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CONTROL THEORY

Computation of nonlinear eigenvalues based on the Ore determinant: preliminary results

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Abstract. The concept of eigenvalues has recently been generalized for nonlinear systems, but the method to find them is missing. Unlike the linear case, now one has to deal with non-commutative polynomials from the Ore ring. In the paper, the Ore determinant of a polynomial matrix, describing generic linearization of the state equations, is used instead of the standard definition of determinant of the polynomial matrix with real coefficients. It is shown how to compute the Ore determinant of a polynomial matrix associated with the nonlinear system and conjectured that the eigenvalues can be found from factorization of the Ore determinants of the corresponding system matrix. Moreover, it is proved that such factorization into the first-order polynomials can always be done. Many examples illustrate the computations and concepts throughout the paper.

Keywords: control theory, nonlinear control systems, skew polynomial ring, Ore determinant, nonlinear eigenvalues.

1. INTRODUCTION

The concept of eigenvalues plays an important role in linear systems theory. It is well known that eigenvalues characterize the stability of a linear system and are also useful for finding various state transformations, for instance, state equations into the diagonal form. The notion of eigenvalues has recently been extended to nonlinear systems [5], and it was shown that such a concept provides generalization of the one known from linear theory. The concept of eigenvalues of the nonlinear system was actually already introduced in [14] under the name of characteristic function. The concept may also be viewed as a direct extension of that from [21], which addresses the linear time-varying systems. The extension becomes obvious once the tangent linear system is associated with the nonlinear system. Similar to the linear case, the eigenvalues of the nonlinear system can be used for various state transformations (see [5]), and recently it was shown in [7] that they play a similar role in the characterization of the stability of nonlinear systems. Moreover, in the papers [13,18], eigenvalues were used to address the structural accessibility problem, and in [6,8], eigenvalues were used in the development of an accessibility/observability criterion. A different (geometric) extension of the eigenvalue was considered in [17] that is useful in model reduction problems.

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However, the open problem is the computation of the eigenvalues of a nonlinear system. In the linear case, eigenvalues can be found as the roots of the characteristic polynomial of a linear control system, that is, the determinant of the matrix $\lambda I - A$, if we consider a linear system $\dot{x} = Ax$. In this paper, we propose that the eigenvalues of a nonlinear system can be found in a similar way if non-commutative generalization of the determinant is applied. In more detail, consider an autonomous nonlinear system $\dot{x} = f(x)$ with which we associate the tangent linear system $\mathrm{d}\dot{x} = A\mathrm{d}x$, where $A = \partial f/\partial x$, and then employ its polynomial description $sI - A$. The polynomials in such a description belong, however, into the (non-commutative) skew polynomial ring, often called the Ore ring. The paper suggests the idea that the eigenvalues of a nonlinear system can be found from factorization of the Ore determinants[1] of the matrix $sI - A$. The claim is not formally proven in the paper but presented as a conjecture. However, the numerous examples run by us support the idea, and one such example is also presented in the paper. Moreover, a novel method for computation of the Ore determinants of the matrix $sI - A$ is given, and it is also proven that the Ore determinants of $sI - A$ are always factorizable into factors of degree one over the corresponding skew polynomial ring.

For the sake of simplicity, we restrict the attention to the second-order systems. Such simplification allows to focus on the presented ideas. The general case can be handled, in principle, analogously, but is technically much more involved and requires the help of symbolic computation tools like Mathematica or Maple to complete the computations.

2. ALGEBRAIC SETTING

In this paper, we will use the algebraic setting of [2,3,22] adapted to nonlinear autonomous systems, defined by the differential equations of the form

$$\dot{x} = f(x), \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$ and the components of f are assumed to be from the field $\mathscr{K}$ of meromorphic functions in variables from the set $\{x_1, \ldots, x_n\}$. Define the derivative operator $\mathrm{d}/\mathrm{d}t$ that acts on functions $\varphi(x_1, \ldots, x_n) \in \mathscr{K}$ as follows:

$$\dot{\varphi} = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} \dot{x}_i,$$

where we substitute $\dot{x}_i$ from (1).

Let $\mathscr{E}$ denote the vector space of differential one-forms defined as $\mathscr{E} = \mathrm{span}_{\mathscr{K}}\{\mathrm{d}\xi; \xi \in \mathscr{K}\}$, where d is the standard differential operator. The operator $\mathrm{d}/\mathrm{d}t$ induces the derivative operator that acts on $\mathscr{E}$, and it is denoted by the same symbol. Let $\omega = \sum_i \alpha_i \mathrm{d}\xi_i$ be in $\mathscr{E}$, then

$$\dot{\omega} = \sum_i (\dot{\alpha}_i \mathrm{d}\xi_i + \alpha_i \mathrm{d}\dot{\xi}_i).$$

The operator $\mathrm{d}/\mathrm{d}t$ also induces the left skew polynomial ring $\mathscr{K}[s]$ of the polynomials in s (interpreted as $\mathrm{d}/\mathrm{d}t$) over $\mathscr{K}$ with the standard addition and the non-commutative multiplication defined by the commutation rule

$$s\xi = \xi s + \dot{\xi}, \tag{2}$$

where $\xi \in \mathscr{K}$. The ring $\mathscr{K}[s]$ represents the ring of the derivative operators that act on any $\omega \in \mathscr{E}$ as follows:

$$\left(\sum_{i=0}^{k} \alpha_i s^i \right) \omega = \sum_{i=0}^{k} \alpha_i \omega^{(i)},$$

where $\omega^{(i)} := \mathrm{d}/\mathrm{d}t(\omega^{(i-1)})$ for $i \geq 1$.

[1] Note that the Ore determinant is not unique unlike the classical determinant.

