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FIXED POINT THEORY

Modified shrinking projection methods in CAT(0) space

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Abstract. The aim of this paper is to introduce three modified shrinking projection methods involving two $\mathscr{G}$-nonexpansive mappings. We also prove the convergence of our proposed iterations to obtain the common fixed points of $\mathscr{G}$-nonexpansive mappings in the setting of CAT(0) space. In addition we construct a numerical example which supports our main results and show a comparison of new iterative schemes by using MATLAB2018a.

Keywords: $\mathscr{G}$-nonexpansive mapping, fixed point, CAT(0) space, directed graph.

1. INTRODUCTION

In 1922, Banach proved a classical theorem known as Banach's contraction theorem [3], which ensures the existence of a unique fixed point for a contraction mapping in a complete metric space. Banach's contraction theorem has been generalized in many directions due to its numerous applications.

In 2008, Jachymski [8] proved the Banach contraction theorem for $\mathscr{G}$-contraction mappings, next Tiammee et al. [16] proved Browder's convergence theorem for $\mathscr{G}$-nonexpansive mappings. Toward this direction, many authors dealt with the existence of fixed points of $\mathscr{G}$-contraction mappings, $\mathscr{G}$-nonexpansive mappings and monotone nonexpansive mappings in Banach, Hilbert and hyperbolic metric spaces with directed graph (for more details see [1,2]).

Recently, Tripak [17], Suparatulatorn et al. [13] and Thianwan and Yambangwai [15] proved the convergence analysis of sequences generated by different iteration processes involving $\mathscr{G}$-nonexpansive mappings in Banach space with directed graph. For more details on modified iteration processes and $\mathscr{G}$-nonexpansive mappings we refer our readers to see [18–21].

In 2008, Takahashi et al. [14] proposed a modified hybrid method, the so-called shrinking projection method, as follows: Let C be a non-empty closed convex subset of a Hilbert space H, T be a self-mapping on C and $F(T)$ denote the set of fixed points of T:

$$\begin{aligned}
&x_1 = x,\\
&C_1 = C,\\
&y_n = Tx_n,\\
&C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\
&x_{n+1} = P_{C_{n+1}}x,
\end{aligned}$$

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for each $n \in \mathbb{N}$. Takahashi et al. proved the strong convergence of this sequence to $P_F x$. In 2019, Hammad et al. [7] introduced four new modified shrinking projection methods with two step iterative schemes and proved some convergence results in Hilbert spaces with graph.

Inspired by above work, we introduced three new modified shrinking projection methods involving two $\mathscr{G}$-nonexpansive mappings in CAT(0) spaces with graph. We also provide a numerical example to illustrate the rate of convergence of our proposed methods.

2. PRELIMINARIES

This section includes some well-known lemmas and definitions.

Throughout this paper, we denote by $\mathscr{X}$ a CAT(0) space. Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{X}$.

Kirk introduced fixed point theory in CAT(0) space. For more details on CAT(0) spaces, see [9–11]. Let $\mathscr{X}$ be a complete CAT(0) space, for a bounded sequence $\{x_n\}$ in $\mathscr{X}$ and $x \in \mathscr{X}$, setting

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ is defined by

$$r(\{x_n\}) = \inf\{r(x,x_n) : x \in \mathscr{X}\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{x \in \mathscr{X} : r(x,x_n) = r(\{x_n\})\}.$$

In 2008, Kirk and Panyanak [11] introduced Δ-convergence in CAT(0) spaces which is analogue of weak convergence in Banach spaces and restriction of Lim's concepts of convergence [12] to CAT(0) spaces.
Definition. [11] *Let $\mathscr{X}$ be a complete CAT(0) space, a sequence $\{x_n\}$ in $\mathscr{X}$ is said to Δ-converge to $x \in \mathscr{X}$ if x is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$.*
Proposition 2.1. [5] *If sequence $\{x_n\}$ in $\mathscr{X}$, Δ-converges to $x \in \mathscr{X}$, then*

$$x \in \bigcap_{k=1}^{\infty} \overline{co}\{x_k, x_{k+1}, \ldots\},$$

where $\overline{co}(\mathscr{A}) = \bigcap\{\mathscr{B} : \mathscr{B} \supseteq \mathscr{A}$ and B is closed and convex$\}$.

Berg and Nikolaev [4] introduced the concept of quasi-linearization in CAT(0) spaces. Authors denoted a pair $(a,b) \in \mathscr{X} \times \mathscr{X}$ by $\overrightarrow{ab}$ and called it a vector. Using this, they defined quasi-linearization as a map $\langle .,. \rangle : (\mathscr{X} \times \mathscr{X}) \times (\mathscr{X} \times \mathscr{X}) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), a,b,c,d \in \mathscr{X}.$$

It can be seen that
$\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a,b,c,$ $d,e,f \in \mathscr{X}$.

Let $\mathscr{X}$ be a complete CAT(0) space and $\mathscr{C}$ be a nonempty subset of $\mathscr{X}$. Let $\triangle$ denote the diagonal of the cartesian product $\mathscr{C} \times \mathscr{C}$, i.e., $\triangle = \{(x,x) : x \in \mathscr{C}\}$. Consider a directed graph $\mathscr{G}$ such that the set $V(\mathscr{G})$ of its vertices coincides with $\mathscr{C}$, and the set $E(\mathscr{G})$ of its edges contains all loops, i.e., $E(\mathscr{G}) \supseteq \triangle$. We identify the graph $\mathscr{G}$ with the pair $(V(\mathscr{G}), E(\mathscr{G}))$ and assume that $\mathscr{G}$ has no parallel edge. A set B dominates x_0 if for each $x \in B$, $(x,x_0) \in E(\mathscr{G})$ and B is said to be dominated by x_0 if for each $x \in B$, $(x_0,x) \in E(\mathscr{G})$.

Let $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ be a self map. An edge-preserving mapping, i.e.$((x,y)\in E(\mathscr{G})\Rightarrow(\mathscr{T}x,\mathscr{T}y)\in E(\mathscr{G}))$ is said to be $\mathscr{G}$-nonexpansive if

$$d(\mathscr{T}x,\mathscr{T}y)\leq d(x,y),\ \ \forall(x,y)\in E(\mathscr{G}).$$

Recall that a mapping $\mathscr{T}$ is said to be firmly nonexpansive [9] if

$$d^2(\mathscr{T}x,\mathscr{T}y)\leq\langle\overrightarrow{\mathscr{T}x\mathscr{T}y},\overrightarrow{xy}\rangle,$$

for all $x,y\in\mathscr{X}$.

