N}\;:\;\sup_{m,n>D}L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-a_{kl}|M^{-1/p_{kl}}<\infty$;
(xi) $\exists\alpha\in\mathbb{R}\;:\;\lim_{m,n}|\sum_{k,l}a_{mnkl}-\alpha|^{q_{mn}}=0$.
(b) $A\in(\mathscr{C}_{bp}(p),\mathscr{C}_{bp}(q))$ iff (i)–(xi) *and the following conditions hold:*
(xii) $\forall M,l\in\mathbb{N}\;:\;\sup_{m,n}(\sum_k|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}<\infty$;
(xiii) $\forall M,k\in\mathbb{N}\;:\;\sup_{m,n}(\sum_l|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}<\infty$;
(xiv) $\exists M\in\mathbb{N}\;:\;\sup_{m,n}(\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}})^{q_{mn}}<\infty$.
(c) $A\in(\mathscr{C}_{bp}(p),\mathscr{C}_{r}(q))$ *iff* (i)–(xi) *and the following conditions hold:*
(xv) $\exists(\beta_{mkl})\;\forall k,l\in\mathbb{N}\;:\;\lim_n|a_{mnkl}-\beta_{mkl}|^{q_{mn}}=0$;
(xvi) $\forall M,l,m\in\mathbb{N}\;:\;\sup_n\sum_k|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xvii) $\forall M,l,m\in\mathbb{N}\;:\;\lim_n(\sum_k|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xviii) $\forall M,k,m\in\mathbb{N}\;:\;\sup_n\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xix) $\forall M,k,m\in\mathbb{N}\;:\;\lim_n(\sum_l|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xx) $\exists(\gamma_{nkl})\;\forall k,l\in\mathbb{N}\;:\;\lim_m|a_{mnkl}-\gamma_{nkl}|^{q_{mn}}=0$;
(xxi) $\forall M,l,n\in\mathbb{N}\;:\;\sup_m\sum_k|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xxii) $\forall M,l,n\in\mathbb{N}\;:\;\lim_m(\sum_k|a_{mnkl}-\gamma_{nkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xxiii) $\forall M,k,n\in\mathbb{N}\;:\;\sup_m\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(xxiv) $\forall M,k,n\in\mathbb{N}\;:\;\lim_m(\sum_l|a_{mnkl}-\gamma_{nkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xxv) $\forall m\;\exists M\in\mathbb{N}\;:\;\sup_n\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(xxvi) $\forall L,m\;\exists M\in\mathbb{N}\;:\;\sup_n L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-\beta_{mkl}|M^{-1/p_{kl}}<\infty$;
(xxvii) $\forall n\;\exists M\in\mathbb{N}\;:\;\sup_m\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(xxviii) $\forall L,n\;\exists M\in\mathbb{N}\;:\;\sup_m L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}-\gamma_{nkl}|M^{-1/p_{kl}}<\infty$;
(xxix) $\exists(\alpha_m)\in\mathbb{R}\;m\in\mathbb{N}\;:\;\lim_n|\sum_{k,l}a_{mnkl}-\alpha_m|^{q_{mn}}=0$;
(xxx) $\exists(\delta_n)\in\mathbb{R}\;n\in\mathbb{N}\;:\;\lim_m|\sum_{k,l}a_{mnkl}-\delta_n|^{q_{mn}}=0$.