Lemma 1 (Ore condition [15,16]). *For all non-zero $a, b \in \mathscr{K}[s]$, there exist non-zero $a_1, b_1 \in \mathscr{K}[s]$ such that $a_1 b = b_1 a$.*

The tangent linear system, associated with the nonlinear system (1), is given by

$$\mathrm{d}\dot{x} = A\mathrm{d}x, \tag{3}$$

where $A = (\partial f/\partial x) \in \mathscr{K}^{n\times n}$. Using the tangent linear system description (3), one can now associate with the state equations (1) their polynomial system description

$$(sI - A)\mathrm{d}x = 0.$$

Example 1 ([5]). *Consider the system*

$$\dot{x}_1 = x_1 + x_1 x_2, \quad \dot{x}_2 = x_2^2. \tag{4}$$

The polynomial description of this system can be found as

$$\begin{pmatrix} s-1-x_2 & -x_1 \\ 0 & s-2x_2 \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

3. EIGENVALUES OF THE NONLINEAR SYSTEM

Definition 1 ([5]). *A function $\lambda \in \mathscr{K}$ is said to be an eigenvalue and a non-zero vector $e \in \mathscr{K}^n$ an eigenvector of system* (1) *if they satisfy*

$$\lambda e + \dot{e} = Ae, \tag{6}$$

where $A = \partial f/\partial x$.

Example 2 (continuation of Example 1). *In order to find eigenvalues and eigenvectors for the system* (4)*, we are looking for $\lambda \in \mathscr{K}$ and a non-zero vector $(e_1, e_2)^T \in \mathscr{K}^2$ such that*

$$\lambda \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} 1+x_2 & x_1 \\ 0 & 2x_2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Table 1 shows various solutions for λ, e_1 and e_2 that can be checked by direct substitution.

Table 1. Possible eigenvalues and various eigenvectors for the system in Example 2

λ	e_1	e_2	λ	e_1	e_2
0	x_1	0	$1+x_2$	1	0
	$x_1 x_2$	x_2^2		2	0
1	x_2	0	$2x_2$	x_1/x_2	1
	$2x_2$	0		$2x_1/x_2$	2

3.1. Equivalence of eigenvalues

Example 2 demonstrates that, unlike the linear case, there may be more than n eigenvalues for the nth order nonlinear system. Hence, a natural question to be asked is this: in what sense is Definition 1 a generalization of the results from the linear theory, where the nth order system has exactly n eigenvalues (counted with multiplicity)? We will answer this question below.

First, recall the notion of the so-called $\mathrm{d}/\mathrm{d}t$-conjugacy [11,12], adapted in this paper from pseudo-derivation to standard derivation as a special case. The $\mathrm{d}/\mathrm{d}t$-conjugacy can be understood as an equivalence relation under which the possible eigenvalues of a nonlinear system split into at most n equivalence classes. Let λ and $\tilde{\lambda}$ be two eigenvalues for the system (1) with the eigenvectors e and $\tilde{e}$, respectively. Assume that e and $\tilde{e}$ are dependent, that is, $\tilde{e} = ce$ for some non-zero $c \in \mathscr{K}$. From (6), one has

$$\tilde{\lambda}\tilde{e} + \dot{\tilde{e}} = A\tilde{e} \quad \Leftrightarrow \quad \tilde{\lambda}ce + \dot{c}e + c\dot{e} = Ace \quad \Leftrightarrow \quad (\tilde{\lambda} + \dot{c}c^{-1})e + \dot{e} = Ae,$$

which implies the relation $\lambda = \tilde{\lambda} + \dot{c}c^{-1}$. The above motivates the following definition.

Definition 2. *Elements α and β in $\mathscr{K}$ are said to be $\mathrm{d}/\mathrm{d}t$-conjugate if there exists a non-zero element c in $\mathscr{K}$ such that*

$$\alpha = \beta + \dot{c}c^{-1}.$$

Example 3 (continuation of Example 2). *Consider the eigenvalues listed in Table 1. One can show that, for instance, $\lambda = 1$ and $\tilde{\lambda} = 1 + x_2$ are $\mathrm{d}/\mathrm{d}t$-conjugate. Indeed, for $c = 1/x_2$, $\lambda = \tilde{\lambda} + \dot{c}c^{-1}$. Also, $\lambda = 0$ and $\tilde{\lambda} = 1 + x_2$ are $\mathrm{d}/\mathrm{d}t$-conjugate, since $\tilde{\lambda} = \lambda + \dot{c}c^{-1}$ for $c = x_1$.*

Proposition 1. *Let $\tilde{\lambda}$ be an eigenvalue of the nonlinear system* (1). *Then*

$$\lambda = \tilde{\lambda} + \dot{c}c^{-1}, \qquad c \in \mathscr{K} \tag{7}$$

is also an eigenvalue of the nonlinear system (1).

Proof. Observe that $\tilde{\lambda}$ being an eigenvalue of the nonlinear system (1) implies

$$\tilde{\lambda}\tilde{e} + \dot{\tilde{e}} = A\tilde{e}. \tag{8}$$

From (7), we have $\tilde{\lambda} = \lambda - \dot{c}c^{-1}$. After substituting it into (8), we get

$$(\lambda - \dot{c}c^{-1})\tilde{e} + \dot{\tilde{e}} = A\tilde{e} \quad \Leftrightarrow \quad \lambda\frac{\tilde{e}}{c} - \frac{\dot{c}\tilde{e}}{c^2} + \frac{\dot{\tilde{e}}}{c} = A\frac{\tilde{e}}{c} \quad \Leftrightarrow \quad \lambda\frac{\tilde{e}}{c} + \left(\dot{\frac{\tilde{e}}{c}}\right) = A\frac{\tilde{e}}{c},$$

resulting in $\lambda e + \dot{e} = Ae$, where $e = \tilde{e}/c$. □

Proposition 2. *The $\mathrm{d}/\mathrm{d}t$-conjugacy is an equivalence relation.*

Proof. One has to show that the $\mathrm{d}/\mathrm{d}t$-conjugacy is reflexive, symmetric, and transitive.

