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GROUP AND
GRAPH THEORY

Finite groups whose coprime graph is split, threshold, chordal, or a cograph

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Abstract. Given a finite group G, the coprime graph of G, denoted by $\Gamma(G)$, is defined as an undirected graph with the vertex set G, and for distinct $x, y \in G$, x is adjacent to y if and only if $(o(x), o(y)) = 1$, where $o(x)$ and $o(y)$ are the orders of x and y, respectively. This paper classifies the finite groups with split, threshold and chordal coprime graphs, as well as gives a characterization of the finite groups whose coprime graph is a cograph. As some applications, the paper classifies the finite groups G such that $\Gamma(G)$ is a cograph if G is a nilpotent group, a dihedral group, a generalized quaternion group, a symmetric group, an alternating group, or a sporadic simple group.

Keywords: coprime graphs, split graphs, cographs, threshold graphs, chordal graphs, finite group.

1. INTRODUCTION

In algebraic graph theory, a popular and interesting research topic is graph representations of an algebraic structure. For example, for the algebraic structure 'group', Cayley graphs defined on a group are very famous and have a long history. On the other hand, graphs from algebraic structures have been actively investigated in the literature since they have valuable applications. For example, Cayley graphs can be used as classifiers for data mining (see [12]). Moreover, graph algebras and automata are closely related (see [11]).

Given a finite group G, one can define a variety of graphs on G by some properties of G, for example, the power graph of a group [1] and the commuting graph of a group [3]. Distance Laplacian spectra of power graphs were studied in [17]. Notice that in group theory, the order of an element is one of the most basic and important concepts. In [13], the authors introduced the concept of a coprime graph. Let G be a finite group. The *coprime graph* of G, denoted by $\Gamma(G)$, is an undirected graph with the vertex set G; two distinct vertices x, y are adjacent in $\Gamma(G)$ if and only if $o(x)$ and $o(y)$ are relatively prime, namely $(o(x), o(y)) = 1$, where $o(x)$ and $o(y)$ are the orders of x and y, respectively. Note that the identity element is always adjacent to any other vertex in $\Gamma(G)$. Ma et al. [13] then studied some relationships between coprime graphs and groups. Dorbidi [7] showed that the clique number of $\Gamma(G)$ is always equal to the chromatic number of $\Gamma(G)$ for any

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finite group G, and classified the finite groups whose coprime graph is a complete r-partite graph or a planar graph. Selvakumar and Subajini [16] classified the finite groups whose coprime graph has (non)orientable genus one. Hamm and Way [8] studied the coprime graph of a dihedral group and gave the exact value of the independence number of $\Gamma(G)$, where G is dihedral. Moreover, they also studied perfect coprime graphs. Alraqad et al. [2] obtained the finite groups whose coprime graph has precisely three end-vertices. Zahidah et al. [18] investigated some connectivity indices of $\Gamma(G)$, where G is a generalized quaternion group.

In this paper, any graph means a simple graph, which is an undirected graph without loops and multiple edges. Assume that Γ and Δ are two graphs. If Γ has no induced subgraphs isomorphic to Δ, then Γ is said to be a Δ-*free* graph. This is equivalent to saying that Δ is a forbidden subgraph of Γ. In the literature, some important graphs can be defined by using graph structures or forbidden subgraphs. For example, by using graph structures for the famous *split graph*, it can be defined as the vertex set partitioned into the disjoint union of an independent set and a clique. On the other hand, by using forbidden subgraphs, the split graph can be defined as one that has no induced subgraphs isomorphic to C_4, C_5 and $2K_2$, where C_n denotes the cycle of length n and $2K_2$ denotes two independent edges.

If a graph has no induced subgraphs isomorphic to the four vertices path P_4, then the graph is called a *cograph*. If a graph has no induced subgraphs isomorphic to P_4, C_4 and $2K_2$, then the graph is called a *threshold graph*. A graph is said to be a *chordal graph* provided that it contains no induced cycles of length greater than 3. Namely, in a chordal graph, every cycle of length 4 or more has a chord. Note that if a graph is C_4-free and P_4-free, then it is chordal. Moreover, it is clear that any threshold graph is also a cograph. Threshold graphs have some applications in computer science (see [9]).