Theorem 3.12. *Let $A=(a_{mnkl})$ be a 4-dimensional matrix. Then,*
(a) $A\in(\mathscr{C}_{bp}(p),\mathscr{C}_{p0}(q))$ *iff the following conditions hold:*
(i) $\forall k,l\in\mathbb{N}\;:\;\lim_{m,n}|a_{mnkl}|^{q_{mn}}=0$;
(ii) $\forall M,m,n,l\in\mathbb{N}\;:\;\sum_k|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(iii) $\forall M,l\in\mathbb{N}\;:\;\lim_{m,n}(\sum_k|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(iv) $\forall M,m,n,k\in\mathbb{N}\;:\;\sum_l|a_{mnkl}|M^{1/p_{kl}}<\infty$;
(v) $\forall M,k\in\mathbb{N}\;:\;\lim_{m,n}(\sum_l|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(vi) $\forall L\;\exists M,D\in\mathbb{N}\;:\;\sup_{m,n>D}L^{1/q_{mn}}\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(vii) $\forall m,n\in\mathbb{N}\;\exists M\in\mathbb{N}\;:\;\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}}<\infty$;
(viii) $\lim_{m,n}|\sum_{k,l}a_{mnkl}|^{q_{mn}}=0$.
(b) $A\in(\mathscr{C}_{bp}(p),\mathscr{C}_{bp0}(q))$ iff (i)–(viii) *and the following conditions hold:*
(ix) $\forall M,l\in\mathbb{N}\;:\;\sup_{m,n}(\sum_k|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}<\infty$;
(x) $\forall M,k\in\mathbb{N}\;:\;\sup_{m,n}(\sum_l|a_{mnkl}|M^{1/p_{kl}})^{q_{mn}}<\infty$;
(xi) $\exists M\in\mathbb{N}\;:\;\sup_{m,n}(\sum_{k,l}|a_{mnkl}|M^{-1/p_{kl}})^{q_{mn}}<\infty$.
(c) $A\in(\mathscr{C}_{bp}(p),\mathscr{C}_{r0}(q))$ *iff* (i)–(viii) *and the following conditions hold:*
(xii) $\exists(\beta_{mkl})\;\forall k,l\in\mathbb{N}\;:\;\lim_n|a_{mnkl}-\beta_{mkl}|^{q_{mn}}=0$;
(xiii) $\forall M,l,m\in\mathbb{N}\;:\;\lim_n(\sum_k|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{q_{mn}}=0$;
(xiv) $\forall M,k,m\in\mathbb{N}\;:\;\lim_n(\sum_l|a_{mnkl}-\beta_{mkl}|M^{1/p_{kl}})^{q_{mn}}=0$;

(xv) $\exists(\gamma_{nkl})\ \forall k,l \in \mathbb{N}\ \lim_m |a_{mnkl} - \gamma_{nkl}|^{q_{mn}} = 0$;
(xvi) $\forall M,l,n \in \mathbb{N}\ :\ \lim_m (\sum_k |a_{mnkl} - \gamma_{nkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(xvii) $\forall M,k,n \in \mathbb{N}\ :\ \lim_m (\sum_l |a_{mnkl} - \gamma_{nkl}| M^{1/p_{kl}})^{q_{mn}} = 0$;
(xviii) $\forall L,m\ \exists M \in \mathbb{N}\ :\ sup_n L^{1/q_{mn}} \sum_{k,l} |a_{mnkl} - \beta_{mkl}| M^{-1/p_{kl}} < \infty$;
(xix) $\forall L,n\ \exists M \in \mathbb{N}\ :\ \sup_m L^{1/q_{mn}} \sum_{k,l} |a_{mnkl} - \gamma_{nkl}| M^{-1/p_{kl}} < \infty$;
(xx) $\lim_m |\sum_{k,l} a_{mnkl}|^{q_{mn}} = 0\ (n \in \mathbb{N})$;
(xxi) $\lim_n |\sum_{k,l} a_{mnkl}|^{q_{mn}} = 0\ (m \in \mathbb{N})$.

The diagram gives the number of the theorem where the characterization of (E,F) (of $(E,F)_v$) is to be found.

