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FUZZY GRAPHS

Novel concepts in linear Diophantine fuzzy graphs with an application

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Abstract. The linear Diophantine fuzzy graph (LDFG) notion serves as a new mathematical approach for the ambiguity and uncertainty modeling in decision-making issues. An LDFG eliminates the strict limitations of various existing graphs. The energy concept in graph theory is one of the most attractive topics that is very important in biological and chemical sciences. The article aims at developing the notion of fuzzy graphs (FG) towards LDFGs, and, also, we extend the energy notion of an FG to the energy of an LDFG and use the concept of energy to model problems linked to the LDFG. To fulfill such a purpose, we make an LDFG and investigate the effectiveness of that part by calculating the concept of energy on this LDFG. We define the LDFG adjacency matrix (AM) concept and the energy of an LDFG. Also, we introduce the new Laplacian energy (LE) concept of an LDFG and investigate its properties. Finally, an application of the LDFG energy to find the most effective component in the hospital information system has been presented.

Keywords: fuzzy graph, spectrum, eigenvalues, energy of fuzzy graph, linear Diophantine fuzzy graph.

1. INTRODUCTION

The fuzzy set concept was introduced by Zadeh [1]. In 1973, Kaufmann [2] was the first to propose the definitions of fuzzy graphs (FG). Gutman [3] defined the graph energy notion. After that, several energy concepts on FGs were discussed in [4–9]. Shi et al. [10] extended the energy on picture fuzzy graphs in 2022. Yager [11,12] suggested the notion of Pythagorean fuzzy sets (PFS). The linear Diophantine fuzzy set (LDFS) and its application to multi-attribute decision-making was suggested by Riaz and Hashmi [13]. In the LDFS theory, the application of reference or control parameters corresponding to membership and non-membership grades makes it most agreeable for modeling ambiguities in real-life problems. Recently, linear Diophantine fuzzy graphs (LDFG) have been developed into LDF soft rough sets [14], algebraic structures of LDFGs [15] and LDF relations with decision-making [16]. The LDFG concept was introduced by Hanif et al. [17]. Gutman et al. [18] investigated the eigenvalues of the Laplacian energy (LE) of its Laplacian matrix (LM). Kosari et al. [19–24] studied novel domination concepts in vague graphs. Akram et al. [25] studied a new approach to decision-making. LDF Einstein aggregation operators for multi-criteria decision-making problems were introduced by Iampan et al. [26]. Izatmand et al. [27] studied

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generalized Hamacher aggregation operators based on the linear Diophantine uncertain linguistic setting and their applications in decision-making problems. Riaz et al. [28] expressed the notion of interval-valued linear Diophantine fuzzy Frank aggregation operators with multi-criteria decision-making. Mohammad et al. [29] presented some LDF similarity measures. Muhiuddin [30] introduced the LDFS theory concept applied to BCK/BCI-algebras. Rashmanlou et al. [31] introduced the ring sum in product intuitionistic FGs. Talebi et al. [32,33] explained the new concepts of irregular-intuitionistic FGs and the novel properties of edge-irregular single-valued neutrosophic graphs. Kosari et al. [34] studied new concepts in vague graphs. Akram et al. [35,36] considered the energy of Pythagorean FGs. Bipolar fuzzy information [37–39] was introduced by Poulik et al. Some papers of FGs were studied in [40–43]. The LDFG theory becomes superior to FG theories due to the broader space for membership and non-membership values. FGs are useful tools to explain objects and the relationships among them. An LDFG belongs to the FG family and has good capabilities. The concept of LDFG is a new mathematical tool for optimization, artificial intelligence, soft computing, and decision analysis and process modeling. The LDFG theory widens the area of fuzzy information via reference parameters due to its wonderful characteristic of a broad depiction zone for allowed doublets. Because the actual world is not exact, and there is a dearth of knowledge, determining and selecting the optimal choice is a tough and unforeseen decision-making dilemma. The primary aim is to guide decision-makers through the process of selecting the best option inside an LDFG. In this work, we introduced certain new notions, including the LDFG energy and LE. Correspondingly, we presented some of their interesting properties with examples and wanted to solve real problems by using energy applications. New concepts, such as graph energy on LDFG, were introduced. This article developed through four sections. In Section 2, we gave all the essential definitions related to FG, LD and the FG energy. In Section 3, we defined the energy of an LDFG. In Section 4, we presented an application of the energy on FG. Finally, we gave a summary of the paper.

2. PRELIMINARIES

In this part, we study some essential notions of an LDFG.

Definition 2.1. A graph is an ordered pair $G^* = (X,E)$, where X is the set of vertices of G^* and $E \subseteq X \times X$ is the set of edges of G^*.
An FG on a graph $G^* = (X,E)$ is a pair $G = (\chi,\lambda)$, where χ is an FS on X and λ is an FS on E, such that

$$\lambda(ab) \leq \min\{\chi(a),\chi(b)\},$$

for all $ab \in E$.

Definition 2.2. Suppose that W is the universe. An LDFS ψ_S on W is defined by

$\psi_S = \{< v,(M_S^\tau(v),N_S^\nu(v)),(\delta,\gamma) >: v \in W\}$, where $M_S^\tau(v),N_S^\nu(v),\delta,\gamma \in [0,1]$, such that

$$0 \leq \delta M_S^\tau(v) + \gamma N_S^\nu(v) \leq 1, \forall v \in W,$$

$$0 \leq \delta + \gamma \leq 1;$$

the hesitation part can be written as

$$\eta = 1 - (\delta M_S^\tau(v) + \gamma N_S^\nu(v)),$$

where η is the reference parameter (RP). The value of $\psi_S =< (M_S^\tau(v),N_S^\nu(v)),(\delta,\gamma) >$ is introduced as the linear Diophantine fuzzy number (LDFN).

Definition 2.3. An absolute LDFS on W is of the form

$$1_{\psi_S} = \{< v,(1,0),(1,0) : v \in W\},$$

and empty LDFS on W is of the form

$$0_{\psi_S} = \{< v, (0,1), (0,1) : v \in W\}.$$

Definition 2.4. [17] Suppose that $\psi_S =< (M_S^\tau, N_S^\nu), (\delta, \gamma) >$ and $\psi_R =< (M_R^\tau, N_R^\nu), (\mu, \theta) >$ are two LDFSs on the reference set W and $v \in W$, then

* $\psi_S^c =< (N_S^\nu, M_S^\tau), (\gamma, \delta) >$.
* $\psi_S = \psi_R \Rightarrow M_S^\tau = M_R^\tau, \quad N_S^\nu = N_R^\nu, \quad \delta = \mu, \quad \gamma = \theta.$
* $\psi_S \subseteq \psi_R \Rightarrow M_S^\tau \leq M_R^\tau, \quad N_S^\nu \geq N_R^\nu, \quad \delta \leq \mu, \quad \gamma \geq \theta.$
* $\psi_S \cup \psi_R =< (M_{S\cup R}^\tau, N_{S\cap R}^\nu), (\delta \vee \mu, \gamma \wedge \theta) >$.
* $\psi_S \cap \psi_R =< (M_{S\cap R}^\tau, N_{S\cup R}^\nu), (\delta \wedge \mu, \gamma \vee \theta) >$,

where

$$M_{S\cup R}^\tau(v) = M_S^\tau(v) \vee M_R^\tau(v),$$

$$M_{S\cap R}^\tau(v) = M_S^\tau(v) \wedge M_R^\tau(v),$$

$$N_{S\cup R}^\nu(v) = N_S^\nu(v) \vee N_R^\nu(v),$$

$$N_{S\cap R}^\nu(v) = N_S^\nu(v) \wedge N_R^\nu(v).$$

Definition 2.5. [17] An LDFG is explained by $G = (\psi_S, \psi_R)$, where ψ_S is an LDFS on X and ψ_R is an LDFS on $E \subseteq X \times X$ as follows:

$$M_R^\tau(ab) \leq \min\{M_S^\tau(a), M_S^\tau(b)\},$$

$$N_R^\nu(ab) \leq \max\{N_S^\nu(a), N_S^\nu(b)\},$$

$$\mu^{ab} \leq \min\{\delta^a, \delta^b\},$$

$$\theta^{ab} \leq \max\{\gamma^a, \gamma^b\}.$$

For all $a, b \in X$, where $\delta^a, \delta^b, \gamma^a, \gamma^b$ are the RPs associated with the vertices a, b, and μ^{ab}, θ^{ab} are the RPs associated with the edge ab.