Recently, Cameron [4] surveyed various graphs from a group G, where the vertex set is G and edges of these graphs reflect the structure of G in various ways, such as the power graph, enhanced power graph, commuting graph, and others. In this paper, Cameron proposed the question: for which groups is some graph of a group a perfect graph, a cograph, a split graph, or a threshold graph (see [4, Question 14])? This question regarding the power graph and enhanced power graph has been studied by Manna et al. [15] and Ma et al. [14], respectively. Motivated by Cameron's question, we try to solve the following question in this paper.

Question 1.1. *For which finite groups is the coprime graph a cograph, a split graph, or a threshold graph?*

In 2021, Hamm and Way [8] gave a characterization of the finite groups whose coprime graph is perfect, and classified the finite abelian groups with perfect coprime graphs. In this paper, we classify all finite groups with split, threshold, and chordal coprime graphs. We also give a characterization of the finite groups whose coprime graph is a cograph. As applications, we find the finite groups G for which $\Gamma(G)$ is a cograph if G is a nilpotent group, a dihedral group, a generalized quaternion group, a symmetric group, an alternating group, or a sporadic simple group.

2. PRELIMINARIES

This section will introduce some definitions and notations in group theory and graph theory.

Every group considered in our paper is finite. For convenience, we always use G to denote a finite group with the identity e. Denote by $\pi_e(G)$ and $\pi(G)$ the set of orders of elements of G and the set of prime divisors of $|G|$, respectively. For some element $g \in G$, the *order* of g, denoted by $o(g)$, is the size of the set of all elements belonging to the cyclic subgroup $\langle g\rangle$, which is generated by the element g. Particularly, an element is called an *involution* if its order is 2. As usual, denote by $\mathbb{Z}_n$ the cyclic group of order n. We note that a finite group is a *nilpotent* group if and only if the finite group is the direct product of its Sylow subgroups.

Given a graph, say Γ, denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of Γ, respectively. We use P_n to denote a path with n vertices. Furthermore, denote by $2K_2$ a matching with four vertices, namely, two independent edges. If $\{x,y\} \in E(\Gamma)$, then we denote this by $x-y$. In a graph, we use $x_1 - x_2 - \cdots - x_n$ to denote a path isomorphic to P_n.

The following observation can be obtained directly from the definition of the coprime graph of a group.

Observation 2.1. *If H is a subgroup of G, then $\Gamma(H)$ is an induced subgraph of $\Gamma(G)$.*

From Observation 2.1, we get the following result.

Observation 2.2. *$\Gamma(G)$ is a split graph (resp. a threshold graph, a chordal graph, or a cograph) if and only if for any subgroup H of G, $\Gamma(H)$ is a split graph (resp. a threshold graph, a chordal graph, or a cograph).*

In this paper, Observations 2.1 and 2.2 will be used frequently and at times without explicit reference to them.

3. C_4-FREE COPRIME GRAPHS

This section classifies the finite groups whose coprime graph is C_4-free. Our main theorem is the following.

Theorem 3.1. *$\Gamma(G)$ is C_4-free if and only if G is isomorphic to either a p-group or $\mathbb{Z}_2 \times Q$, where p is a prime and Q is a q-group for some odd prime q.*

We first give two results before giving the proof of Theorem 3.1.

Lemma 3.2. ([14, Proposition 2.6]) *Let G be a p-group of order n. Then $\Gamma(G) \cong K_{1,n-1}$. In particular, $\Gamma(G)$ is C_4-free.*

Lemma 3.3. *Let G be a group with $\pi(G) = \{2,q\}$, where q is an odd prime. Then $\Gamma(G)$ is C_4-free if and only if $G \cong \mathbb{Z}_2 \times Q$, where Q is a q-group.*

Proof. We first prove the sufficiency of our result. Suppose that $G \cong \mathbb{Z}_2 \times Q$ with a q-group Q. Note that G has a unique involution, say u, and so u must belong to the center of G. It follows that q is a prime divisor of the order of any element in $G \setminus \{e,u\}$, and so $G \setminus \{e,u\}$ induces an empty graph in $\Gamma(G)$. Moreover, since e is adjacent to every other vertex in $\Gamma(G)$, we have that $\Gamma(G)$ is C_4-free, as desired.