	F						
E	$\mathscr{M}_u(q)$	$\mathscr{C}_p(q)$	$\mathscr{C}_{p0}(q)$	$\mathscr{C}_{bp}(q)$	$\mathscr{C}_{bp0}(q)$	$\mathscr{C}_r(q)$	$\mathscr{C}_{r0}(q)$
$\mathscr{M}_u(p)$	3.4	3.5	3.6	3.5	3.6	3.5	3.6
$\mathscr{C}_{bp0}(p)$	3.7	3.8	3.9	3.8	3.9	3.8	3.8
$\mathscr{C}_{bp}(p)$	3.10	3.11	3.12	3.11	3.12	3.11	3.12

4. CONCLUSIONS

Maddox [10] generalized the spaces c_0, c, ℓ_∞ by adding the powers p_k $(k \in \mathbb{N})$ to the elements (x_k) in the definitions of these spaces. Gökhan and Çolak [4–6] defined the corresponding double sequence spaces for the Pringsheim convergence and the bounded Pringsheim convergence. In the article of Gökhan et al. [7], the authors characterized some classes of matrix transformations involving these double sequence spaces with powers with transformed double sequences uniformly bounded. However, many of their results turned out to be incorrect. In this article, we have provided counterexamples to these claims and proven the correct results for a broader class of matrices. In the next paper, we will characterize the corresponding matrix transformations without assuming uniform boundedness.

ACKNOWLEDGEMENTS

The publication costs of the article were covered by Tallinn University, School of Digital Technologies, and the Estonian Academy of Sciences.

REFERENCES

1. Başar, F. and Sever, Y. The space $\mathscr{L}_q$ of double sequences. *Math. J. Okayama Univ.*, 2009, **51**, 149–157.
2. Boos, J. *Classical and Modern Methods in Summability*. Oxford University Press, Oxford, 2000.
3. Boos, J. and Seydel, D. Theorems of Toeplitz–Silverman type for maps defined by sequences of matrices. *J. Anal.*, 2001, **9**, 149–181.
4. Gökhan, A. and Çolak, R. The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$. *Appl. Math. Comput.*, 2004, **157**(2), 491–501.
5. Gökhan, A. and Çolak, R. Double sequence space $\ell_\infty^2(p)$. *Appl. Math. Comput.*, 2005, **160**, 147–153.
6. Gökhan, A. and Çolak, R. On double sequence spaces ${}_0c_2^P(p)$, ${}_0c_2^{PB}(p)$ and $\ell_2(p)$. *Int. J. Pure Appl. Math.*, 2006, **30**(3), 309–321.
7. Gökhan, A., Çolak, R. and Mursaleen, M. Some matrix transformations and generalized core of double sequences. *Math. Comput. Model.*, 2009, **49**(7–8), 1721–1731.
8. Grosse-Erdmann, K.-G. Matrix transformations between the sequence spaces of Maddox. *J. Math. Anal. Appl.*, 1993, **180**(1), 223–238.
9. Lascarides, C. G. and Maddox, I. J. Matrix transformations between some classes of sequences. *Math. Proc. Cambridge Philos. Soc.*, 1970, **68**, 99–104.
10. Maddox, I. J. Spaces of strongly summable sequences, *Quarterly J. Math.*, 1967, **18**(1), 345–355.

11. Nakano, H. Modulared sequence spaces. *Proc. Japan Acad.*, 1951, **27**(9), 508–512.
12. Zeltser, M., Boos, J. and Leiger, T. Sequences of 0's and 1's: new results via double sequence spaces. *J. Math. Anal. Appl.*, 2002, **275**(2), 883–899.
13. Zeltser, M. *Investigation of double sequence spaces by soft and hard analytical methods.* PhD thesis. University of Tartu, Estonia, 2001.

Astmetega koonduvate topeltjadade maatriksteisendused

Maria Zeltser ja Şeyda Sezgek

Maddox (1967) üldistas ruume c_0, c, ℓ_∞, lisades astmed p_k $(k \in \mathbb{N})$ jadade (x_k) elementidele ruumide definitsioonides. Gökhan ja Çolak (2004–2006) defineerisid Pringsheimi ja tõkestatud Pringsheimi koonduvuse jaoks vastavad topeltjadade ruumid. Artiklis [7] iseloomustasid autorid mõningaid maatriksteisenduste klasse, mis hõlmavad neid astmetega topeltjadade ruume. Paljud nende tulemused osutusid aga valeks. Selles artiklis anname vastavaid vastunäiteid ja tõestame õigeid tulemusi laiema maatriksite klassi jaoks.