Definition 2.6. [17] Suppose that $G = (\psi_S, \psi_R)$ is an LDFG.

The order of G is described by

$$O(G) =< (\sum_{a\in X} M_S^\tau(a), \sum_{a\in X} N_S^\nu(a)), (\sum_{a\in X} \delta^a, \sum_{a\in X} \gamma^a) >.$$

The degree of a vertex a in G is described by

$$d(a) =< (\sum_{ab\in E} M_S^\tau(ab), \sum_{ab\in E} N_S^\nu(ab)), (\sum_{ab\in E} \mu^{ab}, \sum_{ab\in E} \theta^{ab}) >.$$

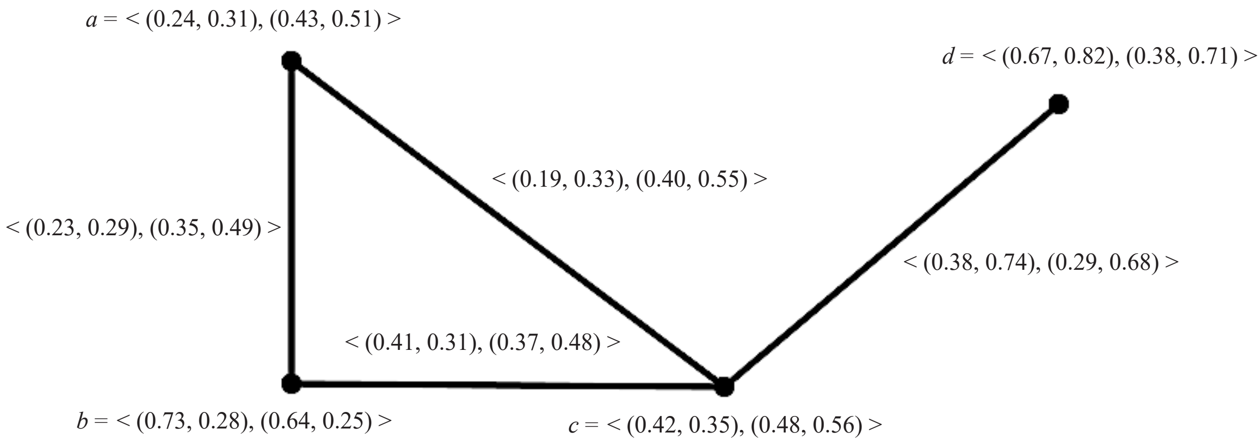

Fig. 1. An LDFG.

Table 1. $<(M_S^\tau, N_S^\nu), (\delta, \gamma)>$

ψ_S	$<(M_S^\tau, N_S^\nu), (\delta, \gamma)>$
a	$<(0.24, 0.31), (0.43, 0.51)>$
b	$<(0.73, 0.28), (0.64, 0.25)>$
c	$<(0.42, 0.35), (0.48, 0.56)>$
d	$<(0.67, 0.82), (0.38, 0.71)>$

Table 2. $<(M_R^\tau, N_R^\nu), (\mu, \theta)>$

ψ_R	$<(M_R^\tau, N_R^\nu), (\mu, \theta)>$
ab	$<(0.23, 0.29), (0.35, 0.49)>$
ac	$<(0.19, 0.33), (0.40, 0.55)>$
ad	$<(0.41, 0.31), (0.37, 0.48)>$
cd	$<(0.38, 0.74), (0.29, 0.68)>$

Example 2.7. Consider a graph $G^* = (X, E)$, where $X = \{a, b, c, d\}$ and $E = \{ab, ac, bc, cd\}$. Suppose that ψ_S is an LDF-subset of X, and ψ_R is an LDF-subset of E, as explained in Table 1 and Table 2.

Graph G in Fig. 1 is an LDFG. Also, the order of the LDFG G is $O(G) =<(2.06, 1.76), (1.93, 2.03)>$. The degree of each vertex in the LDFG G is

$$d(a) = <(0.42,0.62),(0.75,1.04)>,$$
$$d(b) = <(0.64,0.60),(0.72,0.97)>,$$
$$d(c) = <(0.98,1.38),(1.06,1.71)>,$$
$$d(d) = <(0.38,0.74),(0.29,0.68)>.$$

Definition 2.8. Two vertices that are connected by an edge are named adjacent. The adjacency matrix (AM) $A = [v_{lk}]$ for a graph $G^* = (X,E)$ is a matrix with n rows and m columns, $n = |V|$ and its entries defined by

$$v_{lk} = \begin{cases} 1 & if\ (u_l,u_k) \in E, \\ 0 & if\ otherwise. \end{cases}$$

Definition 2.9. The spectrum of a matrix is defined as a set of its eigenvalues, and we denoted it with $Spec(G)$. The eigenvalues λ_l, $l = 1,2,...,n$ of the AM of G are the eigenvalues of G. The spectrum $\lambda_1,\lambda_2,...,\lambda_n$ of the AM of G is the $Spec(G)$, the eigenvalues of the graph satisfy the following relations:

$$\sum_{l=1}^{n} \lambda_l = 0, \quad \sum_{l=1}^{n} \lambda_l^2 = 2m.$$

Definition 2.10. The energy of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of A, that is

$$E(G) = \sum_{l=1}^{n} |\lambda_l|,$$

where λ_l is an eigenvalue of A.

Theorem 2.11. Suppose that G is a simple graph with n vertices and m edges, and A is the AM of G, then

$$\sqrt{2m+n(n-1)|A|^{\frac{2}{n}}} \leq E(G) \leq \sqrt{2mn}.$$

Definition 2.12. The AM $A(G)$ of an LDFG, $G = (M,N)$ is defined as a square matrix $A(G) = [v_{lk}]$, $v_{lk} = <(M_S^\tau,N_S^\nu),(\delta,\gamma)>$, where $M_S^\tau(u_lu_k), N_S^\nu(u_lu_k), \delta(u_lu_k)$ and $\gamma(u_lu_k)$ represent the strength of relationship between u_l and u_k, respectively:

$$A(G) = <A(M_S^\tau(u_lu_k)), A(N_S^\nu(u_lu_k)), A(\delta(u_lu_k)), A(\gamma(u_lu_k)>.$$

Definition 2.13. The energy of an LDFG, $G = (M,N)$ is defined as follows:

$$E(G) = <E(M_S^\tau(u_lu_k)), E(N_S^\nu(u_lu_k)), E(\delta(u_lu_k)), E(\gamma(u_lu_k)>,$$

in other words,

$$E(G) = <\sum_{l=1}^{n}|\alpha_l|, \sum_{l=1}^{n}|\beta_l|, \sum_{l=1}^{n}|\pi_l|, \sum_{l=1}^{n}|\varepsilon_l|>,$$

where α_l, β_l, π_l and ε_l are the eigenvalues of $A(M_S^\tau(u_lu_k)), A(N_S^\nu(u_lu_k)), A(\delta(u_lu_k))$ and $A(\gamma(u_lu_k))$, respectively.

