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TOPOLOGICAL ALGEBRAS

On the topological product of galbed algebras

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Abstract. We generalize some results about the topological product and the topological projective limit of galbed algebras. We also prove that the topological product is an (α_n)-galbed algebra if and only if its each factor is an (α_n)-galbed algebra.

Keywords: galbed algebra, (α_n)-galbed algebra, topological product, topological projective limit.

1. INTRODUCTION

Throughout this paper, the set $\mathbb{N}$ of natural numbers includes also 0, all topological spaces are Hausdorff spaces, and $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. All topological algebras will be associative algebras over the field $\mathbb{K}$ (with separately continuous multiplication).

Let $s_{\mathbb{K}}$ denote the set of all $\mathbb{K}$-valued sequences (α_n), i.e. $(\alpha_n) \in s_{\mathbb{K}}$ if and only if $\alpha_n \in \mathbb{K}$ for each $n \in \mathbb{N}$.

As usual, the set l^0 will consist of all such sequences $(\alpha_n) \in s_{\mathbb{K}}$ for which the set $\{n \in \mathbb{N} : \alpha_n \neq 0\}$ is finite. Similarly, the set l^1 will consist of all such sequences $(\alpha_n) \in s_{\mathbb{K}}$ for which

$$\sum_{\nu=0}^{\infty} |\alpha_\nu| < \infty.$$

Set $l = l^1 \setminus l^0$.

Let us recall that *an* (α_n)*-galbed algebra* for $(\alpha_n) \in l$ is a topological algebra (A, τ_A) for which, for each neighbourhood O of zero in (A, τ_A), there exists a neighbourhood U of zero in (A, τ_A) such that

$$\left\{ \sum_{k=0}^{n} \alpha_k a_k : n \in \mathbb{N}, a_0, \ldots, a_n \in U \right\} \subseteq O.$$

A topological algebra (A, τ_A) is *a galbed algebra* if there exists $(\alpha_n) \in l$ such that (A, τ_A) is an (α_n)-galbed algebra.

One can read a short history of galbed algebras in [2], p. 14.

The main aim of this paper is to generalize Proposition 1 from [2], p. 16, about the topological product of galbed algebras by substituting the condition of the finiteness of the number of certain sequences $(\alpha_n) \in l$ by a weaker condition, where finiteness is no more needed. After doing that, we will also generalize Proposition 3 from [2], p. 23, about the projective limit of galbed algebras, using the same technique as in the generalization of Proposition 1 of [2].

In [2], p. 15, the following result (called Lemma 1 in [2]) was proved.

Lemma 1. *Let (A,τ_A) be an (α_n)-galbed algebra and (B,τ_B), where $\tau_B=\tau_A|_B$, a subalgebra of (A,τ_A). Then (B,τ_B) is also an (α_n)-galbed algebra.*

Next, let us recall a result (rewording of Lemma 7 from [1], p. 14), which says the following.

Lemma 2. *Let (A,τ_A) be an (α_n)-galbed topological algebra. Then there exists a sequence $(\beta_n)\in l$ such that $\beta_k\in\mathbb{R}$, $\beta_k>0$ for each $k\in\mathbb{N}$, and (A,τ_A) is a (β_n)-galbed algebra.*

In the proof of this lemma, the sequence (β_n) is constructed as follows.

Let (A,τ_A) be an (α_n)-galbed algebra. Define $i_0:=\min\{k:\alpha_k\neq 0\}$, $i_j:=\min\{k:k>i_{j-1},\alpha_k\neq 0\}$ for each $j\in\mathbb{N}\setminus\{0\}$, and set $\beta_k:=|\alpha_{i_k}|$ for each $k\in\mathbb{N}$. Then (β_n) satisfies the conditions of the lemma, and (A,τ_A) is a (β_n)-galbed algebra.

Definition 1. *Let $(\alpha_n)\in l$. The sequence (β_n), defined by $\beta_k:=|\alpha_{i_k}|$ for each $k\in\mathbb{N}$, where $i_0:=\min\{k:\alpha_k\neq 0\}$ and $i_j:=\min\{k:k>i_{j-1},\alpha_k\neq 0\}$ for each $j\in\mathbb{N}\setminus\{0\}$, is called the* **skeleton sequence of the sequence** (α_n).