We next prove the necessity of this lemma. Suppose that $\Gamma(G)$ is C_4-free. Let $a \in G$ with $o(a) = q$. If G has two distinct involutions u,v, then $u - a - v - a^{-1} - u$ is an induced cycle of length 4, a contradiction. It follows that G has a unique involution belonging to the center of G. If G has an element w of order 4, then $w - a - w^{-1} - a^{-1} - w$ is an induced cycle isomorphic to C_4, which is also a contradiction. Thus, G has a unique Sylow 2-subgroup isomorphic to $\mathbb{Z}_2$. Now, let Q be a Sylow q-subgroup. Then we have that Q has index 2 in G, which implies that Q is normal in G. Also, since $\pi(G) = \{2,q\}$, it follows that $G \cong \mathbb{Z}_2 \times Q$, as desired. □

We are now ready to prove Theorem 3.1.

Proof. The sufficiency follows trivially from Lemmas 3.2 and 3.3. For the converse, suppose that $\Gamma(G)$ is C_4-free. Assume, to the contrary, that G has two elements a,b such that $o(a) = q_1$ and $o(b) = q_2$, where q_1 and q_2 are two distinct odd primes. Then $a - b - a^{-1} - b^{-1} - a$ is an induced cycle isomorphic to C_4, which is impossible. We conclude that $\pi(G) \subseteq \{2,q\}$, where q is an odd prime. If $\pi(G) = \{2,q\}$, then Lemma 3.3 completes our proof. Otherwise, G is a p-group for some prime p, and the required result follows from Lemma 3.2. □

4. GROUPS WHOSE COPRIME GRAPH IS A COGRAPH

The family of cographs is the smallest class of graphs that possesses the 1-vertex graphs and is closed under two operations: complementation and disjoint union. Furthermore, we know that every cograph is perfect.

In the following, we first define a family of finite groups. A finite group G is called *an* Ω*-group* provided that G satisfies the following two conditions:

(i) $o(x) = p^m q^n$ for each $x \in G$, where p and q are two different primes, and m and n are two non-negative integers;

(ii) there are no pairwise distinct primes p,q,r such that $\{pq, pr\} \subseteq \pi_e(G)$.

We get the following result from the definition of an Ω-group.

Observation 4.1. *Any subgroup of an* Ω*-group is also an* Ω*-group.*

The following result is our main theorem of this section and characterizes the finite groups whose coprime graph is a cograph.

Theorem 4.2. $\Gamma(G)$ *is* P_4*-free if and only if* G *is an* Ω*-group, which in turn is true if and only if* $\Gamma(G)$ *is a cograph.*

Proof. We first show that the sufficiency is valid. Suppose that G is an Ω-group. Assume, by way of contradiction, that $\Gamma(G)$ has an induced path P_4, say $a-b-c-d$. Note that $e \notin \{a,b,c,d\}$. Now, considering (i) and the order of b, we have the following two cases:

Case 1. $\pi(\langle b\rangle) = \{p,q\}$, where p,q are distinct primes.

Note that $(o(b), o(d)) \neq 1$. If $\pi(\langle d\rangle) = \{p,q\}$, then, since $(o(a), o(b)) = 1$, it must be $(o(a), o(d)) = 1$, which is impossible. It follows that only one of p and q must belong to $\pi(\langle d\rangle)$. Without loss of generality, we may assume $p \in \pi(\langle d\rangle)$. Now, by ($ii$), we must have that $o(d) = p^m$ with $m \geq 1$, and so $p \mid o(a)$ as $(o(a), o(d)) \neq 1$. As a result, we have $p \mid (o(a), o(b))$, which implies that a and b are non-adjacent, a contradiction.

Case 2. $\pi(\langle b\rangle) = \{p\}$, where p is a prime.