Example 2.14. Suppose a graph $G^* = (X,E)$, where $X = \{a_1,a_2,a_3,a_4\}$ and $E = \{a_1a_2, a_2a_3, a_3a_4, a_1a_4, a_2a_4\}$. Suppose $G = (M,N)$ is an LDFG of G^*, as shown in Fig. 2. Suppose ψ_S is an LDF-subset of X, and suppose ψ_R is an LDF-subset of E, as expressed in Table 3 and Table 4.

$$A(G) = \begin{bmatrix} 0 & (0.6,0.5),(0.2,0.7) & 0 & (0.4,0.6),(0.2,0.8) \\ (0.6,0.5),(0.2,0.7) & 0 & (0.4,0.8),(0.3,0.5) & (0.5,0.6),(0.3,0.6) \\ 0 & (0.4,0.8),(0.3,0.5) & 0 & (0.3,0.8),(0.5,0.4) \\ (0.4,0.6),(0.2,0.8) & (0.5,0.6),(0.3,0.6) & (0.3,0.8),(0.5,0.4) & 0 \end{bmatrix}.$$

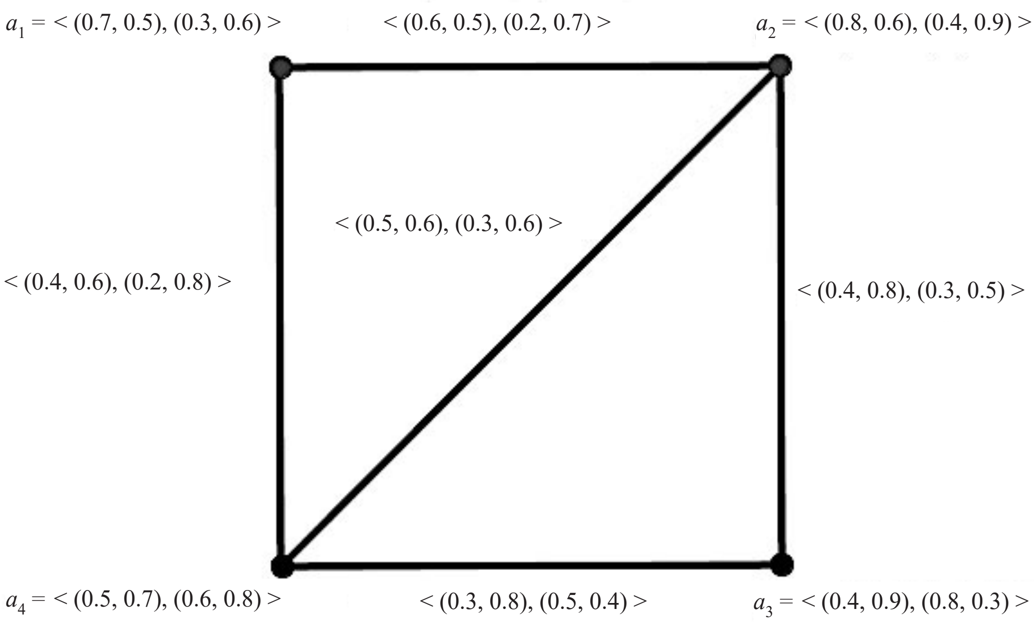

Fig. 2. An LDFG.

Table 3. $<(M_S^\tau, N_S^\nu),(\delta,\gamma)>$

ψ_S	$<(M_S^\tau, N_S^\nu),(\delta,\gamma)>$
a_1	$<(0.7,0.5),(0.3,0.6)>$
a_2	$<(0.8,0.6),(0.4,0.9)>$
a_3	$<(0.4,0.9),(0.8,0.3)>$
a_4	$<(0.5,0.7),(0.6,0.8)>$

Table 4. $<(M_R^\tau, N_R^\nu),(\mu,\theta)>$

ψ_R	$<(M_R^\tau, N_R^\nu),(\mu,\theta)>$
a_1a_2	$<(0.6,0.5),(0.2,0.7)>$
a_2a_3	$<(0.4,0.8),(0.3,0.5)>$
a_3a_4	$<(0.3,0.8),(0.5,0.4)>$
a_1a_4	$<(0.4,0.6),(0.2,0.8)>$
a_2a_4	$<(0.5,0.6),(0.3,0.6)>$

Now, we obtain the AMs and eigenvalues of each degree of G as follows:

$$A(M_S^{\tau}(a_l a_k)) = \begin{bmatrix} 0 & 0.6 & 0 & 0.4 \\ 0.6 & 0 & 0.4 & 0.5 \\ 0 & 0.4 & 0 & 0.3 \\ 0.4 & 0.5 & 0.3 & 0 \end{bmatrix}$$

$$Spec(A(M_S^{\tau}(a_l a_k))) = (-0.721, -0.434, 0.001, 1.154).$$

$$A(N_S^{\nu}(a_l a_k)) = \begin{bmatrix} 0 & 0.5 & 0 & 0.6 \\ 0.5 & 0 & 0.8 & 0.6 \\ 0 & 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0.8 & 0 \end{bmatrix}$$

$$Spec(A(N_S^{\nu}(a_l a_k))) = (-1.107, -0.604, 0.006, 1.706).$$

$$A(\delta(a_i a_j)) = \begin{bmatrix} 0 & 0.2 & 0 & 0.2 \\ 0.2 & 0 & 0.3 & 0.3 \\ 0 & 0.3 & 0 & 0.5 \\ 0.2 & 0.3 & 0.5 & 0 \end{bmatrix}$$

$$Spec(A(\delta(a_l a_k))) = (-0.543, -0.276, 0.013, 0.806).$$

$$A(\gamma(a_l a_k)) = \begin{bmatrix} 0 & 0.7 & 0 & 0.8 \\ 0.7 & 0 & 0.5 & 0.6 \\ 0 & 0.5 & 0 & 0.4 \\ 0.8 & 0.6 & 0.4 & 0 \end{bmatrix}$$

$$Spec(A(\gamma(a_l a_k))) = (-0.974, -0.614, 0.015, 1.573).$$

$$E(A(M_S^{\tau}(a_l a_k))) = \sum_{l=1}^{n} |\alpha_l| = 2.31,$$

$$E(A(N_S^{\nu}(a_l a_k))) = \sum_{l=1}^{n} |\beta_l| = 3.423,$$

$$E(A(\delta(a_l a_k))) = \sum_{l=1}^{n} |\pi_l| = 1.638,$$

$$E(A(\gamma(a_l a_k))) = \sum_{l=1}^{n} |\varepsilon_l| = 3.176.$$

Therefore, the energy of an LDFG, $G = (M, N)$ is equal to $E(G) = (2.31, 3.423, 1.638, 3.176)$.

All the essential notations are shown in Table 5.

Table 5. Some essential notations

Notation	Meaning
FS	Fuzzy set
FG	Fuzzy graph
LDFS	Linear Diophantine fuzzy set
LDFG	Linear Diophantine fuzzy graph
LD	Linear Diophantine
LDFN	Linear Diophantine fuzzy number
AM	Adjacency matrix
LE	Laplacian energy
LM	Laplacian matrix
RP	Reference parameter
SM	Symmetric matrix
DM	Degree matrix

3. ENERGY OF A LINEAR DIOPHANTINE FUZZY GRAPH

In this section, we define the notion of the energy of an LDFG, which can be used in real science.

Theorem 3.1. Consider that $G = (M,N)$ is an LDFG and $A(G)$ is its AM. If $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$, $\pi_1 \geq \pi_2 \geq ... \geq \pi_n$ and $\varepsilon_1 \geq \varepsilon_2 \geq ... \geq \varepsilon_n$ are the eigenvalues of $A(M_S^\tau(u_l u_k))$, $A(N_S^\nu(u_l u_k))$, $A(\delta(u_l u_k))$ and $A(\gamma(u_l u_k))$, respectively, then

$$(i) \sum_{l=1}^{n} \alpha_l = 0, \quad \sum_{l=1}^{n} \beta_l = 0, \quad \sum_{l=1}^{n} \pi_l = 0, \quad \sum_{l=1}^{n} \varepsilon_l = 0.$$

$$(ii) \sum_{l=1}^{n} \alpha_l^2 = 2 \sum_{1\leq l\leq k\leq n} (M_S^\tau(u_l u_k))^2, \quad \sum_{l=1}^{n} \beta_l^2 = 2 \sum_{1\leq l\leq k\leq n} (N_S^\nu(u_l u_k))^2,$$

$$\sum_{l=1}^{n} \pi_l^2 = 2 \sum_{1\leq l\leq k\leq n} (\delta(u_l u_k))^2, \quad \sum_{l=1}^{n} \varepsilon_l^2 = 2 \sum_{1\leq l\leq k\leq n} (\gamma(u_l u_k))^2.$$

Proof. (i) Since $A(G)$ is a symmetric matrix (SM) with zero trace, its eigenvalues are real with a sum equal to zero.