- Reflexivity: one has to prove that any $\alpha \in \mathscr{K}$ is $\mathrm{d}/\mathrm{d}t$-conjugate to itself. Indeed, $\alpha = \alpha + \dot{a}a^{-1}$ for any non-zero $a \in \mathbb{R}$.
- Symmetry: one has to prove that $\alpha \in \mathscr{K}$ is $\mathrm{d}/\mathrm{d}t$-conjugate to $\beta \in \mathscr{K}$ if and only if β is $\mathrm{d}/\mathrm{d}t$-conjugate to α. That is, for $\alpha, \beta \in \mathscr{K}$, we have $\alpha = \beta + \dot{a}a^{-1}$ for some non-zero $a \in \mathscr{K}$ if and only if $\beta = \alpha + \dot{b}b^{-1}$ for some non-zero $b \in \mathscr{K}$. Set $b = 1/a$, then $\beta = \alpha - \dot{a}(a^{-1})^2 a = \alpha - \dot{a}a^{-1}$, which implies $\alpha = \beta + \dot{a}a^{-1}$.
- Transitivity: one has to prove that if $\alpha \in \mathscr{K}$ is $\mathrm{d}/\mathrm{d}t$-conjugate to $\beta \in \mathscr{K}$ and $\beta \in \mathscr{K}$ is $\mathrm{d}/\mathrm{d}t$-conjugate to $\gamma \in \mathscr{K}$, then $\alpha \in \mathscr{K}$ is $\mathrm{d}/\mathrm{d}t$-conjugate to $\gamma \in \mathscr{K}$. That is, if

$$\alpha = \beta + \dot{a}a^{-1} \tag{9}$$

for some non-zero $a \in \mathscr{K}$ and

$$\beta = \gamma + \dot{b}b^{-1} \tag{10}$$

for some non-zero $b \in \mathscr{K}$, then $\alpha = \gamma + \dot{c}c^{-1}$ for some non-zero $c \in \mathscr{K}$. After substituting (10) into (9), we get $\alpha = \gamma + \dot{b}b^{-1} + \dot{a}a^{-1} = \gamma + (\dot{ba})(ba)^{-1}$. Set $c = ba$, and the result follows.

□

4. EIGENVALUES AND ORE DETERMINANTS

In this section, in subsection 4.1, we give the definition of the Ore determinant of a matrix with elements from the ring $\mathscr{K}[s]$. Then, in subsection 4.2, we propose an alternative way of computing the Ore determinants of the matrix $sI-A$, which describes the tangent linear system (3) of the nonlinear state equations (1). It is shown in subsection 4.3 that the Ore determinants of the matrix $sI-A$, associated with the nonlinear system, can always be factorized into the first-order polynomials. Finally, in subsection 4.4, we suggest to compute the nonlinear eigenvalues of the system (1) from the factorization of the Ore determinant of the matrix $sI-A$. The latter is given as a conjecture (i.e. without a proof), and the example supports this hypothesis.

Unlike the classical case, the entries of the matrix $sI-A$ are now not polynomials with real coefficients but non-commutative polynomials from the skew polynomial ring $\mathscr{K}[s]$. Therefore, one has to use non-commutative determinants. To define a 'good' non-commutative notion of a determinant is not a trivial task. Some definitions have been introduced in [15,20]. Since we use the Ore (skew) polynomial ring, the natural choice is to use the concept of the Ore determinant, although in the literature, the Dieudonne determinant has also been introduced and applied [9,20] for nonlinear systems.

4.1. Computation of the Ore determinant by definition

For the sake of simplicity and clarity of presentation, we will consider below only the case of 2×2 matrices. The notion of the Ore determinant is closely related to the problem of finding the solution of the set of linear equations, defined over a non-commutative polynomial ring $\mathscr{K}[s]$:

$$k_{11}x_1+k_{12}x_2=l_1, \quad k_{21}x_1+k_{22}x_2=l_2 \tag{11}$$

or given in a matrix form as

$$K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \quad K=\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}\in\mathscr{K}[s]^{2\times 2}, \quad l_1,l_2\in\mathscr{K}[s].$$

To eliminate the variable x_2, we multiply the first equation by $\kappa_{22}\in\mathscr{K}[s]$ and the second equation by $\kappa_{12}\in\mathscr{K}[s]$ and require that $\kappa_{22}k_{12}=\kappa_{12}k_{22}$, which is the Ore condition from Lemma 1:

$$\kappa_{22}k_{11}x_1+\kappa_{22}k_{12}x_2=\kappa_{22}l_1, \quad \kappa_{12}k_{21}x_1+\kappa_{12}k_{22}x_2=\kappa_{12}l_2.$$

Now, subtracting the second equation from the first yields $(\kappa_{22}k_{11}-\kappa_{12}k_{21})x_1=\kappa_{22}l_1-\kappa_{12}l_2$. Since $\mathscr{K}[s]$ is the Ore ring, then it can be embedded into a skew field $\mathscr{F}(s)$, called the field of left fractions. Thus, in $\mathscr{F}(s)$ there exists $(\kappa_{22}k_{11}-\kappa_{12}k_{21})^{-1}$, which results in

$$x_1=(\kappa_{22}k_{11}-\kappa_{12}k_{21})^{-1}(\kappa_{22}l_1-\kappa_{12}l_2).$$