From Case 1 and (i), we see that $o(c) = q^m$ with $m \geq 1$, where q is a prime, which is different from p. Since $a \in N(b)$, $a \notin N(d)$, and $(o(b), o(d)) \neq 1$, it follows that $\pi(\langle d\rangle) = \{p,r\}$, where r is a prime with $r \notin \{p,q\}$. As a consequence, it must have $\pi(\langle a\rangle) = \{q,r\}$. Namely, there exist distinct primes p,q,r such that $\{pr, qr\} \subseteq \pi_e(G)$, contrary to ($ii$).

We next prove the necessity of this theorem. Suppose that $\Gamma(G)$ is P_4-free. If G has an element x of order pqr with pairwise distinct primes p,q,r, then it is easy to verify that $x^p - x^{qr} - x^{pr} - x^q$ is an induced path isomorphic to P_4, a contradiction. It follows that (i) holds. In order to prove that G is an Ω-group, it suffices to show that (ii) is valid. Assume, for the sake of contradiction, that there exist pairwise distinct primes p,q,r such that $\{pq, ps\} \subseteq \pi_e(G)$. Let $u, v \in G$ with $o(u) = pq$ and $o(v) = ps$. Then one can obtain easily that $u - v^p - u^p - v$ is an induced path isomorphic to P_4, a contradiction. Thus, (ii) holds. We conclude that G is an Ω-group. □

Note that if G is a group with $|\pi(G)| \leq 2$, then G must be an Ω-group. Thus, we have the following result:

Corollary 4.3. *Let* G *be a group with* $|\pi(G)| \leq 2$*. Then* $\Gamma(G)$ *is a cograph.*

As we know, a finite group is *nilpotent* if and only if this group is the direct product of its Sylow subgroups. Particularly, in a finite nilpotent group, an element of order p and an element of order q can commute, where p,q are distinct primes. In the following, as a corollary of Theorem 4.2, we will characterize all finite nilpotent groups whose coprime graph is a cograph.

Corollary 4.4. *Let G be a nilpotent group. Then $\Gamma(G)$ is a cograph if and only if G is either a p-group or isomorphic to $P \times Q$, where P and Q are a p-group and a q-group, respectively.*

Given two non-trivial groups H and K, for which direct product $H \times K$ is the coprime graph a cograph? Next, we will characterize the direct products $H \times K$ whose coprime graph is a cograph.

Corollary 4.5. *Let H and K be two non-trivial groups. Then $\Gamma(H \times K)$ is a cograph if and only if $\pi(H) \subseteq \{p,q\}$ and $\pi(K) \subseteq \{p,q\}$, where p,q are distinct primes.*

Proof. The sufficiency follows trivially from Corollary 4.3. For the necessity, suppose that $\Gamma(H \times K)$ is a cograph. Suppose for a contradiction that $\pi(H) = \{p,q,r\}$, where p,q,r are pairwise distinct primes. Take an element of prime order, say s. Without loss of generality, we may assume that $s \notin \{p,q\}$. Then we deduce that $H \times K$ has elements of order ps and qs. By (*ii*), we see that $H \times K$ is not an Ω-group, and so $\Gamma(H \times K)$ is not a cograph by Theorem 4.2, a contradiction. Thus, we may assume that $\pi(H) \subseteq \{p,q\}$, which also implies that $|\pi(K)| \leq 2$. If $|\pi(H)| = 1$; then, similarly, we must have that either $|\pi(K)| = 1$ or $\pi(K) = \{p,q\}$, and so, in this case, we may assume that $\pi(K) \subseteq \{p,q\}$. If $\pi(H) = \{p,q\}$, then there is no prime in $\pi(K)$ such that $r \notin \{p,q\}$, which implies that $\pi(K) \subseteq \{p,q\}$, as desired. □

For a group G, if its every non-trivial element is of prime power order, then G is called a *CP-group* [6]. Given a prime p, it is clear that every p-group is also a CP-group. Delgado and Wu characterized all finite CP-groups (see [6, Theorem 4]). Now, the following result holds.