(ii) Based on the properties of the matrix, we have

$$tr((A(M_S^\tau(u_l u_k)))^2) = \sum_{l=1}^{n} \alpha_l^2,$$

where

$$tr((A(M_S^\tau(u_l u_k)))^2) = (0 + (M_S^\tau(u_1 u_2))^2 + ... + (M_S^\tau(u_1 u_n))^2$$

$$+(M_S^\tau(u_2 u_1))^2 + 0 + ... + (M_S^\tau(u_2 u_n))^2$$

$$\vdots$$

$$+(M_S^\tau(u_n u_1))^2 + (M_S^\tau(u_n u_2))^2 + ... + 0) = 2 \sum_{1\leq l\leq k\leq n} (M_S^\tau(u_l u_k))^2.$$

Hence,

$$\sum_{l=1}^{n} \alpha_l^2 = 2 \sum_{1\leq l\leq k\leq n} (M_S^\tau(u_l u_k))^2.$$

Also, we have

$$tr((A(N_S^{\nu}(u_lu_k)))^2) = \sum_{l=1}^{n} \beta_l^2,$$

where

$$tr((A(N_S^{\nu}(u_lu_k)))^2) = (0 + (N_S^{\nu}(u_1u_2))^2 + ... + (N_S^{\nu}(u_1u_n))^2$$

$$+(N_S^{\nu}(u_2u_1))^2 + 0 + ... + (N_S^{\nu}(u_2u_n))^2$$

$$\vdots$$

$$+(N_S^{\nu}(u_nu_1))^2 + (N_S^{\nu}(u_nu_2))^2 + ... + 0) = 2 \sum_{1\leq l\leq k\leq n} (N_S^{\nu}(u_lu_k))^2.$$

Hence,

$$\sum_{l=1}^{n} \beta_l^2 = 2 \sum_{1\leq l\leq k\leq n} (N_S^{\nu}(u_lu_k))^2.$$

Also, we can write

$$\sum_{l=1}^{n} \pi_l^2 = 2 \sum_{1\leq l\leq k\leq n} (\delta(u_lu_k))^2, \quad \sum_{l=1}^{n} \varepsilon_l^2 = 2 \sum_{1\leq l\leq k\leq n} (\gamma(u_lu_k))^2.$$

Theorem 3.2. Assume $G = (M,N)$ is an LDFG and $A(G)$ is the AM of G. Then,

$$(i)\sqrt{2 \sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_lu_k))^2 + n(n-1)|det(A(M_S^{\tau}(u_lu_k)))|^{\frac{2}{n}}}$$
$$\leq E(M_S^{\tau}(u_lu_k)) \leq \sqrt{2 \sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_lu_k))^2}.$$

$$(ii)\sqrt{2 \sum_{1\leq l\leq k\leq n} (N_S^{\nu}(u_lu_k))^2 + n(n-1)|det(A(N_S^{\nu}(u_lu_k)))|^{\frac{2}{n}}}$$
$$\leq E(N_S^{\nu}(u_lu_k)) \leq \sqrt{2 \sum_{1\leq l\leq k\leq n} (N_S^{\nu}(u_lu_k))^2}.$$

$$(iii)\sqrt{2 \sum_{1\leq l\leq k\leq n} (\delta(u_lu_k))^2 + n(n-1)|det(A(\delta(u_lu_k)))|^{\frac{2}{n}}}$$
$$\leq E(\delta(u_lu_k)) \leq \sqrt{2 \sum_{1\leq l\leq k\leq n} (\delta(u_lu_k))^2}.$$

$$(iv)\sqrt{2 \sum_{1\leq l\leq k\leq n} (\gamma(u_lu_k))^2 + n(n-1)|det(A(\gamma(u_lu_k)))|^{\frac{2}{n}}}$$
$$\leq E(\gamma(u_lu_k)) \leq \sqrt{2 \sum_{1\leq l\leq k\leq n} (\gamma(u_lu_k))^2}.$$

Proof. (i) By using Cauchy-Schwarz unequality to the vectors $(1,1,...,1)$ and $(|\alpha_1|,|\alpha_2|,...,|\alpha_n|)$ with n entries, we get

$$\sum_{l=1}^{n} |\alpha_l| \leq \sqrt{n}\sqrt{\sum_{l=1}^{n} |\alpha_l|^2}, \quad (3.1)$$

$$(\sum_{l=1}^{n} \alpha_l)^2 = \sum_{l=1}^{n} |\alpha_l|^2 + 2\sum_{1\leq l\leq k\leq n} \alpha_l \alpha_k. \quad (3.2)$$

By comparing the coefficients of α^{n-2} in the characteristic polynomial

$$\prod_{l=1}^{n}(\alpha - \alpha_l) = |A(G) - \alpha_l|,$$

we have

$$\sum_{1\leq l\leq k\leq n} \alpha_l \alpha_k = -\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2. \quad (3.3)$$

By replacing (3.3) in (3.2), we obtain

$$\sum_{l=1}^{n} |\alpha_l|^2 = 2\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2. \quad (3.4)$$

Replacing (3.4) in (3.1), we get

$$\sum_{l=1}^{n} |\alpha_l| \leq \sqrt{n}\sqrt{2\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2} = \sqrt{2n\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2}.$$

Therefore,

$$E(M_S^{\tau}(u_l u_k)) \leq \sqrt{2n\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2}.$$

$$(E(M_S^{\tau}(u_l u_k)))^2 = (\sum_{l=1}^{n} |\alpha_l|)^2 = \sum_{l=1}^{n} |\alpha_l|^2 + 2\sum_{1\leq l\leq k\leq n} |\alpha_l \alpha_k|$$

$$= 2\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2 + \frac{2n(n-1)}{2} AM\{|\alpha_l \alpha_k|\}.$$

Since $AM\{|\alpha_l \alpha_k|\} \geq GM\{|\alpha_l \alpha_k|\}, 1 \leq l \leq k \leq n$, so,

$$E(M_S^{\tau}(u_l u_k)) \geq \sqrt{2\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2 + n(n-1)GM\{|\alpha_l \alpha_k|\}},$$

also, since

$$GM\{|\alpha_l \alpha_k|\} = \left(\prod_{1\leq l\leq k\leq n} |\alpha_l \alpha_k|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{l=1}^{n} |\alpha_l|^{n-1}\right)^{\frac{2}{n(n-1)}}$$

$$= \left(\prod_{l=1}^{n} |\alpha_l|\right)^{\frac{2}{n}} = |det(A(M_S^{\tau}(u_l u_k)))|^{\frac{2}{n}},$$

so,

$$E(M_S^{\tau}(u_l u_k)) \geq \sqrt{2n\sum_{1\leq l\leq k\leq n} (M_S^{\tau}(u_l u_k))^2 + n(n-1)|det(A(M_S^{\tau}(u_l u_k))|^{\frac{2}{n}}}.$$

Thus,

$$\sqrt{2nM_S^\tau(u_lu_k)^2+n(n-1)|det(A(M_S^\tau(u_lu_k)))|^{\frac{2}{n}}} \le E(M_S^\tau(u_lu_k))$$

$$\le \sqrt{2n\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2}.$$

Similarly, we can prove the cases *(ii)*, *(iii)*, *(iv)*.