Notice that if (β_n) is the skeleton sequence of (α_n), then

$$\sum_{n=0}^{\infty}\beta_n=\sum_{n=0}^{\infty}|\alpha_n|.$$

The construction of the skeleton sequence for any sequence $(\alpha_n)\in l$ will be a very important tool for proving our main result of this paper.

Let I be any set on indices and $\{(A_i,\tau_i):i\in I\}$ a family of topological algebras. Denote by (A,τ) the topological product of the family $\{(A_i,\tau_i):i\in I\}$. Then A is just the algebraic direct product of algebras $\{A_i:i\in I\}$, and the topology τ is the product topology. Recall that the base of the product topology consists of all sets of the form $\prod_{i\in I}U_i$, where U_i is from the basis of the topology τ_i for each $i\in I$. It means that every neighbourhood O of zero in τ contains a neighbourhood of zero in the form $\prod_{i\in I}O_i$, where O_i is a neighbourhood of zero in τ_i for each $i\in I$.

Next, recall that *a pre-ordered set* is a pair $(I,\leqslant)$, where I is any set and $\leqslant$ is a pre-order, i.e. a reflexive and transitive binary relation, on I.

Let $\{(A_i,\tau_i):i\in I\}$ be a collection of topological algebras indexed by a pre-ordered set $(I,\leqslant)$. The pair (B,τ_B) is the topological (projective) limit[1] of the collection $\{(A_i,\tau_i):i\in I\}$ of topological algebras if there is

1) a family $(\phi_{ij}:A_j\to A_i)_{i,j\in I,i\leqslant j}$ of continuous algebra homomorphisms

and

2) a family $(\pi_i:B\to A_i)_{i\in I}$ of algebra homomorphisms (called projections)

such that

a) the map $\phi_{ii}:A_i\to A_i$ is the identity map for each $i\in I$;

b) for all $i,j,k\in I$, where $i\leqslant j\leqslant k$, holds $\phi_{ij}\circ\phi_{jk}=\phi_{ik}$;

c) for all $i,j\in I$, where $i\leqslant j$, holds $\phi_{ij}\circ\pi_j=\pi_i$;

d) for[2] any topological algebra (C,τ_C) and any family $(\rho_i:C\to A_i)_{i\in I}$ of continuous algebra homomorphisms with $\phi_{ij}\circ\rho_j=\rho_i$ for all $i,j\in I$ with $i\leqslant j$ there exists a unique continuous algebra homomorphism $\rho:C\to B$ such that $\rho_i=\pi_i\circ\rho$ for all $i\in I$.

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1 Several authors demand for a topological projective limit that the pre-order $\leqslant$ on I should be a direct order, i.e. for any $i,j\in I$ there should exist $k\in I$ such that $i\leqslant k$ and $j\leqslant k$. In this paper, we do not need that restriction.

2 This condition is usually omitted in the literature on topological algebras, but it should be included based on the category-theoretical principles. Everything in this paper will remain true also in the case when we do not consider this condition as a part of the definition of a topological (projective) limit.

The topology τ_B is the projective limit topology, i.e. the initial topology with respect to the projections $(\pi_i)_{i\in I}$. In other words, τ_B is the smallest topology that makes all projections continuous.

The topological projective limit of the collection $(A_i,\tau_i)_{i\in I}$ of topological algebras is often denoted by $(B,\tau_B) = (\varprojlim A_i, \tau_{\varprojlim A_i})$.

It is known that the (projective) limit of a collection of topological algebras is a subalgebra of the direct product of the same collection of topological algebras.

2. RESULTS

Now we can generalize Proposition 1 from [2], p. 16.

Proposition 1. *Let I be any set of indices, $\{(A_i,\tau_i) : i \in I\}$ a collection of topological algebras, (A,τ_A) the topological product of topological algebras $\{(A_i,\tau_i) : i \in I\}$ and $\{\alpha(i) = (\alpha(i)_n) : i \in I\}$ a collection of sequences from l such that, for each $i \in I$, the algebra (A_i,τ_i) is an $(\alpha(i)_n)$-galbed algebra. For each $i \in I$, let $\beta(i) = (\beta(i)_n)$ be the skeleton sequence of the sequence $\alpha(i)$. If $\inf\{\beta(i)_k : i \in I\} > 0$ for each $k \in \mathbb{N}$, then*