Let $\mathscr{C}$ be a non-empty closed and convex subset of a CAT(0) space. The metric projection $\mathscr{P}_{\mathscr{C}}:\mathscr{X}\to\mathscr{C}$ maps each point $x\in\mathscr{X}$ to the unique point $\mathscr{P}_{\mathscr{C}}x\in\mathscr{C}$ such that

$$d(x,\mathscr{P}_{\mathscr{C}}x)=\inf\{d(x,y):y\in\mathscr{C}\}.$$

We also know that $\mathscr{P}_{\mathscr{C}}$ is firmly nonexpansive, i.e.,

$$d^2(\mathscr{P}_{\mathscr{C}}x,\mathscr{P}_{\mathscr{C}}y)\leq\langle\overrightarrow{\mathscr{P}_{\mathscr{C}}x\mathscr{P}_{\mathscr{C}}y},\overrightarrow{xy}\rangle,$$

for all $x,y\in\mathscr{X}$. Furthermore, $\langle\overrightarrow{x\mathscr{P}_{\mathscr{C}}x},\overrightarrow{y\mathscr{P}_{\mathscr{C}}x}\rangle\leq 0$, for all $x\in\mathscr{X}$ and $y\in\mathscr{C}$.

Lemma 2.1. [6] *Let $\mathscr{C}$ be a closed, convex and nonempty subset of a CAT(0) space and $\mathscr{P}_{\mathscr{C}}:\mathscr{X}\to\mathscr{C}$ be the metric projection. Then, we have the following inequality:*

$$d^2(x,\mathscr{P}_{\mathscr{C}}x)+d^2(y,\mathscr{P}_{\mathscr{C}}x)\leq d^2(x,y),\ forall x\in\mathscr{X}\, and y\in\mathscr{C}.$$

Lemma 2.2. *Let $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ be a nonexpansive mapping, where $\mathscr{C}$ is a nonempty subset of a complete CAT(0) space. Let $\mathscr{G}=(V(\mathscr{G}),E(\mathscr{G}))$ be a directed graph such that $V(\mathscr{G})=\mathscr{C}$. Then for any $\varepsilon>0$, there exists a positive $\xi(\varepsilon)>0$, such that $d(x,\mathscr{T}x)<\varepsilon$ for all $x\in co\{x_0,x_1\}$, whenever $x_0,x_1\in\mathscr{C}$ with $(x_0,x),(x_1,x)\in E(\mathscr{G})$, $d(x_0,\mathscr{T}x_0)\leq\xi(\varepsilon)$ and $d(x_1,\mathscr{T}x_1)\leq\xi(\varepsilon)$.*

Proof. Let $x=(1-\lambda)x_0\oplus\lambda x_1$ for $\lambda\in[0,1]$ and $\varepsilon>0$.

We assume two cases which are as follows:

Case 1. If $d(x_0,x_1)<\frac{\varepsilon}{3}$, then

$$d(x,x_0)=\lambda d(x_0,x_1)<\frac{\varepsilon}{3}.$$

If $\xi(\varepsilon)<\frac{\varepsilon}{3}$, then we have

$$\begin{aligned}d(\mathscr{T}x,x)&\leq d(\mathscr{T}x,\mathscr{T}x_0)+d(\mathscr{T}x_0,x_0)+d(x_0,x)\\&\leq 2d(x,x_0)+d(\mathscr{T}x_0,x_0)\\&< 2(\tfrac{\varepsilon}{3})+\xi(\varepsilon)\\&<\varepsilon.\end{aligned}$$

Case 2. If $d(x_0,x_1)\geq\frac{\varepsilon}{3}$, then for any non-negative number $\lambda<\frac{\varepsilon}{3d(x_0,x_1)}$, we have

$$d(x,x_0)=\lambda d(x_0,x_1)<\frac{\varepsilon}{3}.$$

If $\xi(\varepsilon)<\frac{\varepsilon}{3}$ and $\lambda<\frac{\varepsilon}{3d(x_0,x_1)}$, then we have

$$\begin{aligned}d(\mathscr{T}x,x)&\leq d(\mathscr{T}x,\mathscr{T}x_0)+d(\mathscr{T}x_0,x_0)+d(x_0,x)\\&\leq 2d(x,x_0)+d(\mathscr{T}x_0,x_0)\\&< 2(\tfrac{\varepsilon}{3})+\xi(\varepsilon)\\&<\varepsilon.\end{aligned}\tag{2.1}$$

We may assume that $\lambda \in [\frac{\varepsilon}{3d(x_0,x_1)},1]$ and $d(x_0,x_1) \geq \frac{\varepsilon}{3}$. Then, we obtain

$$\begin{aligned} d(\mathscr{T}x,x_0) &\leq d(\mathscr{T}x,\mathscr{T}x_0)+d(\mathscr{T}x_0,x_0) \\ &\leq d(x,x_0)+d(\mathscr{T}x_0,x_0) \\ &\leq d(x,x_0)+\xi(\varepsilon) \\ &= \lambda d(x_1,x_0)+\xi(\varepsilon) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(\mathscr{T}x,x_1) &\leq d(\mathscr{T}x,\mathscr{T}x_1)+d(\mathscr{T}x_1,x_1) \\ &\leq d(x,x_1)+d(\mathscr{T}x_1,x_1) \\ &\leq d(x,x_1)+\xi(\varepsilon) \\ &= (1-\lambda)d(x_1,x_0)+\xi(\varepsilon). \end{aligned} \tag{2.3}$$

From (2.2), (2.3) and $\lambda \in [\frac{\varepsilon}{3d(x_0,x_1)},1]$, in the case of $(1-\lambda) < \frac{\varepsilon}{3d(x_0,x_1)}$ we obtain

$$\begin{aligned} d(\mathscr{T}x,x) &\leq (1-\lambda)d(\mathscr{T}x,x_0)+\lambda d(\mathscr{T}x,x_1) \\ &\leq 2(1-\lambda)\lambda d(x_1,x_0)+\xi(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

For the case of $(1-\lambda) \geq \frac{\varepsilon}{3d(x_0,x_1)}$, we set $u = \frac{d(Tx,x_0)}{\lambda d(x_1,x_0)}$, $v = \frac{d(Tx,x_1)}{(1-\lambda)d(x_1,x_0)}$. By the uniform convexity of CAT(0) space [5], we have $d(u,v) \leq \frac{\varepsilon}{d(x_0,x_1)}$. This implies that

$$d(Tx,x) = \lambda(1-\lambda)d(u,v)d(x_1,x_0) < \varepsilon.$$ □

3. CONVERGENCE RESULTS

In this section, first we prove demiclosedness principle of a $\mathscr{G}$-nonexpansive mapping in CAT(0) space which will be used in our main results.