Corollary 4.6. $\Gamma(G)$ *is a cograph if G is a CP-group.*

4.1. Dihedral and generalized quaternion groups

For positive integer $n \geq 3$, the *dihedral group* D_{2n} with order $2n$ is defined as the group of symmetries of a regular polygon on n points, including all rotations and reflections. It is clear that D_{2n} is a class of non-abelian groups. In general, D_{2n} is defined by the following presentation:

$$D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle. \tag{1}$$

Also, $a^i b$ is an involution for any $1 \leq i \leq n$, and D_{2n} has a partition

$$\{\{ab, a^2b, \ldots, b\}, \langle a \rangle\}. \tag{2}$$

Theorem 4.7. *Let D_{2n} be the dihedral group as presented in* (1)*. Then $\Gamma(D_{2n})$ is a cograph if and only if $n = p^m q^n$, where p,q are two distinct primes and m,n are two non-negative integers.*

Proof. We first suppose that $\Gamma(D_{2n})$ is a cograph. Note that $\Gamma(\langle a \rangle)$ is also a cograph and a cyclic group is nilpotent. It follows from Corollary 4.4 that $o(a) = n = p^m q^n$, where p,q are two distinct primes and m,n are two non-negative integers, as desired. For the converse, let $n = p^m q^n$, where p,q are two distinct primes and m,n are two non-negative integers. Then (2) implies that

$$\pi_e(D_{2n}) = \{2, d : d \mid p^m q^n\}.$$

If $2 \in \{p,q\}$, then $|\pi(D_{2n})| \leq 2$, and so $\Gamma(D_{2n})$ is a cograph by Corollary 4.3, as desired. If $2 \notin \{p,q\}$, since D_{2n} has no elements of order $2p$ and $2q$, we have that D_{2n} is an Ω-group, and so $\Gamma(D_{2n})$ is a cograph by Theorem 4.2, as desired. □

Johnson introduced the *generalized quaternion group* of order $4n$ with $n \geq 2$, which is denoted by Q_{4n} [10]. In general, Q_{4n} has a presentation as follows:

$$Q_{4n} = \langle a, b : a^n = b^2, a^{2n} = b^4 = e, b^3ab = a^{-1} \rangle.$$

Clearly, Q_{4n} is a family of non-abelian groups. Note that $o(a^ib) = 4$ for each $1 \leq i \leq 2n$, and Q_{4n} has a partition $\{\{a^ib : 1 \leq i \leq 2n\}, \langle a \rangle\}$. Thus, we have

$$\pi_e(Q_{4n}) = \{4, d : d \mid 2n\}. \tag{3}$$

Now, by (3) and a similar argument as in the proof of Theorem 4.7, we get the following result.

Theorem 4.8. $\Gamma(Q_{4n})$ *is a cograph if and only if* $n = p^m$ *or* $2^m q^n$, *where* p *is a prime,* q *is an odd prime, and* m, n *are positive integers.*

4.2. Symmetric groups and alternating groups

The *symmetric group* of order $n!$, denoted by $\mathbf{S}_n$, is the group consisting of all permutations on n letters. As we know, the symmetric group is important in many different areas of mathematics, including combinatorics and group theory, since every finite group is a subgroup of some symmetric group. Note that $\mathbf{S}_n$ is abelian if and only if $n \leq 2$.

Theorem 4.9. $\Gamma(\mathbf{S}_n)$ *is a cograph if and only if* $n \leq 6$.

Proof. Note first the fact that $\pi_e(\mathbf{S}_6) = \{1,2,3,4,5,6\}$. Then it is easy to see that $\mathbf{S}_6$ is an Ω-group, and so by Observation 4.1 and Theorem 4.2, we have that $\Gamma(\mathbf{S}_n)$ is a cograph for any $n \leq 6$. Now, consider $\mathbf{S}_7$. Note that $(12)(234), (12)(34567) \in \mathbf{S}_7$. Since $o((12)(234)) = 6$ and $o((12)(34567)) = 10$, we have that $\mathbf{S}_7$ is not an Ω-group, and so $\Gamma(\mathbf{S}_7)$ is not a cograph by Theorem 4.2. Now, Observation 4.1 implies the desired result. □

In symmetric group $\mathbf{S}_n$, the set of all even permutations is a subgroup, which is called the *alternating group* on n letters and is denoted by $\mathbf{A}_n$. Note that for any $n \geq 5$, $\mathbf{A}_n$ is a simple group.