1) *the sequence (γ_n), where $\gamma_n = \inf\{\beta(i)_n : i \in I\}$ for each $n \in \mathbb{N}$, belongs to l;*

2) *(A,τ_A) is a (γ_n)-galbed algebra.*

Proof. 1) Since all elements of $(\beta(i)_n)$ are positive for each $i \in I$, then also all the elements of (γ_n) are positive, which means that $(\gamma_n) \notin l^0$. Fix any $j \in I$. By the definition, $(\alpha(j)_n) \in l$, which means that

$$\sum_{n=0}^{\infty} |\alpha(j)_n| < \infty.$$

Notice that

$$\sum_{n=0}^{\infty} |\gamma_n| = \sum_{n=0}^{\infty} \gamma_n \leqslant \sum_{n=0}^{\infty} \beta(j)_n = \sum_{n=0}^{\infty} |\alpha(j)_n| < \infty.$$

Hence, $(\gamma_n) \in l$.

2) For each $j \in I$, let $\mathscr{B}_j$ be the base of neighbourhoods of zero in A_j for τ_j, and $p_j : A \to A_j$ be the projection, sending an element $b = (a_i)_{i\in I} \in A$ to $a_j \in A_j$.

Let O be an arbitrary neighbourhood of zero in (A,τ_A). From the definition of the base of the product topology, it is known that there exist $l \in \mathbb{N}$, $i_1,\dots,i_l \in I$, neighbourhoods $O_{i_1} \in \mathscr{B}_{i_1},\dots,O_{i_l} \in \mathscr{B}_{i_l}$ and a neighbourhood

$$O' = \bigcap_{u=1}^{l} p_{i_u}^{-1}(O_{i_u})$$

from the base of neighbourhoods of zero in τ_A such that $O' \subseteq O$.

Remember that each (A_{i_u},τ_{i_u}) is an $(\alpha(i_u)_n)$-galbed algebra. This means that there exist balanced neighbourhoods U_{i_1} in (A_{i_1},τ_{i_1}),..., U_{i_l} in (A_{i_l},τ_{i_l}) of zero such that

$$\sum_{v=0}^{q} \alpha(i_u)_v a(i_u)_v \in O_{i_u}$$

for each $q \in \mathbb{N}$, each $u \in \{1,\dots,l\}$ and all $a(i_u)_0,\dots,a(i_u)_q \in U_{i_u}$.

Let

$$U = \bigcap_{u=1}^{l} p_{i_u}^{-1}(U_{i_u}),$$

$q \in \mathbb{N}$ and $b_0, \ldots, b_q \in U$ be arbitrary elements. Then U is a neighbourhood of zero in A in the product topology, $p_{i_u}(b_v) \in U_{i_u}$ and

$$0 < \left| \frac{\gamma_v}{\alpha(i_u)_v} \right| \leqslant 1$$

for all $u \in \{1, \ldots, l\}$, $v \in \{0, \ldots, q\}$ and $q \in \mathbb{N}$.

As the neighbourhoods $U_{i_1}, \ldots, U_{i_l}$ are balanced, we obtain for all $u \in \{1, \ldots, l\}$ and $n \in \mathbb{N}$ that

$$\frac{\gamma_n}{\alpha(i_u)_n} p_{i_u}(b_n) \in U_{i_u}.$$

Since

$$p_{i_u}\Big(\sum_{v=0}^{q} \gamma_v b_v\Big) = \sum_{v=0}^{q} \gamma_v p_{i_u}(b_v) = \sum_{v=0}^{q} \alpha(i_u)_v \Big(\frac{\gamma_v}{\alpha(i_u)_v} p_{i_u}(b_v)\Big) \in O_{i_u}$$

for each $u \in \{1, \ldots, l\}$, then

$$\sum_{v=0}^{q} \gamma_v b_v \in \bigcap_{k=1}^{l} p_{i_u}^{-1}(O_{i_u}) \subset O$$

for all $b_0, \ldots, b_q \in U$ and each $q \in \mathbb{N}$. Hence, (A, τ_A) is a (γ_n)-galbed algebra in the product topology. □

Using this result, we can generalize also Proposition 3 from [1], p. 23, as follows.