Theorem 3.1. *Let $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ be a $\mathscr{G}$-nonexpansive mapping, where $\mathscr{C}$ is a closed non-empty convex subset of a complete CAT(0) space $\mathscr{X}$. Let $\mathscr{G}=(V(\mathscr{G}),E(\mathscr{G}))$ be a directed graph such that $V(\mathscr{G})=\mathscr{C}$ and the sequence $\{x_n\}$ in $\mathscr{C}$ Δ-converges to $x\in\mathscr{C}$. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x)\in E(\mathscr{G})$ for all $k\in\mathbb{N}$ and $d(x_n,\mathscr{T}x_n)\to 0$, then $\mathscr{T}x=x$.*

Proof. Let the sequence $\{x_n\}$ in $\mathscr{C}$ Δ-converge to $x\in\mathscr{C}$ and $d(x_n,\mathscr{T}x_n)\to 0$. By the hypothesis, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x)\in E(\mathscr{G})$. By setting, $\varepsilon_{n_k}=d(x_{n_k},\mathscr{T}x_{n_k})$. Let $\varepsilon>0$. Since $\varepsilon_{n_k}\to 0$ as $k\to\infty$ there exists $N\in\mathbb{N}$ such that $\varepsilon_{n_k}<\varepsilon$ for all $k\geq N$.

By the above Lemma 2.2, for each $z\in\overline{co}\{x_{n_k}:k\geq N\}$, $d(z,\mathscr{T}z)<\varepsilon$. By Proposition 2.1, $\overline{co}\{x_{n_k}:k\geq N\}$ contains the Δ-limit, x, of subsequence $\{x_{n_k}\}$. This implies that $d(x,\mathscr{T}x)<\varepsilon$. Hence, $d(x,\mathscr{T}x)=0$, that is $x=\mathscr{T}x$, since ε is arbitrary.

Theorem 3.2. *Let $\mathscr{X}$ be a complete CAT(0) space and $\mathscr{C}$ a closed non-empty convex subset of $\mathscr{X}$ such that a subset $\{z\in\mathscr{X}:d(x,z)\leq d(y,z)\}$ is convex for all $x,y\in\mathscr{X}$. Let $\mathscr{G}=(V(\mathscr{G}),E(\mathscr{G}))$ be a directed graph such that $V(\mathscr{G})=\mathscr{C}$ and $E(\mathscr{G})$ is convex. Let $\mathscr{T}_1,\mathscr{T}_2:\mathscr{C}\to\mathscr{C}$ be two $\mathscr{G}$-nonexpansive mappings such that $\mathscr{F}=\mathscr{F}(\mathscr{T}_1)\cap\mathscr{F}(\mathscr{T}_2)\neq\emptyset$, $\mathscr{F}$ is closed and $\mathscr{F}(\mathscr{T}_1)\times\mathscr{F}(\mathscr{T}_2)\subseteq E(\mathscr{G})$. Let the sequence $\{r_n\}$ be generated by $r_1\in\mathscr{C}$ with $\mathscr{C}_1=\mathscr{C}$,*

$$\begin{aligned} &x_n=(1-\varsigma_n)r_n\oplus\varsigma_n\mathscr{T}_1r_n, \\ &y_n=(1-\eta_n)\mathscr{T}_1r_n\oplus\eta_n\mathscr{T}_1x_n, \\ &z_n=(1-\zeta_n)\mathscr{T}_2y_n\oplus\zeta_n\mathscr{T}_1x_n, \\ &\mathscr{C}_{n+1}=\{z\in\mathscr{C}_n:d(z_n,z)\leq d(r_n,z)\}, \\ &r_{n+1}=\mathscr{P}_{\mathscr{C}_{n+1}}r_1, n\geq 1, \end{aligned} \tag{i}$$

where $\{\zeta_n\}$, $\{\eta_n\}$, $\{\varsigma_n\}\subset[0,1]$. Suppose that the following conditions are satisfied:

(i) $\{r_n\}$ dominates ρ for all $\rho\in\mathscr{F}$ and if there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $\{r_{n_k}\}$ $\Delta-$

converges to $w \in \mathscr{C}$*, then* $(r_{n_k}, w) \in E(\mathscr{G})$*;*
$(ii) \liminf_{n\to\infty} \eta_n > 0$*;*
$(iii)\ 0 < \liminf_{n\to\infty} \zeta_n \le \limsup_{n\to\infty} \zeta_n < 1$;
$(iv)\ 0 < \liminf_{n\to\infty} \varsigma_n \le \limsup_{n\to\infty} \varsigma_n < 1$.
Then, the sequence $\{r_n\}$ *converges strongly to* $\mathscr{P}_{\mathscr{F}} r_1$.

Proof. We will start with proving that $\mathscr{P}_{\mathscr{C}_{n+1}} r_1$ is well defined for each $r_1 \in \mathscr{C}$. As shown in Tiammee et al. [16], $\mathscr{F}(\mathscr{T}_i)$ is convex for all $i = 1,2$. $\mathscr{F}$ is closed and convex by using the assumption. Hence, $\mathscr{P}_{\mathscr{F}} r_1$ is well defined. Let $\rho \in \mathscr{F}$. Since $\{r_n\}$ dominates ρ, we have $(r_n, \rho) \in E(\mathscr{G})$. Using the edge preservingness of $\mathscr{T}_1$ and convexity of $E(\mathscr{G})$, we have $(x_n, \rho) \in E(\mathscr{G})$. We also have $(\mathscr{T}_1 r_n, \rho)$ and $(\mathscr{T}_1 x_n, \rho) \in E(\mathscr{G})$ because $\mathscr{T}_1$ is edge preserving. By convexity of $E(\mathscr{G})$, we have $(y_n, \rho) \in E(\mathscr{G})$. By using the edge preservingness of $\mathscr{T}_1$ and $\mathscr{T}_2$, $(\mathscr{T}_1 x_n, \rho)$, $(\mathscr{T}_2 y_n, \rho) \in E(\mathscr{G})$. Again applying convexity of $E(\mathscr{G})$, we have $(z_n, \rho) \in E(\mathscr{G})$.

$$\begin{aligned}
d(z_n, \rho) &= d((1-\zeta_n)\mathscr{T}_2 y_n \oplus \zeta_n \mathscr{T}_1 x_n, \rho) \\
&\le (1-\zeta_n) d(\mathscr{T}_2 y_n, \rho) + \zeta_n d(\mathscr{T}_1 x_n, \rho) \\
&\le (1-\zeta_n) d(y_n, \rho) + \zeta_n d(x_n, \rho) \\
&\le (1-\zeta_n)[d((1-\eta_n)\mathscr{T}_1 r_n \oplus \eta_n \mathscr{T}_1 x_n, \rho)] + \zeta_n [d((1-\varsigma_n) r_n \oplus \varsigma_n \mathscr{T}_1 r_n, \rho)] \\
&\le (1-\zeta_n)(1-\eta_n) d(\mathscr{T}_1 r_n, \rho) + (1-\zeta_n)\eta_n d(\mathscr{T}_1 x_n, \rho) + \zeta_n (1-\varsigma_n) d(r_n, \rho) + \zeta_n \varsigma_n d(\mathscr{T}_1 r_n, \rho) \\
&\le (1-\zeta_n)(1-\eta_n) d(r_n, \rho) + (1-\zeta_n)\eta_n d(r_n, \rho) + \zeta_n d(r_n, \rho) \\
&\le d(r_n, \rho).
\end{aligned}$$

By definition of $\mathscr{C}_{n+1}$, we have $\rho \in \mathscr{C}_{n+1}$. Thus $\mathscr{F} \subset \mathscr{C}_{n+1}$. It is easy to see that C_{n+1} is closed and by our assumption it is convex. This implies that $\mathscr{P}_{\mathscr{C}_{n+1}} r_1$ is well defined for all $r_1 \in \mathscr{C}$.