Theorem 3.3. Suppose $G=(M,N)$ is an LDFG and $A(G)$ is the AM of G. If $n\le 2\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2$, $n\le 2\sum_{1\le l\le k\le n}(N_S^\nu(u_lu_k))^2$, $n\le 2\sum_{1\le l\le k\le n}(\delta(u_lu_k))^2$, $n\le 2\sum_{1\le l\le k\le n}(\gamma(u_lu_k))^2$, then

$$(i)\quad E(M_S^\tau(u_lu_k)) \le \frac{2\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2}{n}$$

$$+\sqrt{(n-1)\left\{2\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2-\left(\frac{2\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2}{n}\right)^2\right\}}.$$

$$(ii)\quad E(N_S^\nu(u_lu_k)) \le \frac{2\sum_{1\le l\le k\le n}(N_S^\nu(u_lu_k))^2}{n}$$

$$+\sqrt{(n-1)\left\{2\sum_{1\le l\le k\le n}(N_S^\nu(u_lu_k))^2-\left(\frac{2\sum_{1\le l\le k\le n}(N_S^\nu(u_lu_k))^2}{n}\right)^2\right\}}.$$

$$(iii)\quad E(\delta(u_lu_k)) \le \frac{2\sum_{1\le l\le k\le n}(\delta(u_lu_k))^2}{n}$$

$$+\sqrt{(n-1)\left\{2\sum_{1\le l\le k\le n}(\delta(u_lu_k))^2-\left(\frac{2\sum_{1\le l\le k\le n}(\delta(u_lu_k))^2}{n}\right)^2\right\}}.$$

$$(iv)\quad E(\gamma(u_lu_k)) \le \frac{2\sum_{1\le l\le k\le n}(\gamma(u_lu_k))^2}{n}$$

$$+\sqrt{(n-1)\left\{2\sum_{1\le l\le k\le n}(\gamma(u_lu_k))^2-\left(\frac{2\sum_{1\le l\le k\le n}(\gamma(u_lu_k))^2}{n}\right)^2\right\}}.$$

Proof. *(i)* If $A=[v_{lk}]_{n\times n}$ is an SM with zero trace, then $\alpha_{max}\ge \frac{2\sum_{1\le l\le k\le n}u_lu_k}{n}$, where α_{max} is the maximum eigenvalue of A. If $A(G)$ is the AM of an LDFG G, then $\alpha_1 \ge \frac{2\sum_{1\le l\le k\le n}M_S^\tau(u_lu_k)}{n}$, where $\alpha_1\ge\alpha_2\ge ......\ge\alpha_n$. Moreover,

$$\sum_{l=1}^{n}\alpha_l^2 = 2\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2,$$

$$\sum_{l=2}^{n}\alpha_l^2 = 2\sum_{1\le l\le k\le n}(M_S^\tau(u_lu_k))^2-\alpha_1^2. \quad (3.5)$$

By using Cauchy–Schwarz unequality to the vectors $(1,1,...,1)$ and $(|\alpha_1|,|\alpha_2|,...,|\alpha_n|)$ with $n-1$ entries, we get

$$E(M_S^\tau(u_lu_k))-\alpha_1=\sum_{l=2}^{n}|\alpha_l|\leq\sqrt{(n-1)\sum_{l=2}^{n}|\alpha_l|^2}. \quad (3.6)$$

Replacing (3.5) in (3.6), we must have

$$E(M_S^\tau(u_lu_k))-\alpha_1\leq\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2-\alpha_1^2\right)},$$

$$E(M_S^\tau(u_lu_k))\leq\alpha_1+\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2-\alpha_1^2\right)}. \quad (3.7)$$

Now, the function $H(x)=x+\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2-x^2\right)}$ decreases on the interval $\left(\sqrt{\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}{n}},\sqrt{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}\right)$.

Also, $n\leq 2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2, 1\leq\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}{n}$.
So,

$$\sqrt{\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}{n}}\leq\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}{n}\leq\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))}{n}$$
$$\leq\alpha_1\leq\sqrt{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}.$$

Therefore, (3.7) implies

$$E(M_S^\tau(u_lu_k))\leq\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}{n}$$
$$+\sqrt{(n-1)\left\{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2-\left(\frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2}{n}\right)^2\right\}}.$$

Similarly, we can prove the cases *(ii)*, *(iii)*, *(iv)*.

Theorem 3.4. Suppose $G=(M,N)$ is an LDFG. Then $E(G)\leq\frac{n}{2}(1+\sqrt{n})$.

Proof.

Let $G=(M,N)$ be an LDFG. If $n\leq 2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2=2z$, then it is clear to show that $f(z)=\frac{2z}{n}+\sqrt{(n-1)(2z-(\frac{2z}{n})^2)}$ is maximized, when $z=\frac{n^2+n\sqrt{n}}{4}$. Replacing this value of z in the place of $z=\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2$, we must have $E(M_S^\tau(u_lu_k))\leq\frac{n}{2}(1+\sqrt{n})$.
Similarly, it is easy to show that $E(N_S^\nu(u_lu_k))\leq\frac{n}{2}(1+\sqrt{n})$, $E(\delta(u_lu_k))\leq\frac{n}{2}(1+\sqrt{n})$ and $E(\gamma(u_lu_k))\leq\frac{n}{2}(1+\sqrt{n})$. Hence, $E(G)\leq\frac{n}{2}(1+\sqrt{n})$.

Definition 3.5. [10] Suppose $G=(M,N)$ is an LDFG on n vertices. The DM $Z(G)=[z_{lk}]$ of G is an $n\times n$ diagonal matrix, which is described as

$$z_{lk}=\begin{cases}d_G(u_l) & l=k\\ 0 & l\neq k.\end{cases}$$

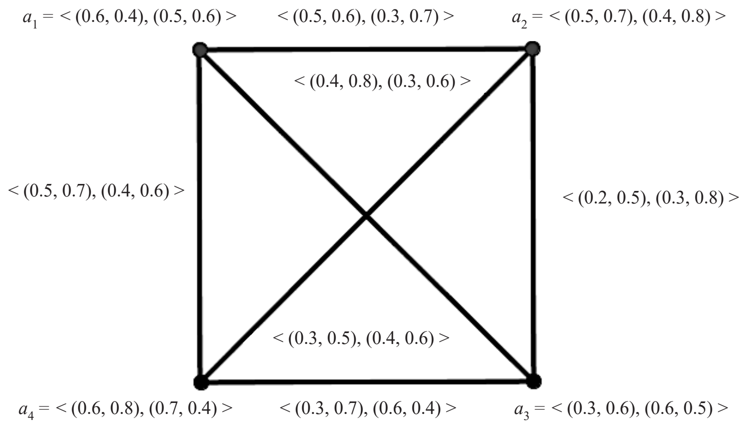

Fig. 3. An LDFG.

Definition 3.6. The LE of an LDFG, $G = (M,N)$ is defned as $L(G) = Z(G) - A(G)$, where $Z(G)$ and $A(G)$ are the DM and AM of an LDFG, respectively.

Definition 3.7. The LM of an LDFG, $G = (M,N)$ is defined as follows:

$$LE(G) =< LE(M_S^{\tau}(u_l u_k)), LE(N_S^{\nu}(u_l u_k)), LE(\delta(u_l u_k)), LE(\gamma(u_l u_k)) >,$$

$$LE(G) =< \sum_{l=1}^{n} |\alpha_l|, \sum_{l=1}^{n} |\beta_l|, \sum_{l=1}^{n} |\pi_l|, \sum_{l=1}^{n} |\varepsilon_l| >,$$

where

$$\alpha_l = \alpha_l^* - \frac{2\sum_{1\leq l\leq k\leq n} M_S^{\tau}(u_l u_k)}{n},$$

$$\beta_l = \beta_l^* - \frac{2\sum_{1\leq l\leq k\leq n} N_S^{\nu}(u_l u_k)}{n},$$

$$\pi_l = \pi_l^* - \frac{2\sum_{1\leq l\leq k\leq n} \delta(u_l u_k)}{n},$$

$$\varepsilon_l = \varepsilon_l^* - \frac{2\sum_{1\leq l\leq k\leq n} \gamma(u_l u_k)}{n},$$

where α_l^*, β_l^*, π_l^* and ε_l^*, $l = 1,2,3,...,n$ are the eigenvalues of $LE(M_S^{\tau}(u_l u_k))$, $LE(N_S^{\nu}(u_l u_k))$, $LE(\delta(u_l u_k))$ and $LE(\gamma(u_l u_k))$, respectively.