The expression $\kappa_{22}k_{11}-\kappa_{12}k_{21}$ is called the Ore determinant. Similarly, one can find κ_{21} and κ_{11} such that $\kappa_{21}k_{11}=\kappa_{11}k_{21}$, which is again the Ore condition. Now, multiplying the first equation of (11) by κ_{21} and the second equation by κ_{11} gives us

$$\kappa_{21}k_{11}x_1+\kappa_{21}k_{12}x_2=\kappa_{21}l_1, \quad \kappa_{11}k_{21}x_1+\kappa_{11}k_{22}x_2=\kappa_{11}l_2.$$

This time, we subtract the first equation from the second to get $(\kappa_{11}k_{22}-\kappa_{21}k_{12})x_2=\kappa_{11}l_2-\kappa_{21}l_1$, resulting in

$$x_2=(\kappa_{11}k_{22}-\kappa_{21}k_{12})^{-1}(\kappa_{11}l_2-\kappa_{21}l_1).$$

The expression $\kappa_{22}k_{11}-\kappa_{12}k_{21}$ is the second Ore determinant. In general, for a $p\times p$ matrix with entries in $\mathscr{K}[s]$, there are p different Ore determinants. Without additional restrictions, the determinants may be even polynomials of different degrees (see [10]). Since we want the Ore determinant to be the generalization of the standard commutative determinant, we have to specify the definition of the Ore determinant further by introducing additional restrictions. Sometimes the result is called the restricted Ore determinant. Note that the definition of the Ore determinant relies on the Ore condition from Lemma 1. In fact, the Ore condition is just a way to overcome the problems caused by non-commutativity. If the elements a and b in Lemma 1 commute, one can take $a_1=a$ and $b_1=b$. Moreover, if one of them, let us say a, would be zero, then it is natural to take $a_1=0$ and $b_1=b$. Therefore, from now on, we restrict the Ore condition in Lemma 1 as follows:

- if $a\neq 0$ and $b\neq 0$, then choose $a_1\neq 0$ and $b_1\neq 0$ such that $a_1b=b_1a$ and $\deg a_1=\deg a$ and $\deg b_1=\deg b$;
- if $a=0$, $b\neq 0$, then choose $a_1=0$ and $b_1=b$;
- if $a\neq 0$, $b=0$, then choose $a_1=a$ and $b_1=0$;
- if $a=b=0$, then choose $a_1=b_1=0$.

Example 4 (continuation of Example 1). *Recall the polynomial description* (5) *and denote*

$$sI-A=\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{pmatrix}=\begin{pmatrix} s-1-x_2 & -x_1 \\ 0 & s-2x_2\end{pmatrix}.$$

Then,

$$\text{Ore det}_1(sI-A)=\kappa_{22}k_{11}-\kappa_{12}k_{21},$$

where $\kappa_{22}k_{12}=\kappa_{12}k_{22}$*, that is,* $\kappa_{22}(-x_1)=\kappa_{12}(s-2x_2)$ *and* κ_{22} *and* κ_{12} *can be found as* $\kappa_{22}=s-1-3x_2$*,* $\kappa_{12}=-x_1$*. Therefore,*

$$\text{Ore det}_1(sI-A)=(s-1-3x_2)(s-1-x_2)=s^2-(4x_2+2)s+2x_2^2+4x_2+1. \tag{12}$$

The second Ore determinant is

$$\text{Ore det}_2(sI-A)=\kappa_{11}k_{22}-\kappa_{21}k_{12},$$

where $\kappa_{21}k_{11}=\kappa_{11}k_{21}$*, that is,* $\kappa_{21}(s-1-x_2)=\kappa_{11}\cdot 0$*. Hence,* $\kappa_{21}=0$*,* $\kappa_{11}=s-1-x_2$*, and one gets the second Ore determinant as*

$$\text{Ore det}_2(sI-A)=(s-1-x_2)(s-2x_2). \tag{13}$$

4.2. Alternative computation of the Ore determinant

There is an alternative way to compute the Ore determinants of the matrix $sI-A$ corresponding to the system of the form (1). Note that there exists an output function for the system (1) such that the polynomial description of the corresponding output differential equation equals the Ore determinant of the matrix $sI-A$ for this system. First, we demonstrate this fact by an example.

Example 5 (continuation of Example 1). *Consider the output function* $y=x_1$*. Note that* $\dot{y}=x_1+x_1x_2$*. Using the state elimination algorithm from* [2]*, one can find the output differential equation of the system*

$$\ddot{y}=\frac{2\dot{y}^2}{y}-2\dot{y}+y. \tag{14}$$

The polynomial description of this equation can be found as follows. First, we apply the differential operator to the equation (14) *and express the time derivatives of* $\mathrm{d}y$ *as* $\mathrm{d}\ddot{y}=s^2\mathrm{d}y$*,* $\mathrm{d}\dot{y}=s\mathrm{d}y$ *to get*

$$\left[s^2-\left(\frac{4\dot{y}}{y}-2\right)s-\left(1-\frac{2\dot{y}^2}{y^2}\right)\right]\mathrm{d}y=0.$$

Then, substituting $y = x_1$ and $\dot{y} = x_1 + x_1x_2$ gives, after rearrangement,

$$\left[s^2 - (4x_2+2)s + 2x_2^2 + 4x_2 + 1\right] \mathrm{d}y = 0. \tag{15}$$

Observe that the polynomial in (15) *is the Ore determinant* (12) *of the system. Our final observation is related to the fact that to get the polynomial description* (15) *of the system, we do not have to find the output differential equation explicitly, as this may not always be a trivial task. One can instead work with the tangent linear system and with the differentials of the system variables. Apply the differential operator to* (4) *to get*

$$\mathrm{d}\dot{x}_1 = (1+x_2)\mathrm{d}x_1 + x_1\mathrm{d}x_2, \qquad \mathrm{d}\dot{x}_2 = 2x_2\mathrm{d}x_2, \qquad \mathrm{d}y = \mathrm{d}x_1,$$

which results in

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}x_1, \\ \mathrm{d}\dot{y} &= (1+x_2)\mathrm{d}x_1 + x_1\mathrm{d}x_2, \\ \mathrm{d}\ddot{y} &= ((1+x_2)^2 + x_2^2)\mathrm{d}x_1 + (2x_1 + 4x_1x_2)\mathrm{d}x_2. \end{aligned}$$