Next, we will show that $\lim_{n\to\infty} d(r_n, r_1)$ exists. Since $\mathscr{F}$ is closed convex and non-empty subset of $\mathscr{X}$, there exists a unique $v \in \mathscr{F}$ such that $v = \mathscr{P}_{\mathscr{F}} r_1$. From $r_n = \mathscr{P}_{\mathscr{C}_n} r_1$ and $r_{n+1} \in \mathscr{C}_n$, for all $n \in \mathbb{N}$,

$$d(r_n, r_1) \le d(r_{n+1}, r_1). \tag{3.1}$$

As we know that $\mathscr{F} \subset \mathscr{C}_n$, we obtain

$$d(r_n, r_1) \le d(v, r_1). \tag{3.2}$$

It follows from (3.1) and (3.2) that the sequence $\{r_n\}$ is nondecreasing and bounded. Therefore, $\lim_{n\to\infty} d(r_n, r_1)$ exists.

To show that $r_n \to w \in \mathscr{C}$ as $n \to \infty$. For $m > n$, we have $r_m = \mathscr{P}_{\mathscr{C}_m} r_1 \in \mathscr{C}_m \subset \mathscr{C}_n$, by definition of $\mathscr{C}_n$. We have

$$\langle \overrightarrow{r_1 \mathscr{P}_{\mathscr{C}_m} r_1}, \overrightarrow{\mathscr{P}_{\mathscr{C}_m} r_1 r_n} \rangle \ge 0,$$

$$d^2(r_m, r_n) \le d^2(r_m, r_1) - d^2(r_n, r_1).$$

Since $\lim_{n\to\infty} d(r_n, r_1)$ exists, we obtain $\{r_n\}$ as a Cauchy sequence. This implies that there exists $w \in \mathscr{C}$ such that $r_n \to w \in \mathscr{C}$ as $n \to \infty$. We also have

$$\lim_{n\to\infty} d(r_{n+1}, r_n) = 0. \tag{3.3}$$

Next, we will have to show that $w \in \mathscr{F}$. Since $r_{n+1} \in \mathscr{C}_{n+1} \subset \mathscr{C}_n$. From (3.3), we have

$$\begin{aligned}
d(z_n, r_n) &\le d(z_n, r_{n+1}) + d(r_{n+1}, r_n) \\
&\le 2\, d(r_{n+1}, r_n) \\
&\to 0\ (as\ n \to \infty).
\end{aligned} \tag{3.4}$$

Since $\{r_n\}$ dominates $\rho \in \mathscr{F}$ and using (2.2), we have

$$\begin{aligned}
d^2(z_n,\rho) &= d^2((1-\zeta_n)\mathscr{T}_2 y_n \oplus \zeta_n \mathscr{T}_1 x_n, \rho) \\
&\leq (1-\zeta_n)d^2(\mathscr{T}_2 y_n,\rho) + \zeta_n d^2(\mathscr{T}_1 x_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) \\
&\leq (1-\zeta_n)d^2(y_n,\rho) + \zeta_n d^2(x_n,\rho) \\
&\leq (1-\zeta_n)d^2((1-\eta_n)\mathscr{T}_1 r_n \oplus \eta_n \mathscr{T}_1 x_n,\rho) + \zeta_n d^2((1-\varsigma_n)r_n \oplus \varsigma_n \mathscr{T}_1 r_n,\rho) \\
&\leq (1-\zeta_n)[(1-\eta_n)d^2(r_n,\rho) + \eta_n d^2(x_n,\rho)] + \zeta_n d^2(r_n,\rho) \\
&\leq (1-\zeta_n)[(1-\eta_n)d^2(r_n,\rho) + \eta_n d^2(r_n,\rho) - \eta_n \varsigma_n(1-\varsigma_n)d^2(\mathscr{T}_1 r_n, r_n)] + \zeta_n d^2(r_n,\rho) \\
&\leq d^2(r_n,\rho) - \eta_n\varsigma_n(1-\zeta_n)(1-\varsigma_n)d^2(r_n,\mathscr{T}_1 r_n)
\end{aligned}$$

and

$$\begin{aligned}
d^2(z_n,\rho) &= d^2((1-\zeta_n)\mathscr{T}_2 y_n \oplus \zeta_n \mathscr{T}_1 x_n, \rho) \\
&\leq (1-\zeta_n)d^2(\mathscr{T}_2 y_n,\rho) + \zeta_n d^2(\mathscr{T}_1 x_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) \\
&\leq (1-\zeta_n)d^2(y_n,\rho) + \zeta_n d^2(x_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) \\
&\leq d^2(r_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n).
\end{aligned}$$

This implies that

$$\eta_n\varsigma_n(1-\zeta_n)(1-\varsigma_n)d^2(\mathscr{T}_1 r_n, r_n) \leq d^2(r_n,\rho) - d^2(z_n,\rho)$$

and

$$\zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) \leq d^2(r_n,\rho) - d^2(z_n,\rho).$$

From our assumptions and (3.4), we have

$$\lim_{n\to\infty} d(\mathscr{T}_1 r_n, r_n) = 0 \tag{3.5}$$

and

$$\lim_{n\to\infty} d(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) = 0. \tag{3.6}$$

From (3.5) we get

$$\begin{aligned}
d(x_n,r_n) &\leq d((1-\varsigma_n)r_n \oplus \varsigma_n \mathscr{T}_1 r_n, r_n) \\
&\leq \varsigma_n d(\mathscr{T}_1 r_n, r_n) \\
&\to 0 \ (as\ n\to\infty).
\end{aligned} \tag{3.7}$$

From (3.5) and (3.7) we get

$$\begin{aligned}
d(y_n,r_n) &\leq d((1-\eta_n)\mathscr{T}_1 r_n \oplus \eta_n \mathscr{T}_1 x_n, r_n) \\
&\leq (1-\eta_n)d(\mathscr{T}_1 r_n, r_n) + \eta_n d(\mathscr{T}_1 x_n, r_n) \\
&\leq (1-\eta_n)d(\mathscr{T}_1 r_n, r_n) + \eta_n d(\mathscr{T}_1 x_n, \mathscr{T}_1 r_n) + \eta_n d(\mathscr{T}_1 r_n, r_n) \\
&\to 0 \ (as\ n\to\infty).
\end{aligned} \tag{3.8}$$