Example 3.8. Assume a graph $G^* = (X,E)$, where $X = \{a_1, a_2, a_3, a_4\}$ and $E = \{a_1a_2, a_2a_3, a_3a_4, a_1a_4, a_2a_4, a_1a_3\}$.

Suppose $G = (M,N)$ is an LDFG of G^*, as shown in Fig. 3. Suppose ψ_S is an LDF-subset of X, and suppose ψ_R is an LDF-subset of $E \subseteq X \times X$, as expressed in Table 6 and Table 7. The AM, DM, and LM are as follows, respectively:

$$A(G) = \begin{bmatrix} 0 & (0.5,0.6),(0.3,0.7) & (0.3,0.5),(0.4,0.6) & (0.5,0.7),(0.4,0.6) \\ (0.5,0.6),(0.3,0.7) & 0 & (0.2,0.5),(0.3,0.8) & (0.4,0.8),(0.3,0.6) \\ (0.3,0.5),(0.4,0.6) & (0.2,0.5),(0.3,0.8) & 0 & (0.3,0.7),(0.6,0.4) \\ (0.5,0.7),(0.4,0.6) & (0.4,0.8),(0.3,0.6) & (0.3,0.7),(0.6,0.4) & 0 \end{bmatrix},$$

$$K(G)=\begin{bmatrix}(1.3,1.8),(1.1,1.9) & 0 & 0 & 0\\ 0 & (1.1,1.9),(0.9,2.1) & 0 & 0\\ 0 & 0 & (0.8,1.7),(1.3,1.8) & 0\\ 0 & 0 & 0 & (1.2,2.2),(1.3,1.6)\end{bmatrix},$$

$$L(G)=\begin{bmatrix}(1.3,1.8),(1.1,1.9) & (-0.5,-0.6),(-0.3,-0.7) & (-0.3,-0.5),(-0.4,-0.6) & (-0.5,-0.7),(-0.4,-0.6)\\ (-0.5,-0.6),(-0.3,-0.7) & (1.1,1.9),(0.9,2.1) & (-0.2,-0.5),(-0.3,-0.8) & (-0.4,-0.8),(-0.3,-0.6)\\ (-0.3,-0.5),(-0.4,-0.6) & (-0.2,-0.5),(-0.3,-0.8) & (0.8,1.7),(1.3,1.8) & (-0.3,-0.7),(-0.6,-0.4)\\ (-0.5,-0.7),(-0.4,-0.6) & (-0.4,-0.8),(-0.3,-0.6) & (-0.3,-0.7),(-0.6,-0.4) & (1.2,2.2),(1.3,1.6)\end{bmatrix}.$$

After computing, we have

$LE(A(M_S^{\tau}(a_ia_j)))=4.401$, $LE(A(N_S^{\nu}(a_ia_j)))=7.599$,

$LE(A(\delta(a_ia_j)))=4.6$ and $LE(A(\gamma(a_ia_j)))=7.401$.

Therefore, the LE of an LDFG, $G=(M,N)$ is equal to $LE(G)=(4.401,7.599,4.6,7.401)$.

Theorem 3.9. Suppose that $G=(M,N)$ is an LDFG and $L(G)$ is the LM of G. If $\alpha_1^*\geq\alpha_2^*\geq...\geq\alpha_n^*$, $\beta_1^*\geq\beta_2^*\geq...\geq\beta_n^*$, $\pi_1^*\geq\pi_2^*\geq...\geq\pi_n^*$ and $\varepsilon_1^*\geq\varepsilon_2^*\geq...\geq\varepsilon_n^*$ are the eigenvalues of $L(M_S^{\tau}(u_lu_k))$, $L(N_S^{\nu}(u_lu_k))$, $L(\delta(u_lu_k))$ and $L(\gamma(u_lu_k))$, respectively, then

$$(i)\sum_{l=1}^{n}\alpha_l^*=2\sum_{1\leq l\leq k\leq n}M_S^{\tau}(u_lu_k)\quad,\quad\sum_{l=1}^{n}\beta_l^*=2\sum_{1\leq l\leq k\leq n}N_S^{\nu}(u_lu_k),$$

$$\sum_{l=1}^{n}\pi_l^*=2\sum_{1\leq l\leq k\leq n}\delta(u_lu_k)\quad,\quad\sum_{l=1}^{n}\varepsilon_l^*=2\sum_{1\leq l\leq k\leq n}\gamma(u_lu_k).$$

$$(ii)\sum_{l=1}^{n}\alpha_l^{*^2}=2\sum_{1\leq l\leq k\leq n}(M_S^{\tau}(u_lu_k))^2+\sum_{l=1}^{n}d_M^2(u_l).$$

$$\sum_{l=1}^{n}\beta_l^{*^2}=2\sum_{1\leq l\leq k\leq n}(N_S^{\nu}(u_lu_k))^2+\sum_{l=1}^{n}d_N^2(u_l).$$

Table 6. $<(M_S^{\tau},N_S^{\nu}),(\delta,\gamma)>$

ψ_S	$<(M_S^{\tau},N_S^{\nu}),(\delta,\gamma)>$
a_1	$<(0.6,0.4),(0.5,0.6)>$
a_2	$<(0.5,0.7),(0.4,0.8)>$
a_3	$<(0.3,0.6),(0.6,0.5)>$
a_4	$<(0.6,0.8),(0.7,0.4)>$

Table 7. $<(M_R^{\tau},N_R^{\nu}),(\mu,\theta)>$

ψ_R	$<(M_R^{\tau},N_R^{\nu}),(\mu,\theta)>$
a_1a_2	$<(0.5,0.6),(0.3,0.7)>$
a_2a_3	$<(0.2,0.5),(0.3,0.8)>$
a_3a_4	$<(0.3,0.7),(0.6,0.4)>$
a_1a_4	$<(0.5,0.7),(0.4,0.6)>$
a_2a_4	$<(0.4,0.8),(0.3,0.6)>$
a_1a_3	$<(0.3,0.5),(0.4,0.6)>$

$$\sum_{l=1}^{n} \pi_l^{*^2} = 2 \sum_{1\le l\le k\le n} (\delta(u_l u_k))^2 + \sum_{l=1}^{n} d_\delta^2(u_l).$$

$$\sum_{l=1}^{n} \varepsilon_l^{*^2} = 2 \sum_{1\le l\le k\le n} (\gamma(u_l u_k))^2 + \sum_{l=1}^{n} d_\gamma^2(u_l).$$

Proof. (i) Since $L(G)$ is an SM with non-negative Laplacian eigenvalues, thus,

$$\sum_{l=1}^{n} \alpha_l^* = tr(L(G)) = \sum_{l=1}^{n} d_M(u_l) = 2 \sum_{1\le l\le k\le n} (M_S^\tau(u_l u_k)).$$

Then, $\sum_{l=1}^{n} \alpha_l^* = 2\sum_{1\le l\le k\le n} M_S^\tau(u_l u_k)$, similarly, $\sum_{l=1}^{n} \beta_l^* = 2\sum_{1\le l\le k\le n} N_S^\nu(u_l u_k)$, $\sum_{l=1}^{n} \pi_l^* = 2\sum_{1\le l\le k\le n} \delta(u_l u_k)$ and $\sum_{l=1}^{n} \varepsilon_l^* = 2\sum_{1\le l\le k\le n} \gamma(u_l u_k)$.