From the first two equations, being a set of linear equations in $\mathrm{d}x_1$ *and* $\mathrm{d}x_2$*, we get* $\mathrm{d}x_1 = \mathrm{d}y$*,* $\mathrm{d}x_2 = 1/x_1\,[\mathrm{d}\dot{y} - (1+x_2)\mathrm{d}y]$*, which, after substituting into the last equation, results in*

$$\mathrm{d}\ddot{y} = (2+4x_2)\mathrm{d}\dot{y} - (2x_2^2 + 4x_2 + 1)\mathrm{d}y.$$

From the above, we get

$$\left[s^2 - (2+4x_2)s + 2x_2^2 + 4x_2 + 1\right] \mathrm{d}y = 0,$$

which is the polynomial description (15).

The ideas depicted in the above example can be generalized as follows. Consider the system (1), where $x = (x_1, x_2)$, and choose the output function $y = x_1$. The tangent linear system reads

$$\mathrm{d}\dot{x}_1 = a_{11}\mathrm{d}x_1 + a_{12}\mathrm{d}x_2, \quad \mathrm{d}\dot{x}_2 = a_{21}\mathrm{d}x_1 + a_{22}\mathrm{d}x_2, \quad \mathrm{d}y = \mathrm{d}x_1, \tag{16}$$

where $a_{ij} = \partial f_i/\partial x_j$, $i, j = 1, 2$. If $a_{12} = 0$, meaning that the system is not observable from $y = x_1$, then one can easily find the Ore determinant of the matrix $sI - A$ from the definition as

$$\text{Ore det}_1(sI - A) = (s - a_{22})(s - a_{11}).$$

If $a_{12} \neq 0$, meaning that the system is observable from $y = x_1$, then the polynomial description of the output differential equation can be found as follows:

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}x_1, \\ \mathrm{d}\dot{y} &= a_{11}\mathrm{d}x_1 + a_{12}\mathrm{d}x_2, \\ \mathrm{d}\ddot{y} &= (\dot{a}_{11} + a_{11}^2 + a_{12}a_{21})\mathrm{d}x_1 + (\dot{a}_{12} + a_{11}a_{12} + a_{12}a_{22})\mathrm{d}x_2. \end{aligned}$$

From the first two equations, we have $\mathrm{d}x_1 = \mathrm{d}y$ and $\mathrm{d}x_2 = \frac{1}{a_{12}}(\mathrm{d}\dot{y} - a_{11}\mathrm{d}y)$, which, after substituting into the last equation and using $\mathrm{d}\ddot{y} = s^2\mathrm{d}y$ and $\mathrm{d}\dot{y} = s\mathrm{d}y$, results in the polynomial description of the output differential equation of the system

$$\left[s^2 - (a_{11} + a_{22} + \dot{a}_{12}a_{12}^{-1})s - (\dot{a}_{11} + a_{12}a_{21} - a_{11}a_{22} - a_{11}\dot{a}_{12}a_{12}^{-1})\right] \mathrm{d}y = 0. \tag{17}$$

In what follows, we will show that the polynomial in (17) is the Ore determinant of the matrix $sI - A$, defined by the system (1) for $n = 2$. The polynomial description of the system (1) reads for $n = 2$ as

$$(sI - A)\mathrm{d}x = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} := \begin{pmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then,

$$\text{Ore}\det{}_1(sI-A) = \kappa_{22}k_{11} - \kappa_{12}k_{21},$$

where $\kappa_{22}k_{12} = \kappa_{12}k_{22}$, that is, $\kappa_{22}(-a_{12}) = \kappa_{12}(s-a_{22})$ and κ_{22} and κ_{12} can be found as $\kappa_{22} = s - a_{22} - \dot{a}_{12}a_{12}^{-1}$, $\kappa_{12} = -a_{12}$. Hence,

$$\begin{aligned}\text{Ore}\det{}_1(sI-A) &= (s-a_{22}-\dot{a}_{12}a_{12}^{-1})(s-a_{11}) - (-a_{12})(-a_{21}) \\ &= s^2 - (a_{11}+a_{22}+\dot{a}_{12}a_{12}^{-1})s - (\dot{a}_{11}+a_{12}a_{21}-a_{11}a_{22}-a_{11}\dot{a}_{12}a_{12}^{-1}).\end{aligned}$$

Similarly, it can be shown that $\text{Ore}\det_2(sI-A)$ is either

$$\text{Ore}\det{}_2(sI-A) = (s-a_{11})(s-a_{22})$$

if $a_{21} = 0$, or, if $a_{21} \neq 0$, it can be found from the polynomial description of the output differential equation of the system for the output function $y = x_2$, from which the system is observable.