Proposition 2. *Let $(I, \leqslant)$ be a pre-ordered set of indices, $\{(A_i, \tau_i) : i \in I\}$ a collection of topological algebras, (B, τ_B) the topological projective limit of the collection $\{(A_i, \tau_i) : i \in I\}$ and $\{\alpha(i) = (\alpha(i)_n) : i \in I\}$ a collection of sequences from l such that, for each $i \in I$, the algebra (A_i, τ_i) is an $(\alpha(i)_n)$-galbed algebra. For each $i \in I$, let $\beta(i) = (\beta(i)_n)$ be the skeleton sequence of the sequence $\alpha(i)$. If $\inf\{\beta(i)_k : i \in I\} > 0$ for each $k \in \mathbb{N}$, then (B, τ_B) is a (γ_n)-galbed algebra, where $\gamma_n = \inf\{\beta(i)_n : i \in I\}$ for each $n \in \mathbb{N}$.*

Proof. By Proposition 1, the topological product (A, τ_A) of the collection $\{(A_i, \tau_i) : i \in I\}$ is a (γ_n)-galbed algebra. Projective limit of this collection is a topological subalgebra of the topological product of the same collection. Hence, by Lemma 1, (B, τ_B) is a (γ_n)-galbed algebra. □

We finish this paper with another result about the topological product of topological algebras.

Theorem 1. *Let I be any set on indices, $\{(A_i, \tau_i) : i \in I\}$ a collection of topological algebras, (A, τ_A) the topological product of topological algebras $\{(A_i, \tau_i) : i \in I\}$ and $(\alpha_n) \in l$. Then (A, τ_A) is an (α_n)-galbed algebra if and only if (A_i, τ_i) is an (α_n)-galbed algebra for each $i \in I$.*

Proof. Suppose that (A, τ) is an (α_n)-galbed algebra and take an arbitrary $i_0 \in I$. Let O_{i_0} be any neighbourhood of zero in (A_{i_0}, τ_{i_0}). Then $O = \prod_{i \in I} U_i$, where

$$U_i = \begin{cases} O_{i_0}, \text{if } i = i_0, \\ A_i, \text{otherwise} \end{cases}$$

is a neighbourhood of zero in (A, τ).

Hence, there exists a neighbourhood U of zero in (A, τ) such that

$$\Big\{\sum_{k=0}^{n} \alpha_k b_k : n \in \mathbb{N}, b_0, \ldots, b_n \in U\Big\} \subseteq O.$$

Now, there exist neighbourhoods $\{U_i : i \in I\}$ of zero such that U_i is a neighbourhood of zero in (A_i, τ_i) for each $i \in I$ and $\prod_{i\in I} U_i \subseteq U$. As the scalar multiplication and addition are performed 'coordinate-wise' in the topological product, we obtain that the condition

$$\left\{\sum_{k=0}^{n} \alpha_k a_k : n \in \mathbb{N}, a_0, \ldots, a_n \in U_{i_0}\right\} \subseteq O_{i_0}$$

must hold. But this means that (A_{i_0}, τ_{i_0}) is an (α_n)-galbed algebra. As i_0 was chosen arbitrarily from I, we see that, for each $i \in I$, (A_i, τ_i) is an (α_n)-galbed algebra.

Suppose that, for each $i \in I$, the algebra (A_i, τ_i) is an (α_n)-galbed algebra. Then, by Corollary 2 from [2], p. 19, (A, τ) is an (α_n)-galbed algebra.

Therefore, (A, τ_A) is an (α_n)-galbed algebra if and only if (A_i, τ_i) is an (α_n)-galbed algebra for each $i \in I$. □

3. CONCLUSION

In the present paper, we found some sufficient conditions on a collection of galbed algebras such that their topological product and topological projective limit would also be galbed algebras. We additionally showed that the topological product of a collection of topological algebras is an (α_n)-galbed algebra if and only if each member of this collection is an (α_n)-galbed algebra.

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Gälbalgebrate topoloogilisest korrutisalgebrast

Mart Abel

Artiklis leitakse piisavad tingimused selleks, et gälbalgebrate pere topoloogiline korrutisalgebra ja topoloogiline projektiivne piir oleksid samuti gälbalgebrad. Samuti näidatakse, et topoloogiliste algebrate pere topoloogiline korrutisalgebra on gälbalgebra siis ja ainult siis kui iga pere liige on samuti gälbalgebra.