(ii) We have

$$tr((L(M_S^\tau(u_l u_k)))^2) = \sum_{l=1}^{n} \alpha_l^*,$$

where

$$\begin{aligned} tr((L(M_S^\tau(u_l u_k)))^2) &= \Big(d^2M(u_1) + M_S^{\tau^2}(u_1u_2) + ... + M_S^{\tau^2}(u_1u_n)\Big) \\ &+ \Big(M_S^{\tau^2}(u_2u_1) + d^2M(u_2) + ... + M_S^{\tau^2}(u_2u_n)\Big) \\ &\vdots \\ &+ \Big(M_S^{\tau^2}(u_nu_1) + M_S^{\tau^2}(u_nu_2) + ... + d^2M(u_n)\Big) = 2 \sum_{1\le l\le k\le n} (M_S^\tau(u_l u_k))^2 + \sum_{l=1}^{n} d_M^2(u_l). \end{aligned}$$

Hence,

$$\sum_{l=1}^{n} \alpha_l^{*^2} = 2 \sum_{1\le l\le k\le n} (M_S^\tau(u_l u_k))^2 + \sum_{l=1}^{n} d_M^2(u_l).$$

In the same way, the other relations are fixed.

Theorem 3.10. Suppose $G = (M,N)$ is an LDFG on n vertices and $L(G)$ is the LM of G, then

$$(i)\ LE(M_S^\tau(u_l u_k)) \le \sqrt{2n \sum_{1\le l\le k\le n} (M_S^\tau(u_l u_k))^2 + n \sum_{l=1}^{n} \left(d_M(u_l) - \frac{2\sum_{1\le l\le k\le n}(M_S^\tau(u_l u_k))}{n}\right)^2}.$$

$$(ii)\ LE(N_S^\nu(u_l u_k)) \le \sqrt{2n \sum_{1\le l\le k\le n} (N_S^\nu(u_l u_k))^2 + n \sum_{l=1}^{n} \left(d_N(u_l) - \frac{2\sum_{1\le l\le k\le n}(N_S^\nu(u_l u_k))}{n}\right)^2}.$$

$$(iii)\ LE(\delta(u_l u_k)) \le \sqrt{2n \sum_{1\le l\le k\le n} (\delta(u_l u_k))^2 + n \sum_{l=1}^{n} \left(d_\delta(u_l) - \frac{2\sum_{1\le l\le k\le n}(\delta(u_l u_k))}{n}\right)^2}.$$

$$(iv)\ LE(\gamma(u_lu_k)) \leq \sqrt{2n\sum_{1\leq l\leq k\leq n}(\gamma(u_lu_k))^2 + n\sum_{l=1}^{n}\left(d_\gamma(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}(\gamma(u_lu_k))}{n}\right)^2}.$$

Proof. (i) Applying Cauchy–Schwarz unequality to the vectors $(1,1,...,1)$ and $(|\alpha_1|,|\alpha_2|,...,|\alpha_n|)$ with n entries, we get

$$\sum_{l=1}^{n}|\alpha_l| \leq \sqrt{n}\sqrt{\sum_{l=1}^{n}|\alpha_l|^2}$$

$$LE(M_S^\tau(u_lu_k)) \leq \sqrt{n}\sqrt{2A_{M_S^\tau}} = \sqrt{2nA_{M_S^\tau}},$$

since

$$A_{M_S^\tau} = 2n\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2 + \frac{1}{2}\sum_{l=1}^{n}\left(d_M(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}M_S^\tau(u_lu_k)}{n}\right)^2.$$

Therefore, we have

$$LE(M_S^\tau(u_lu_k)) \leq \sqrt{2n\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2 + n\sum_{l=1}^{n}\left(d_M(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))}{n}\right)^2}.$$

In the same way, we can prove the cases (*ii*), (*iii*), (*iv*).

Theorem 3.11. Suppose $G=(M,N)$ is an LDFG and $L(G)$ is the LM of G. Then

$(i)\ LE(M_S^\tau(u_lu_k)) \leq |\alpha_l|$

$$+\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))^2 + \sum_{l=1}^{n}\left(d_M(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}(M_S^\tau(u_lu_k))}{n}\right)^2 - \alpha_l^2\right)}.$$

$(ii)\ LE(N_S^\nu(u_lu_k)) \leq |\beta_l|$

$$+\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(N_S^\nu(u_lu_k))^2 + \sum_{l=1}^{n}\left(d_N(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}(N_S^\nu(u_lu_k))}{n}\right)^2 - \beta_l^2\right)}.$$

$(iii)\ LE(\delta(u_lu_k)) \leq |\pi_i|$

$$+\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(\delta(u_lu_k))^2 + \sum_{l=1}^{n}\left(d_\delta(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}(\delta(u_lu_k))}{n}\right)^2 - \pi_l^2\right)}.$$

$(iv)\ LE(\gamma(u_lu_k)) \leq |\varepsilon_l|$

$$+\sqrt{(n-1)\left(2\sum_{1\leq l\leq k\leq n}(\gamma(u_lu_k))^2 + \sum_{l=1}^{n}\left(d_\gamma(u_l) - \frac{2\sum_{1\leq l\leq k\leq n}(\gamma(u_lu_k))}{n}\right)^2 - \varepsilon_l^2\right)}.$$

Proof. By using the Caushy–Schwarz unequality, we write

$$(i)\sum_{l=1}^{n}|\alpha_l| \le \sqrt{n\sum_{l=1}^{n}|\alpha_l|^2},$$

$$\sum_{l=2}^{n}|\alpha_l| \le \sqrt{(n-1)\sum_{l=2}^{n}|\alpha_l|^2}.$$

$$LE(M_S^\tau(u_l u_k)) - |\alpha_1| \le \sqrt{(n-1)(2A_{M_S^\tau} - \alpha_1^2)},$$

$$LE(M_S^\tau(u_l u_k)) \le |\alpha_1| + \sqrt{(n-1)(2A_{M_S^\tau} - \alpha_1^2)},$$

since

$$A_{M_S^\tau} = \sum_{1\le l\le k\le n}(M_S^\tau(u_l u_k))^2 + \frac{1}{2}\sum_{l=1}^{n}\left(d_M(u_l) - \frac{2\sum_{1\le l\le k\le n}M_S^\tau(u_l u_k)}{n}\right)^2.$$

Therefore, $LE(M_S^\tau(u_l u_k)) \le |\alpha_l|$

$$+\sqrt{(n-1)\left(2\sum_{1\le l\le k\le n}(M_S^\tau(u_l u_k))^2 + \sum_{l=1}^{n}\left(d_M(u_l) - \frac{2\sum_{1\le l\le k\le n}(M_S^\tau(u_l u_k)}{n}\right)^2 - \alpha_l^2\right)}.$$

In the same way, we can prove the cases (*ii*), (*iii*), (*iv*).

4. APPLICATIONS OF THE ENERGY OF LDFGS TO FIND THE MOST EFFECTIVE COMPONENT IN THE HOSPITAL INFORMATION SYSTEM

One of the most important areas of the information technology application is the health and treatment field. The hospital information system is the first and most basic system in providing health care. Hospital information systems are computer systems designed to easily manage medical and hospital information and to improve the health care quality. So, considering the importance of the hospital information system and its role in improving medical and health services, we intended to specify the most effective component in the field of technology and information of a hospital in terms of registering information about patients, medicine, finance, laboratory, etc. In the hospital, it is crucial to run the communication smoothly. Thus, the performance of the communication facilities plays an essential role in providing health care. According to the research that was done about different environments, we selected four departments for evaluation in the hospital as $x_i (i = 1,2,3,4)$.

We selected four points from the hospital departments and we invited a decision-making team of four people to evaluate the sections to investigate the effectiveness of each department.