From (3.5), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
d(\mathscr{T}_2 r_n, r_n) &\leq d(\mathscr{T}_2 r_n, \mathscr{T}_2 y_n) + d(\mathscr{T}_2 y_n, r_n) \\
&\leq d(r_n,y_n) + d(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) + d(\mathscr{T}_1 x_n, x_n) + d(x_n,r_n) \\
&\leq d(r_n,y_n) + d(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) + d(\mathscr{T}_1 x_n, \mathscr{T}_1 r_n) + d(\mathscr{T}_1 r_n, r_n) + 2d(x_n,r_n) \\
&\leq d(r_n,y_n) + d(\mathscr{T}_2 y_n, \mathscr{T}_1 x_n) + d(x_n,r_n) + d(\mathscr{T}_1 r_n, r_n) + 2d(x_n,r_n) \\
&\to 0 \ (as\ n\to\infty).
\end{aligned} \tag{3.9}$$

From (3.5), (3.9) and Theorem 3.1, we have $w \in \mathscr{F}$.
Next, we show that $w = v = \mathscr{P}_{\mathscr{F}} r_1$. Since $r_n = \mathscr{P}_{\mathscr{C}_n} r_1$, we have

$$\langle \overrightarrow{r_1 \mathscr{P}_{\mathscr{C}_n} r_1}, \overrightarrow{\mathscr{P}_{\mathscr{C}_n} r_1 \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n,$$

$$\langle \overrightarrow{r_1 r_n}, \overrightarrow{r_n \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n. \tag{3.10}$$

Taking limit in (3.10), we obtain

$$\langle \overrightarrow{r_1 w}, \overrightarrow{w \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n. \tag{3.11}$$

Since $\mathscr{F} \subset \mathscr{C}_n$, so $w = \mathscr{P}_{\mathscr{F}} r_1$. □

Theorem 3.3. *Let $\mathscr{X}$, $\mathscr{C}$, $\mathscr{T}_1$ and $\mathscr{T}_2$ be defined as in Theorem 3.2. Let $\mathscr{G} = (V(\mathscr{G}), E(\mathscr{G}))$ be a directed graph such that $V(\mathscr{G}) = \mathscr{C}$, $E(\mathscr{G})$ is convex and $\mathscr{F} = \mathscr{F}(\mathscr{T}_1) \cap \mathscr{F}(\mathscr{T}_2) \neq \emptyset$, $\mathscr{F}$ is closed and $\mathscr{F}(\mathscr{T}_1) \times \mathscr{F}(\mathscr{T}_2) \subseteq E(\mathscr{G})$. Let the sequence $\{t_n\}$ be generated by $t_1 \in \mathscr{C}$ with $\mathscr{C}_1 = \mathscr{C}$,*

$$\begin{aligned} x_n &= (1-\varsigma_n)t_n \oplus \varsigma_n \mathscr{T}_1 t_n, \\ y_n &= (1-\eta_n)\mathscr{T}_1 t_n \oplus \eta_n \mathscr{T}_1 x_n, \\ z_n &= (1-\zeta_n)\mathscr{T}_2 x_n \oplus \zeta_n \mathscr{T}_1 y_n, \\ \mathscr{C}_{n+1} &= \{z \in \mathscr{C}_n : d(z_n, z) \leq d(t_n, z)\}, \\ t_{n+1} &= \mathscr{P}_{\mathscr{C}_{n+1}} t_1, n \geq 1, \end{aligned} \tag{ii}$$

where $\{\zeta_n\}$, $\{\eta_n\}$, $\{\varsigma_n\} \subset (0,1)$. Suppose that the following conditions are satisfied:
(i) $\{t_n\}$ dominates ρ for all $\rho \in \mathscr{F}$ and if there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\{t_{n_k}\}$ Δ-converges to $w \in \mathscr{C}$, then $(t_{n_k}, w) \in E(\mathscr{G})$;
(ii) $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 1$.
Then, the sequence $\{t_n\}$ converges strongly to $\mathscr{P}_{\mathscr{F}} t_1$.

Proof. We set $t_n = r_n$, by the same proof of Theorem 3.2, then $\mathscr{P}_{\mathscr{F}} t_1$ is well defined and $\mathscr{C}_{n+1}$ is convex and closed. Let $\rho \in \mathscr{F}$. Since $\{t_n\}$ dominates ρ, we have $(t_n, \rho) \in E(\mathscr{G})$. By using the edge preservingness of $\mathscr{T}_1$ and convexity of $E(\mathscr{G})$, we have $(x_n, \rho) \in E(\mathscr{G})$. We also have $(\mathscr{T}_1 x_n, \rho) \in E(\mathscr{G})$ and $(\mathscr{T}_1 t_n, \rho) \in E(\mathscr{G})$ as $\mathscr{T}_1$ is edge-preserving. Again, by using the convexity of $E(\mathscr{G})$, we have $(y_n, \rho) \in E(\mathscr{G})$. We have $(y_n, \rho), (x_n, \rho) \in E(\mathscr{G})$ and using the edge preservingness of $\mathscr{T}_1$ and $\mathscr{T}_2$ and convexity of $E(\mathscr{G})$, we have $(z_n, \rho) \in E(\mathscr{G})$.

$$\begin{aligned} d(z_n,\rho) &= d((1-\zeta_n)\mathscr{T}_2 x_n \oplus \zeta_n \mathscr{T}_1 y_n, \rho) \\ &\leq (1-\zeta_n) d(\mathscr{T}_2 x_n, \rho) + \zeta_n d(\mathscr{T}_1 y_n, \rho) \\ &\leq (1-\zeta_n) d(x_n, \rho) + \zeta_n d(y_n, \rho) \\ &\leq (1-\zeta_n)[d((1-\varsigma_n)t_n \oplus \varsigma_n \mathscr{T}_1 t_n, \rho)] + \zeta_n [d((1-\eta_n)\mathscr{T}_1 t_n \oplus \eta_n \mathscr{T}_1 x_n, \rho)] \\ &\leq (1-\zeta_n)(1-\varsigma_n) d(t_n,\rho) + (1-\zeta_n)\varsigma_n d(\mathscr{T}_1 t_n, \rho) + \zeta_n (1-\eta_n) d(\mathscr{T}_1 t_n, \rho) + \zeta_n \eta_n d(\mathscr{T}_1 x_n, \rho) \\ &\leq (1-\zeta_n) d(t_n,\rho) + \zeta_n (1-\eta_n) d(t_n,\rho) + \zeta_n \eta_n d((1-\varsigma_n)t_n \oplus \varsigma_n \mathscr{T}_1 t_n, \rho) \\ &\leq (1-\zeta_n) d(t_n,\rho) + \zeta_n (1-\eta_n) d(t_n,\rho) + \zeta_n \eta_n (1-\varsigma_n) d(t_n,\rho) + \zeta_n \eta_n \varsigma_n d(\mathscr{T}_1 t_n, \rho) \\ &\leq (1-\zeta_n) d(t_n,\rho) + \zeta_n (1-\eta_n) d(t_n,\rho) + \zeta_n \eta_n (1-\varsigma_n) d(t_n,\rho) + \zeta_n \eta_n \varsigma_n d(t_n,\rho) \\ &\leq ((1-\zeta_n) + \zeta_n(1-\eta_n) + \zeta_n \eta_n (1-\varsigma_n) + \zeta_n \eta_n \varsigma_n) d(t_n,\rho) \\ &\leq d(t_n,\rho). \end{aligned}$$