Example 6. *Consider the system*

$$\dot{x}_1 = x_1x_2, \quad \dot{x}_2 = x_1 - x_2$$

with its polynomial description as

$$(sI-A)\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} := \begin{pmatrix} s-x_2 & -x_1 \\ -1 & s+1 \end{pmatrix}\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $\text{Ore}\det_1(sI-A)$ *can be found from the polynomial description of the output differential equation of the system for the output function* $y = x_1$. *We get*

$$y = x_1, \quad \dot{y} = x_1x_2, \quad \ddot{y} = x_1x_2^2 + x_1^2 - x_1x_2.$$

From the first two equations, we have $x_1 = y$ *and* $x_2 = \dot{y}/y$, *which we substitute into the last equation:*

$$\ddot{y} = \frac{\dot{y}^2}{y} + y^2 - \dot{y}. \tag{18}$$

The polynomial description of the output differential equation (18) *can be found as follows:*

$$d\ddot{y} = \left(\frac{2\dot{y}}{y} - 1\right)d\dot{y} + \left(2y - \frac{\dot{y}^2}{y^2}\right)dy,$$

which can be rewritten as

$$\left[s^2 - \left(\frac{2\dot{y}}{y} - 1\right)s - \left(2y - \frac{\dot{y}^2}{y^2}\right)\right]dy = 0.$$

After substituting $y = x_1$ *and* $\dot{y} = x_1x_2$, *we get* $\left[s^2 + (1-2x_2)s + x_2^2 - 2x_1\right]dy = 0$. *Therefore,*

$$\text{Ore}\det{}_1(sI-A) = s^2 + (1-2x_2)s + x_2^2 - 2x_1. \tag{19}$$

Similarly, $\text{Ore}\det_2(sI-A)$ *can be found from the polynomial description of the output differential equation of the system for the output function* $y = x_2$. *This time we get*

$$y = x_2, \quad \dot{y} = x_1 - x_2, \quad \ddot{y} = x_1x_2 - x_1 + x_2.$$

From the first two equations, we have $x_1 = \dot{y} + y$ *and* $x_2 = y$, *which we substitute into the last equation:*

$$\ddot{y} = \dot{y}y + y^2 - \dot{y}. \tag{20}$$

The polynomial description of the output differential equation (20) *can be found as follows:* $\mathrm{d}\ddot{y} = (y-1)\mathrm{d}\dot{y} + (\dot{y}+2y)\mathrm{d}y$*, which can be rewritten as*

$$(s^2 - (y-1)s - \dot{y} - 2y)\mathrm{d}y = 0.$$

After substituting $y = x_2$ *and* $\dot{y} = x_1 - x_2$*, we get* $\left[s^2 + (1-x_2)s - x_1 - x_2\right]\mathrm{d}y = 0$*. Therefore,*

$$\text{Ore}\ \det{}_2(sI - A) = s^2 + (1-x_2)s - x_1 - x_2. \tag{21}$$

This method of finding the Ore determinants of the matrix $sI - A$, using the state elimination algorithm of [2], is more straightforward and can also be applied for the higher-order systems of the form (1).

4.3. Factorization of the Ore determinant

It is commonly known that factorization into the first-order polynomials is not guaranteed even for ordinary polynomials. However, we will prove that, for the polynomial that results from the Ore determinant of the matrix $sI - A$ associated with nonlinear systems, such factorization exists. The natural question is: what makes the Ore determinant of the matrix $sI - A$ so special compared to the other Ore (skew) polynomials? The main key element is the application of the straightening-out theorem for an autonomous nonlinear system. The Ore determinant of the matrix $sI - A$ associated with the transformed system is equal to s^n, which is obviously factorizable into the first-order polynomials. This determinant in the new coordinates is related to the Ore determinant of $sI - A$ in the original system coordinates. The second key factor results from the observation that the Ore determinant of $sI - A$ can be alternatively computed from a polynomial description of the differential output equation of the system, associated with some output function. The third key element is that the differential output equation can always be transformed into the feedforward form, meaning that its polynomial description can be factorized into the first-order polynomials.

Consider the system (1), where $x = (x_1, x_2)$, and its tangent linear system (16). If $a_{12} = 0$, then the first Ore determinant of $sI - A$ is $(s - a_{22})(s - a_{11})$, that is, it is factorizable into the first-order polynomials. If $a_{12} \neq 0$, the first Ore determinant of $sI - A$ can be found from the polynomial description of the output differential equation of the system for the output function $y = x_1$, as described in subsection 4.2. Let us say this output differential equation is

$$\ddot{y} = F_1(\dot{y}, y) \tag{22}$$

for some $F_1 \in \mathscr{K}$. Similarly, if $a_{21} = 0$, then the second Ore determinant of $sI - A$ is $(s - a_{11})(s - a_{22})$, that is, it is factorizable into the first-order polynomials. If $a_{21} \neq 0$, the second Ore determinant can be found from the polynomial description of the output differential equation of the system for the output function $y = x_2$, as described in subsection 4.2. Let us say this output differential equation is

$$\ddot{y} = F_2(\dot{y}, y) \tag{23}$$

for some $F_2 \in \mathscr{K}$.

To show that the Ore determinants of $sI - A$ are always factorizable into the first-order polynomials, it remains to show that the polynomial descriptions of the output differential equations (22) and (23) are always factorizable into the first-order polynomials. This can be shown in general for an output differential equation

$$\ddot{y} = F(\dot{y}, y), \tag{24}$$

where $F \in \mathscr{K}$. The so-called straightening-out theorem plays a key role here.

Theorem 1 (straightening-out [1,19]). *For a nonlinear system of the form* (1)*, where* f *is analytic, there exist local coordinates* $\xi_1, \ldots, \xi_n$ *such that in the new coordinates one gets*

$$\dot{\xi}_1 = 0, \ \cdots, \quad \dot{\xi}_{n-1} = 0, \quad \dot{\xi}_n = 1$$

in a neighbourhood of a point where f *is non-zero.*

Note that the local properties are a subset of the generic properties. In what follows, we are not interested in the generic properties but in the local ones. Moreover, any meromorphic function is locally analytic on some open subset of $\mathbb{R}^n$. Of course, there may exist different factorizations of Ore determinants around different points because the straightening-out transformation might not be the same for all points. However, this is not an uncommon situation in nonlinear control. This also happens if one applies the Frobenius theorem (in integrating only locally exact one-forms) or in state elimination to find the input-output equation.