The experts gave their opinions based on their evaluation of the selected places and compared the selected options. Their conclusion used the following LD preference relations:
$A_j(G) =< (M_{lk}^{\tau(j)}, N_{lk}^{\nu(j)}), (\delta_{lk}^{(j)}, \gamma_{lk}^{(j)}) >, (j = 1,2,3,4).$

$$A_1 = \begin{bmatrix} (0.3,0.8),(0.4,0.9) & (0.5,0.7),(0.5,0.7) & (0.6,0.5),(0.4,0.6) & (0.5,0.7),(0.4,0.6) \\ (0.5,0.6),(0.3,0.7) & (0.8,0.6),(0.9,0.5) & (0.2,0.5),(0.3,0.8) & (0.4,0.8),(0.3,0.6) \\ (0.3,0.5),(0.4,0.6) & (0.2,0.5),(0.3,0.8) & (0.8,0.7),(0.3,0.8) & (0.3,0.7),(0.6,0.4) \\ (0.5,0.7),(0.4,0.6) & (0.4,0.8),(0.3,0.6) & (0.3,0.7),(0.6,0.4) & (0.8,0.2),(0.3,0.6) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} (0.4,0.7),(0.3,0.9) & (0.7,0.8),(0.6,0.7) & (0.3,0.3),(0.6,0.5) & (0.4,0.6),(0.3,0.7) \\ (0.5,0.9),(0.6,0.7) & (0.8,0.7),(0.7,0.5) & (0.4,0.5),(0.6,0.8) & (0.4,0.3),(0.2,0.6) \\ (0.5,0.2),(0.9,0.6) & (0.4,0.9),(0.6,0.8) & (0.7,0.7),(0.2,0.8) & (0.6,0.5),(0.4,0.7) \\ (0.5,0.6),(0.3,0.6) & (0.3,0.8),(0.3,0.5) & (0.3,0.3),(0.5,0.4) & (0.9,0.7),(0.4,0.6) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} (0.4,0.2),(0.4,0.9) & (0.5,0.7),(0.5,0.7) & (0.8,0.5),(0.4,0.6) & (0.5,0.9),(0.4,0.7) \\ (0.5,0.9),(0.3,0.9) & (0.6,0.6),(0.9,0.5) & (0.3,0.5),(0.3,0.8) & (0.6,0.8),(0.2,0.6) \\ (0.3,0.7),(0.4,0.3) & (0.4,0.7),(0.3,0.8) & (0.8,0.7),(0.3,0.8) & (0.1,0.4),(0.9,0.4) \\ (0.6,0.7),(0.1,0.6) & (0.7,0.8),(0.7,0.6) & (0.3,0.7),(0.7,0.4) & (0.9,0.5),(0.3,0.5) \end{bmatrix},$$

$$A_4 = \begin{bmatrix} (0.3,0.4),(0.4,0.7) & (0.5,0.4),(0.5,0.7) & (0.6,0.5),(0.2,0.6) & (0.3,0.7),(0.2,0.8) \\ (0.1,0.5),(0.9,0.1) & (0.2,0.6),(0.9,0.5) & (0.2,0.7),(0.3,0.7) & (0.4,0.6),(0.3,0.7) \\ (0.8,0.5),(0.8,0.6) & (0.2,0.8),(0.3,0.7) & (0.2,0.7),(0.5,0.8) & (0.3,0.7),(0.9,0.4) \\ (0.6,0.7),(0.4,0.9) & (0.4,0.3),(0.9,0.6) & (0.3,0.7),(0.6,0.3) & (0.9,0.7),(0.5,0.7) \end{bmatrix}.$$

The energy of each LDFG is:
$E(A_1) =< (2.963, 3.337),(2.499, 3.298) >, E(A_2) =< (2.87, 2.83),(2.78, 2.882) >$,
$E(A_3) =< (2.756, 3.154),(2.85, 3.25) >$ and $E(A_4) =< (2.4, 2.912),(2.862, 3.124) >$.

Then, the weights can be calculated as

$$w_j =< ((w_{M_S^\tau})_j, (w_{N_S^\nu})_j), ((w_\delta)_j, (w_\gamma)_j) > \quad , \quad j = 1,2,3,4,$$

$$w_j =< \left(\frac{E(A_{M_S^\tau})_j}{\sum_{k=1}^4 E(A_{M_S^\tau})_k}, \frac{E(A_{N_S^\nu})_j}{\sum_{k=1}^4 E(A_{N_S^\nu})_k} \right), \left(\frac{E(A_\delta)_j}{\sum_{k=1}^4 E(A_\delta)_k}, \frac{E(A_\gamma)_j}{\sum_{k=1}^4 E(A_\gamma)_k} \right) > .$$

Here,

$$w_1 =< (0.269, 0.272),(0.227, 0.262) >,$$
$$w_2 =< (0.261, 0.231),(0.252, 0.229) >,$$
$$w_3 =< (0.250, 0.257),(0.259, 0.258) >,$$
$$w_4 =< (0.218, 0.238),(0.260, 0.248) > .$$

By summing four LD fuzzy preference relations, we obtain the collective LD fuzzy preference relation, which is defined as follows:

$$A = \sum_{j=1}^{4} w_j A_j =$$

$$\begin{bmatrix} (0.35,0.52),(0.37,0.84) & (0.55,0.65),(0.52,0.69) & (0.57,0.45),(0.39,0.57) & (0.42,0.72),(0.32,0.86) \\ (0.41,0.72),(0.53,0.60) & (0.61,0.62),(0.84,0.49) & (0.27,0.54),(0.37,0.77) & (0.44,0.63),(0.24,0.76) \\ (0.46,0.48),(0.62,0.56) & (0.30,0.87),(0.37,0.77) & (0.64,0.69),(0.32,0.84) & (0.32,0.57),(0.70,0.46) \\ (0.54,0.67),(0.29,0.67) & (0.44,0.67),(0.55,0.57) & (0.29,0.60),(0.59,0.37) & (0.87,0.51),(0.37,0.59) \end{bmatrix}.$$

Using Definition 2.3 and $\eta = 1-(\delta M_S^\tau(v)+\gamma N_S^\nu(v))$, we have

$$\begin{bmatrix} 0.4337 & 0.2655 & 0.5212 & 0.2464 \\ 0.3507 & 0.1838 & 0.4843 & 0.4156 \\ 0.446 & 0.2191 & 0.2156 & 0.5138 \\ 0.3945 & 0.3761 & 0.6069 & 0.33772 \end{bmatrix}.$$

If x_i is the information about patients, medicine, finance and laboratory, respectively, the degree of preference of x_i over the other alternatives is

$$\psi(x_i) = \sum_{j=1}^{m} w_j \Big(\sum_{k=1, l\neq j}^{n} (M_{lk}^{\tau(j)} - (M_{kl}^{\tau(j)}) \Big), l = 1, 2, ..., n.$$

Therefore, the net flow of the four alternatives is

$$\psi(x_1) = 0.185, \psi(x_2) = -0.1633, \psi(x_3) = -0.0566, \psi(x_4) = 0.0871,$$

which gives the ranking of $x_2 < x_3 < x_4 < x_1$. Consistent with the information that the decision-making team obtained from four departments of the hospital, they came to the conclusion that the most effective component of the information system is related to patients. Thus, the best choice is x_1.

5. CONCLUSIONS

FGs have many applications in solving different problems in several aspects. Since many parameters in real-world networks are specifically bound to the energy concept, this concept has become one of the most widely used concepts in graph theory. However, the energy in an FG is so important because of the confrontation with uncertain and ambiguous issues. This concept becomes more interesting when we know that we are dealing with an FG called the LDFG. The LDFG is a significant real-world decision issue, and its most fundamental and essential research is the expression of information. The LDFS is a novel method for addressing uncertainty in decision-making issues. We have used LDFGs to assess the validity of decision-making knowledge in the basic framework and to remove any distortion in the decision analysis. This led us to examine the energy in LDFG. So, in this work, we extended the notion of the energy of an FG to the energy of an LDFG. Also, we presented the notion of the energy and LE on an LDFG and investigated some of its properties and used its results in modeling and solving the problems ahead. Finally, an application of graph energy to find the most effective component in the hospital information system was presented. In our future work, we will investigate the concepts of domination set, vertex covering, and independent set in the LDFG and give applications of different types of domination in the LDFG and other sciences.

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