By definition of $\mathscr{C}_{n+1}$, we have $\rho \in \mathscr{C}_{n+1}$. Thus $\mathscr{F} \subset \mathscr{C}_{n+1}$. This implies that $\mathscr{P}_{\mathscr{C}_{n+1}} t_1$ is well defined. Next, we show that $\lim_{n\to\infty} d(t_n, t_1)$ exists. Since $\mathscr{F}$ is non-empty convex closed subset of $\mathscr{X}$, there exists a unique $v \in \mathscr{F}$ such that $v = \mathscr{P}_{\mathscr{F}} t_1$. By $t_n = \mathscr{P}_{\mathscr{C}_n} t_1$ and $t_{n+1} \in \mathscr{C}_n$, for all $n \in \mathbb{N}$,

$$d(t_n, t_1) \leq d(t_{n+1}, t_1). \tag{3.12}$$

As we know that $\mathscr{F} \subset \mathscr{C}_n$, we obtain

$$d(t_n, t_1) \leq d(v, t_1). \tag{3.13}$$

By using (3.12) and (3.13), we get that the sequence $\{t_n\}$ is non-decreasing and bounded. Therefore $\lim_{n\to\infty} d(t_n, t_1)$ exists.

To show that $t_n \to w \in \mathscr{C}$ as $n \to \infty$. For $m > n$, we have $t_m = \mathscr{P}_{\mathscr{C}_m} t_1 \in \mathscr{C}_m \subset \mathscr{C}_n$, by definition of $\mathscr{C}_n$. We have

$$\langle \overrightarrow{t_1 \mathscr{P}_{\mathscr{C}_m} t_1}, \overrightarrow{\mathscr{P}_{\mathscr{C}_m} t_1 t_n} \rangle \geq 0,$$
$$d^2(t_m, t_n) \leq d^2(t_m, t_1) - d^2(t_n, t_1).$$

Since $\lim_{n\to\infty} d(t_n, t_1)$ exists, we obtain that $\{t_n\}$ is a Cauchy sequence. This implies that there exists $w \in \mathscr{C}$ such that $t_n \to w \in \mathscr{C}$ as $n \to \infty$. We also have

$$\lim_{n\to\infty} d(t_{n+1}, t_n) = 0. \tag{3.14}$$

Next, we show that $w \in \mathscr{F}$. Since $t_{n+1} \in \mathscr{C}_{n+1} \subset \mathscr{C}_n$. From (3.14), we have

$$\begin{aligned} d(z_n, t_n) &\leq d(z_n, t_{n+1}) + d(t_{n+1}, t_n) \\ &\leq 2d(t_{n+1}, t_n) \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.15}$$

Since $\{t_n\}$ dominates $\rho \in \mathscr{F}$ and using (2.2), we have

$$\begin{aligned} d^2(z_n, \rho) &= d^2((1-\zeta_n)\mathscr{T}_2 x_n \oplus \zeta_n \mathscr{T}_1 y_n, \rho) \\ &\leq (1-\zeta_n) d^2(\mathscr{T}_2 x_n, \rho) + \zeta_n d^2(\mathscr{T}_1 y_n, \rho) - \zeta_n (1-\zeta_n) d^2(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) \\ &\leq (1-\zeta_n) d^2(x_n, \rho) + \zeta_n d^2(y_n, \rho) \\ &\leq (1-\zeta_n) d^2((1-\varsigma_n) t_n \oplus \varsigma_n \mathscr{T}_1 t_n, \rho) + \zeta_n d^2((1-\eta_n)\mathscr{T}_1 t_n \oplus \eta_n \mathscr{T}_1 x_n, \rho) \\ &\leq (1-\zeta_n)[d^2(t_n, \rho) - \varsigma_n(1-\varsigma_n) d^2(t_n, \mathscr{T}_1 t_n)] + \\ &\quad \zeta_n[(1-\eta_n) d^2(t_n, \rho) + \eta_n d^2((1-\varsigma_n) t_n \oplus \varsigma_n \mathscr{T}_1 t_n, \rho)] \\ &\leq (1-\zeta_n)[d^2(t_n, \rho) - \varsigma_n(1-\varsigma_n) d^2(t_n, \mathscr{T}_1 t_n)] + \\ &\quad \zeta_n[(1-\eta_n) d^2(t_n, \rho) + \eta_n (1-\varsigma_n) d^2(t_n, \rho) + \varsigma_n d^2(\mathscr{T}_1 t_n, \rho)] \\ &\leq (1-\zeta_n)[d^2(t_n, \rho) - \varsigma_n(1-\varsigma_n) d^2(t_n, \mathscr{T}_1 t_n)] \\ &\quad + \zeta_n[(1-\eta_n) d^2(t_n, \rho) + \eta_n d^2(t_n, \rho)] \\ &\leq (1-\zeta_n)[d^2(t_n, \rho) - \varsigma_n(1-\varsigma_n) d^2(t_n, \mathscr{T}_1 t_n)] + \zeta_n[d^2(t_n, \rho)] \\ &\leq d^2(t_n, \rho) - \varsigma_n(1-\varsigma_n)(1-\zeta_n) d^2(t_n, \mathscr{T}_1 t_n) \end{aligned}$$

and

$$\begin{aligned} d^2(z_n, \rho) &= d^2((1-\zeta_n)\mathscr{T}_2 x_n \oplus \zeta_n \mathscr{T}_1 y_n, \rho) \\ &\leq (1-\zeta_n) d^2(\mathscr{T}_2 x_n, \rho) + \zeta_n d^2(\mathscr{T}_1 y_n, \rho) - \zeta_n(1-\zeta_n) d^2(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) \\ &\leq (1-\zeta_n) d^2(x_n, \rho) + \zeta_n d^2(y_n, \rho) - \zeta_n(1-\zeta_n) d^2(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) \\ &\leq d^2(t_n, \rho) - \zeta_n(1-\zeta_n) d^2(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n). \end{aligned}$$