Theorem 4.10. $\Gamma(\mathbf{A}_n)$ *is a cograph if and only if* $n \leq 7$.

Proof. Note the fact that $\pi_e(\mathbf{A}_7) = \{1,2,3,4,5,6,7\}$. It follows that $\mathbf{A}_7$ is an Ω-group. As a result, we have that $\Gamma(\mathbf{A}_n)$ is a cograph for every $n \leq 7$ by Observation 4.1 and Theorem 4.2. Now, considering $\mathbf{A}_8$, we have that $(123)(45)(67), (123)(45678) \in \mathbf{A}_8$. Since $o((123)(45)(67)) = 6$ and $o((123)(45678)) = 15$, we see that $\mathbf{A}_8$ is not an Ω-group, and so $\Gamma(\mathbf{A}_8)$ is not a cograph by Theorem 4.2. This also implies that $\Gamma(\mathbf{A}_n)$ is not a cograph for each $n \geq 8$, as required. □

4.3. Sporadic simple groups

Theorem 4.11. *Let* G *be a sporadic simple group. Then* $\Gamma(G)$ *is a cograph if and only if* G *is isomorphic to either* M_{11} *or* M_{22}.

Proof. It is well known that there exist precisely 26 sporadic simple groups. For the Mathieu group M_{11}, we have that $\pi_e(M_{11}) = \{1,2,3,4,5,6,8,11\}$, and so M_{11} is an Ω-group, which implies that $\Gamma(M_{11})$ is a cograph by Theorem 4.2. For the Mathieu group M_{12}, we have that $\pi_e(M_{12}) = \{1,2,3,4,5,6,8,10,11\}$, and so M_{12} is not an Ω-group, that is, $\Gamma(M_{12})$ is not a cograph. For the Mathieu group M_{22}, we have that $\pi_e(M_{22}) = \{1,2,3,4,5,6,7,8,11\}$, and similarly, we have that $\Gamma(M_{22})$ is a cograph. Note that Corollary 4.5 implies that $\mathbb{Z}_3 \times \mathbf{A}_5$ is not an Ω-group. Since $\mathbb{Z}_3 \times \mathbf{A}_5 \leq M_{23}$ by the $\mathbb{ATLAS}$ of finite groups [5], we have that $\Gamma(M_{23})$ is not a cograph. Now, since the Mathieu group M_{24} has a maximal subgroup isomorphic to M_{23}, it follows that $\Gamma(M_{24})$ is also not a cograph.

By [5], we first have that the McLaughlin group McL has a subgroup isomorphic to $\mathbf{A}_8$. As a result of Theorem 4.10, we see that $\Gamma(McL)$ is not a cograph. Now, note that for each $1 \leq i \leq 3$, the Conway group Co_i has a subgroup isomorphic to McL [5]. Consequently, $\Gamma(Co_i)$ is not a cograph for each $1 \leq i \leq 3$. By [5] again, it follows that the Janko group J_1, Hall-Janko group J_2, Janko group J_3, Janko group J_4, Held group He, Harada-Norton group HN, Thompson group Th, Baby Monster group B, Monster group M, O'Nan group $O'N$, Lyons group Ly, Rudvalis group Ru, and Higman-Sims group HS contain $D_6 \times D_{10}$, $\mathbf{A}_5 \times D_{10}$, $\mathbb{Z}_3 \times \mathbf{A}_6$, M_{24}, $\mathbf{S}_4 \times L_3(2)$, $\mathbf{A}_{12}$, $\mathbf{A}_9$, Co_2, $\mathbf{A}_9$, $D_6 \times D_{10}$, $\mathbf{A}_{11}$, $\mathbf{A}_8$, and $\mathbb{Z}_4 \times \mathbf{A}_5$ as subgroups, respectively. Therefore, by Corollary 4.5 and Theorem 4.10, we have that every graph of the coprime graphs of these 13 groups above is not a cograph. Finally, since every group of Suz, Fi_{22}, Fi_{23} and Fi'_{24} contains M_{12} as a subgroup, the coprime graphs of these four groups are not cographs. □

5. SPLIT, THRESHOLD AND CHORDAL COPRIME GRAPHS

This section will classify the finite groups whose coprime graph is split, threshold, or chordal. The following is the main result of this section.