An easy observation yields the following conclusion.

Lemma 2. *For the equation* (24), *there exists a choice of states* $z = \varphi(y, \dot{y})$, $\varphi \in \mathscr{K}^2$ *that results in the state equations*

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = 0, \quad y = h(z_1, z_2), \quad h \in \mathscr{K}. \tag{25}$$

Proof. Define $(x_1, x_2) = (y, \dot{y})$ to get $\dot{x}_1 = x_2$, $\dot{x}_2 = F(x_2, x_1)$, $y = x_1$. Then, applying the straightening-out theorem, there exists a change of coordinates $\xi = \phi(x)$, $\phi \in \mathscr{K}^2$ such that

$$\dot{\xi}_1 = 0, \quad \dot{\xi}_2 = 1, \quad y = g(\xi_1, \xi_2) \tag{26}$$

for some $g \in \mathscr{K}$. Finally, the change of coordinates $(z_1, z_2) = (\xi_1 \xi_2, \xi_1)$ transforms the equations (26) into the form (25). □

Note that the equations (25) are in the so-called feedforward form. It was shown in [4] that the polynomial description of the output differential equation of such a system can be factorized over $\mathscr{K}[s]$ into the first-order polynomials.

Theorem 2. *The polynomial* $a(s)$ *in the polynomial description* $a(s)\mathrm{d}y = 0$ *of the nonlinear system* (24) *can always be factorized as*

$$a(s) = (s - \alpha_2)(s - \alpha_1),$$

where α_1, α_2 *are in* $\mathscr{K}$.

Proof. First, observe that for every nonlinear system in the form (24), there exists, by Lemma 2, a change of coordinates that transforms the system equations into the feedforward form (25). Then, for the equations (25) one has

$$\mathrm{d}y = c_1 \mathrm{d}z_1 + c_2 \mathrm{d}z_2,$$

where $c_i = \partial h / \partial z_i$, $i = 1, 2$. Next, eliminate $\mathrm{d}z_1$. Applying the Ore condition from Lemma 1, there exist polynomials $s - \alpha_1$ and γ_1 in $\mathscr{K}[s]$ such that $(s - \alpha_1)c_1 = \gamma_1 s$. Actually, $\alpha_1 = \dot{c}_1 c_1^{-1}$ and $\gamma_1 = c_1$, both of which are elements of $\mathscr{K}$. Therefore,

$$(s - \alpha_1)\mathrm{d}y = \gamma_1 s \mathrm{d}z_1 + (s - \alpha_1)c_2 \mathrm{d}z_2.$$

However, from (25), it follows that $s\mathrm{d}z_1 = \mathrm{d}z_2$ and $s\mathrm{d}z_2 = 0$. Thus, after substitution and rearrangement, one gets

$$(s - \alpha_1)\mathrm{d}y = \varepsilon_2 \mathrm{d}z_2$$

for some $\varepsilon_2 \in \mathscr{K}$. Finally, we eliminate $\mathrm{d}z_2$. Applying the Ore condition, there exist polynomials $s - \alpha_2$ and γ_2 in $\mathscr{K}[s]$ such that $(s - \alpha_2)\varepsilon_2 = \gamma_2 s$. Actually, $\alpha_2 = \dot{\varepsilon}_2 \varepsilon_2^{-1}$ and $\gamma_2 = \varepsilon_2$, both of which are elements of $\mathscr{K}$. Then,

$$(s - \alpha_2)(s - \alpha_1)\mathrm{d}y = \gamma_2 s \mathrm{d}z_2.$$

From (25), it follows that $s\mathrm{d}z_2 = 0$. Hence,

$$(s - \alpha_2)(s - \alpha_1)\mathrm{d}y = 0,$$

which completes the proof. □

Example 7 (continuation of Example 6). *The Ore determinants of the system matrix $sI - A$ in Example 6 are given by* (19) *and* (21) *and can be factorized as*

$$\begin{aligned}\text{Ore}\det_1(sI-A) &= \left(s+\tfrac{x_1}{x_2}-x_2\right)\left(s+1-x_2-\tfrac{x_1}{x_2}\right),\\ \text{Ore}\det_2(sI-A) &= \left(s+\tfrac{x_2^2}{x_1-x_2}\right)\left(s+1-x_2-\tfrac{x_2^2}{x_1-x_2}\right).\end{aligned}$$

Example 8. *Consider the system*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1.$$

The Ore determinants of the matrix $sI - A$, defined by the polynomial description of the system, are

$$\text{Ore}\det_1(sI-A) = \text{Ore}\det_2(sI-A) = s^2+1,$$

which is an irreducible polynomial over $\mathbb{R}[s]$. However, it is reducible over $\mathscr{K}[s]$ as

$$s^2+1 = \left(s-\frac{x_1}{x_2}\right)\left(s+\frac{x_1}{x_2}\right).$$

4.4. Computation of eigenvalues from the factorization of the Ore determinant

The definition of nonlinear eigenvalues (and eigenvectors) is not suitable for their calculation because the definition results in a system of nonlinear partial differential equations, depending on the eigenvalue as an unknown variable. Once the eigenvalues are known, the eigenvectors are much easier to find. However, the computation methods to find nonlinear eigenvalues are still missing. Researchers who have used nonlinear eigenvalues in their studies have found them either by the trial-and-error method or from other heuristic considerations. In this paper, we suggest to find nonlinear eigenvalues from the factorization of the Ore determinant of the non-commutative polynomial matrix $sI - A$, describing the system. Since we have not succeeded in justifying our computational method formally, we present it below as a conjecture. Although there is no formal proof, numerous examples demonstrate that the results satisfy the definition of eigenvalues. So far, we have not come across any counterexamples.