This implies that

$$\varsigma_n(1-\varsigma_n)(1-\zeta_n) d^2(t_n, \mathscr{T}_1 t_n) \leq d^2(t_n, \rho) - d^2(z_n, \rho)$$

and

$$\zeta_n(1-\zeta_n) d^2(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) \leq d^2(t_n, \rho) - d^2(z_n, \rho).$$

From our assumption and (3.15), we have

$$\lim_{n\to\infty} d(\mathscr{T}_1 t_n, t_n) = 0 \tag{3.16}$$

and

$$\lim_{n\to\infty} d(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) = 0. \tag{3.17}$$

It follows from (3.16)

$$\begin{aligned} d(x_n,t_n) &\leq d((1-\varsigma_n)t_n \oplus \varsigma_n \mathscr{T}_1 t_n, t_n) \\ &\leq \varsigma_n d(\mathscr{T}_1 t_n, t_n) \\ &\to 0 \; (as\ n \to \infty). \end{aligned} \tag{3.18}$$

From (3.16) and (3.18)

$$\begin{aligned} d(y_n,t_n) &\leq d((1-\eta_n)\mathscr{T}_1 t_n \oplus \eta_n \mathscr{T}_1 x_n, t_n) \\ &\leq (1-\eta_n)d(\mathscr{T}_1 t_n, t_n) + \eta_n d(\mathscr{T}_1 x_n, t_n) \\ &\leq (1-\eta_n)d(\mathscr{T}_1 t_n, t_n) + \eta_n d(\mathscr{T}_1 x_n, \mathscr{T}_1 t_n) + \eta_n d(\mathscr{T}_1 t_n, t_n) \\ &\leq (1-\eta_n)d(\mathscr{T}_1 t_n, t_n) + \eta_n d(x_n, t_n) + \eta_n d(\mathscr{T}_1 t_n, t_n) \\ &\to 0 \; (as\ n \to \infty). \end{aligned} \tag{3.19}$$

From (3.16), (3.17), (3.18) and (3.19) we have

$$\begin{aligned} d(\mathscr{T}_2 t_n, t_n) &\leq d(\mathscr{T}_2 t_n, \mathscr{T}_2 x_n) + d(\mathscr{T}_2 x_n, t_n) \\ &\leq d(t_n, x_n) + d(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) + d(\mathscr{T}_1 y_n, t_n) \\ &\leq d(t_n, x_n) + d(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) + d(\mathscr{T}_1 y_n, \mathscr{T}_1 t_n) + d(\mathscr{T}_1 t_n, t_n) \\ &\leq d(t_n, x_n) + d(\mathscr{T}_2 x_n, \mathscr{T}_1 y_n) + d(y_n, t_n) + d(\mathscr{T}_1 t_n, t_n) \\ &\to 0 \; (as\ n \to \infty). \end{aligned} \tag{3.20}$$

From (3.16), (3.20) and Theorem 3.1, we have $w \in \mathscr{F}$.
Next, we show that $w = v = \mathscr{P}_{\mathscr{F}} t_1$. Since $t_n = \mathscr{P}_{\mathscr{C}_n} t_1$, we have

$$\langle \overrightarrow{t_1 \mathscr{P}_{\mathscr{C}_n} t_1}, \overrightarrow{\mathscr{P}_{\mathscr{C}_n} t_1 \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n,$$

$$\langle \overrightarrow{t_1 t_n}, \overrightarrow{t_n \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n. \tag{3.21}$$

Taking limit in (3.21), we obtain

$$\langle \overrightarrow{t_1 w}, \overrightarrow{w \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n. \tag{3.22}$$

Since $\mathscr{F} \subset \mathscr{C}_n$, so $w = \mathscr{P}_{\mathscr{F}} t_1$. □

Theorem 3.4. *Let $\mathscr{X}$, $\mathscr{C}$, $\mathscr{T}_1$ and $\mathscr{T}_2$ be defined as in Theorem 3.2. Let $\mathscr{G} = (V(\mathscr{G}), E(\mathscr{G}))$ be a directed graph such that $V(\mathscr{G}) = \mathscr{C}$, $E(\mathscr{G})$ is convex and $\mathscr{F} = \mathscr{F}(\mathscr{T}_1) \cap \mathscr{F}(\mathscr{T}_2) \neq \emptyset$, $\mathscr{F}$ is closed and $\mathscr{F}(\mathscr{T}_1) \times \mathscr{F}(\mathscr{T}_2) \subseteq E(\mathscr{G})$. Let the sequence $\{s_n\}$ be generated by $s_1 \in \mathscr{C}$ with $\mathscr{C}_1 = \mathscr{C}$,*

$$\begin{aligned} &x_n = (1-\varsigma_n)s_n \oplus \varsigma_n \mathscr{T}_1 s_n, \\ &y_n = (1-\eta_n)\mathscr{T}_1 s_n \oplus \eta_n \mathscr{T}_1 x_n, \\ &z_n = (1-\zeta_n)\mathscr{T}_2 s_n \oplus \zeta_n \mathscr{T}_1 y_n, \\ &\mathscr{C}_{n+1} = \{z \in \mathscr{C}_n : d(z_n, z) \leq d(s_n, z)\}, \\ &s_{n+1} = \mathscr{P}_{\mathscr{C}_{n+1}} s_1, n \geq 1, \end{aligned} \tag{iii}$$

where $\{\zeta_n\}$, $\{\eta_n\}$, $\{\varsigma_n\} \subset (0,1)$. Suppose that the following conditions are satisfied:
(i) $\{s_n\}$ dominates ρ for all $\rho \in \mathscr{F}$ and if there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\{s_{n_k}\}$ $\Delta-$converges to $w \in \mathscr{C}$, then $(s_{n_k}, w) \in E(\mathscr{G})$;
(ii) $\liminf_{n\to\infty} \eta_n > 0$;
(iii) $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 1$.
Then, the sequence $\{s_n\}$ converges strongly to $\mathscr{P}_{\mathscr{F}} s_1$.