Theorem 5.1. *For a finite group G, the following statements are equivalent:*
(a) $\Gamma(G)$ *is split;*
(b) $\Gamma(G)$ *is threshold;*
(c) $\Gamma(G)$ *is chordal;*
(d) *G is isomorphic to either a p-group or $\mathbb{Z}_2 \times Q$, where p is a prime and Q is a q-group for some odd prime q.*

We first prove a lemma which gives a sufficient condition for $2K_2$-free coprime graphs.

Lemma 5.2. *If G is a group with* $|\pi(G)| \leq 3$*, then* $\Gamma(G)$ *is* $2K_2$*-free.*

Proof. Suppose for a contradiction that $\Gamma(G)$ has an induced subgraph isomorphic to $2K_2$, say Δ, which has the vertex set $\{a,b,c,d\}$, where $\{a,b\},\{c,d\} \in E(\Delta)$. Note that $e \notin \{a,b,c,d\}$. We first claim that $o(a)$ is not a prime power. In fact, if $o(a) = p^m$ for some prime p and positive integer m, then $p \mid o(c)$ and $p \mid o(d)$, which implies that $p \mid (o(c), o(d))$, a contradiction since c and d are adjacent. As a result, $o(a)$ is not a prime power, and similarly, $o(b)$ is also not a prime power. Note that if $o(a)$ has pairwise distinct three prime divisors, then only the vertex adjacent to a is e. Thus, $o(a)$ has precisely two distinct prime divisors. Since $(o(a), o(b)) = 1$, it follows that $o(b)$ is a prime power, a contradiction. □

We are now ready to prove Theorem 5.1.

Proof. Firstly, by Theorem 3.1 and the definitions of a split graph, a threshold graph and a chordal graph, it is easy to see that any of (a), (b) and (c) can imply (d). Now, suppose that G is isomorphic to either a p-group or $\mathbb{Z}_2 \times Q$, where p is a prime and Q is a q-group for some odd prime q. It suffices to prove that $\Gamma(G)$ is split, threshold and chordal. Firstly, it follows from Lemma 5.2 that $\Gamma(G)$ is $2K_2$-free. Then, by Corollary 4.3, we deduce that $\Gamma(G)$ is P_4-free. This also implies that $\Gamma(G)$ is C_5-free, since in a graph, an induced C_5 must contain an induced C_4. As a result, $\Gamma(G)$ is split, threshold and chordal by Theorem 3.1, as desired. □

6. CONCLUSION

This article focuses on finite groups and their coprime graphs, which are undirected graphs with the vertex set of the group G. Two distinct vertices x and y are adjacent if and only if the orders of x and y are coprime (i.e., $(o(x), o(y)) = 1$). We classified finite groups that have split, threshold and chordal coprime graphs and provided a characterization for finite groups whose coprime graphs are cographs. Furthermore, we applied this classification to groups such as nilpotent groups, dihedral groups, generalized quaternion groups, symmetric groups, alternating groups, and sporadic simple groups, particularly focusing on their coprime graphs as cographs, based on the group's structure.

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Lõplikud rühmad tükelduva, künnisega ja kõõludega kaasneva graafiga

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Olgu G lõplik rühm. Sellega kaasnevaks graafiks nimetatakse graafi, mille tippude hulk on G ning selle erinevad tipud x ja y ($x, y \in G$) on kaasnevad siis, kui nende järgud $o(x)$ ja $o(y)$ on ühistegurita. Artiklis leitakse lõplikud rühmad, millega kaasnevad graafid on tükelduvad (*split*), künnisega (*threshold*) ja kõõludega (*chordal*). Artiklis kirjeldatakse ka lõplikke rühmi, millega kaasnevad graafid on kograafid. Saadud tulemuste rakendustena kirjeldatakse nilpotentseid rühmi, dieedrirühmi, üldistatud kvaternioonide rühmi, sümmeetrilisi rühmi, märgimuudurühmi ja sporaadilisi lihtsaid rühmi, millega kaasnevad graafid on kograafid.