Conjecture 1. *Assume that the Ore determinant of the matrix $sI - A$ can be factorized as*

$$\text{Ore}\det(sI-A) = (s-\lambda_1-\dot{c}_1c_1^{-1})\cdots(s-\lambda_n-\dot{c}_nc_n^{-1}) \tag{27}$$

for some $\lambda_i \in \mathscr{K}$ and non-zero $c_i \in \mathscr{K}$. Then $\lambda_i \in \mathscr{K}$, $i = 1,\ldots,n$, are eigenvalues of the system (1). *Note that if $0 \neq c_i \in \mathbb{R}$, then* (27) *becomes* $\text{Ore}\det(sI-A) = (s-\lambda_1)\cdots(s-\lambda_n)$.

Our idea is based on two aspects. First, in the case of linear systems, the method reduces to the well-known result, that is, finding the roots of the ordinary determinant of the matrix of real numbers. It is, therefore, natural to expect a certain analogy in the case of nonlinear systems. Second, since from the definition of the eigenvalue λ and the eigenvector e,

$$e\lambda + \dot{e} = Ae,$$

then the left hand side of this equation looks a lot like the commutation rule in the skew polynomial ring $\mathscr{K}[s]$:

$$es + \dot{e} = se.$$

We do not suggest to say that one can replace λ by s, but only point out this similarity. The similarity suggests that there should be some relation between the factorization of the Ore determinant of the matrix $sI - A$ and the eigenvalues λ of the system (1).

Example 9 (continuation of Example 1). *The Ore determinants of the matrix* $sI - A$ *were computed in Example 4 as* (12) *and* (13) *for the system* (4). *Thus, following our conjecture, the relations* (12) *and* (13) *suggest that* $1 + 3x_2$, $2x_2$ *and* $1 + x_2$ *are the eigenvalues of the system* (4). *Indeed,* $1 + x_2$ *and* $2x_2$ *are listed in Table 1 as eigenvalues of the system* (4). *Also, one can check by direct computation that* $1 + 3x_2$ *is an eigenvalue corresponding to an eigenvector* $(1/x_2^2, 0)^T$.

Moreover, one can rewrite the Ore determinants as

$$\begin{aligned} \text{Ore det}_1(sI - A) &= (s-1-3x_2)(s-1-x_2) = (s-1-3x_2-\dot{c}_1c_1^{-1})(s-1-\dot{c}_2c_2^{-1}) \text{ and} \\ \text{Ore det}_2(sI - A) &= (s-1-x_2)(s-2x_2) = (s-1-x_2-\dot{\bar{c}}_1\bar{c}_1^{-1})(s-\dot{\bar{c}}_2\bar{c}_2^{-1}), \end{aligned}$$

where $c_1 = 1$, $c_2 = x_2$, $\bar{c}_1 = 1$, *and* $\bar{c}_2 = x_2^2$. *Thus, Conjecture 1 suggests that* 1 *and* 0 *are also eigenvalues of the system* (4). *Indeed, one can see that* 0 *and* 1 *are eigenvalues of the system* (4) *listed in Table 1.*

5. CONCLUSIONS

This paper studies the difficult but neglected problem of computing the eigenvalues of a nonlinear system. It was suggested that these eigenvalues can be found from the factorization of the Ore determinant of the matrix $sI - A$, defined by the polynomial description of the nonlinear system. The paper presents only preliminary ideas on computing the eigenvalues. The proof of the claim is left for future research. An alternative method for computing the Ore determinants of the matrix $sI - A$ was also suggested. In particular, one can find the Ore determinant of $sI - A$ by computing the polynomial description of the respective output function (e.g. $y = x_1$) of the nonlinear control system. Finally, it was shown that the Ore determinant of $sI - A$ can always be factorized into the first-order polynomials over the skew polynomial ring $\mathcal{K}[s]$. Another open problem is to show that all Ore determinants actually belong to the same equivalence class and that a representative of this equivalence class is a polynomial that describes the tangent linear system (3) from any (fully) observable output function $y = h(x)$. This would significantly simplify computation of the Ore determinants compared to finding them from the definition. Yet another problem worth studying is to investigate whether the Dieudonne determinant can be of some help in finding the nonlinear eigenvalues.

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Mittelineaarsete süsteemide omaväärtuste arvutamine Ore determinandi abil: esialgsed tulemused

Miroslav Halás, Arvo Kaldmäe, Ülle Kotta ja Juraj Slačka

Hiljuti üldistati süsteemi omaväärtuste mõiste mittelineaarsetele süsteemidele, kuid puuduvad meetodid selliste omaväärtuste arvutamiseks. Erinevalt lineaarsetest süsteemidest on mittelineaarsel juhul vaja rakendada mittekommutatiivseid polünoome Ore ringist. Artiklis pakutakse välja idee omaväärtuste arvutamiseks süsteemi kirjeldava polünoommaatriksi Ore determinandi abil. Erinevalt lineaarsetest süsteemidest kuuluvad polünoommaatriksi elemendid Ore polünoomide ringi. Artiklis näidatakse, kuidas arvutada süsteemi kirjeldava polünoommaatriksi Ore determinanti ning demonstreeritakse, et süsteemi omaväärtused saab leida vastava Ore determinandi teguriteks lahutamisel esimest järku polünoomide korrutiseks. Lisaks tõestatakse, et selline teguriteks lahutus on alati võimalik. Erinevaid mõisteid ja arvutusi illustreerivad näited.