Proof. We set $s_n = r_n$, by the same proof of Theorem 3.1, then $\mathscr{P}_{\mathscr{F}} s_1$ is well defined and $\mathscr{C}_{n+1}$ is convex and closed. Let $\rho \in \mathscr{F}$. Since $\{s_n\}$ dominates ρ, we have $(s_n, \rho) \in E(\mathscr{G})$. Since $\mathscr{T}_1$ is edge-preserving and $E(\mathscr{G})$ is convex, we get $(x_n, \rho) \in E(\mathscr{G})$. We also have $(\mathscr{T}_1 x_n, \rho) \in E(\mathscr{G})$ and $(\mathscr{T}_1 s_n, \rho) \in E(\mathscr{G})$ as $\mathscr{T}_1$ is edge-preserving. Since $E(\mathscr{G})$ is convex, we get $(y_n,\rho)\in E(\mathscr{G})$. We have $(s_n,\rho),(y_n,\rho)\in E(\mathscr{G})$ and using the edge preservingness of $\mathscr{T}_1$ and $\mathscr{T}_2$ and using the convexity property of $E(\mathscr{G})$, we have $(z_n,\rho)\in E(\mathscr{G})$.

$$\begin{array}{rcl} d(z_n,\rho) & = & d((1-\zeta_n)\mathscr{T}_2 s_n \oplus \zeta_n \mathscr{T}_1 y_n,\rho)\\ & \leq & (1-\zeta_n)d(\mathscr{T}_2 s_n,\rho)+\zeta_n d(\mathscr{T}_1 y_n,\rho)\\ & \leq & (1-\zeta_n)d(s_n,\rho)+\zeta_n d(y_n,\rho)\\ & \leq & (1-\zeta_n)d(s_n,\rho)+\zeta_n[d((1-\eta_n)\mathscr{T}_1 s_n\oplus \eta_n\mathscr{T}_1 x_n,\rho)]\\ & \leq & (1-\zeta_n)d(s_n,\rho)+\zeta_n(1-\eta_n)d(\mathscr{T}_1 s_n,\rho)+\zeta_n\eta_n d(\mathscr{T}_1 x_n,\rho)\\ & \leq & (1-\zeta_n)d(s_n,\rho)+\zeta_n(1-\eta_n)d(s_n,\rho)+\zeta_n\eta_n d((1-\varsigma_n)s_n\oplus\varsigma_n\mathscr{T}_1 s_n,\rho)\\ & \leq & (1-\zeta_n)d(s_n,\rho)+\zeta_n(1-\eta_n)d(s_n,\rho)+\zeta_n\eta_n(1-\varsigma_n)d(s_n,\rho)+\zeta_n\eta_n\varsigma_n d(\mathscr{T}_1 s_n,\rho)\\ & \leq & (1-\zeta_n)d(s_n,\rho)+\zeta_n(1-\eta_n)d(s_n,\rho)+\zeta_n\eta_n(1-\varsigma_n)d(s_n,\rho)+\zeta_n\eta_n\varsigma_n d(s_n,\rho)\\ & \leq & ((1-\zeta_n)+\zeta_n(1-\eta_n)+\zeta_n\eta_n(1-\varsigma_n)+\zeta_n\eta_n\varsigma_n)d(s_n,\rho)\\ & \leq & d(s_n,\rho). \end{array}$$

By definition of $\mathscr{C}_{n+1}$, we have $\rho\in\mathscr{C}_{n+1}$. Thus $\mathscr{F}\subset\mathscr{C}_{n+1}$. This implies that $\mathscr{P}_{\mathscr{C}_{n+1}}s_1$ is well defined. In the next step we show that $\lim_{n\to\infty} d(s_n,s_1)$ exists. We know that $\mathscr{F}$ is non-empty convex closed subset of $\mathscr{X}$; there exists a unique $v\in\mathscr{F}$ such that $v=\mathscr{P}_{\mathscr{F}}s_1$. By $s_n=\mathscr{P}_{\mathscr{C}_n}s_1$ and $s_{n+1}\in\mathscr{C}_n$, for all $n\in\mathbb{N}$,

$$d(s_n,s_1)\leq d(s_{n+1},s_1). \tag{3.23}$$

As we know that $\mathscr{F}\subset\mathscr{C}_n$, we obtain

$$d(s_n,s_1)\leq d(v,s_1). \tag{3.24}$$

It follows from (3.23) and (3.24) that the sequence $\{s_n\}$ is non-decreasing and bounded. This implies that $\lim_{n\to\infty} d(s_n,s_1)$ exists.

To show that $s_n\to w\in\mathscr{C}$ as $n\to\infty$. For $m>n$, we have $s_m=\mathscr{P}_{\mathscr{C}_m}s_1\in\mathscr{C}_m\subset\mathscr{C}_n$, by definition of $\mathscr{C}_n$. We have

$$\langle \overrightarrow{s_1\mathscr{P}_{\mathscr{C}_m}s_1}, \overrightarrow{\mathscr{P}_{\mathscr{C}_m}s_1 s_n}\rangle \geq 0,$$
$$d^2(s_m,s_n)\leq d^2(s_m,s_1)-d^2(s_n,s_1).$$

Since $\lim_{n\to\infty} d(s_n,s_1)$ exists, we obtain that $\{s_n\}$ is a Cauchy sequence. This implies that there exists $w\in\mathscr{C}$ such that $s_n\to w\in\mathscr{C}$ as $n\to\infty$. We also get

$$\lim_{n\to\infty} d(s_{n+1},s_n)=0. \tag{3.25}$$

Next, we will show that $w\in\mathscr{F}$. Because $s_{n+1}\in\mathscr{C}_{n+1}\subset\mathscr{C}_n$. From (3.25), we have

$$\begin{array}{rcl} d(z_n,s_n) & \leq & d(z_n,s_{n+1})+d(s_{n+1},s_n)\\ & \leq & 2\,d(s_{n+1},s_n)\\ & \to & 0\ (as\ n\to\infty). \end{array} \tag{3.26}$$

Since $\{s_n\}$ dominates $\rho\in\mathscr{F}$ and using (2.2), we have

$$\begin{array}{rl} d^2(z_n,\rho) & = d^2((1-\zeta_n)\mathscr{T}_2 s_n\oplus\zeta_n\mathscr{T}_1 y_n,\rho)\\ & \leq (1-\zeta_n)d^2(\mathscr{T}_2 s_n,\rho)+\zeta_n d^2(\mathscr{T}_1 y_n,\rho)-\zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 s_n,\mathscr{T}_1 y_n)\\ & \leq (1-\zeta_n)d^2(s_n,\rho)+\zeta_n d^2(y_n,\rho)\\ & \leq (1-\zeta_n)d^2(s_n,\rho)+\zeta_n d^2((1-\eta_n)\mathscr{T}_1 s_n\oplus\eta_n\mathscr{T}_1 x_n,\rho)\\ & \leq (1-\zeta_n)d^2(s_n,\rho)+\zeta_n[(1-\eta_n)d^2(s_n,\rho)+\eta_n d^2((1-\varsigma_n)s_n\oplus\varsigma_n\mathscr{T}_1 s_n,\rho)]\\ & \leq (1-\zeta_n)d^2(s_n,\rho)+\zeta_n[(1-\eta_n)d^2(s_n,\rho)+\eta_n(1-\varsigma_n)d^2(s_n,\rho)+\\ & \varsigma_n\eta_n d^2(\mathscr{T}_1 s_n,\rho)+\eta_n\varsigma_n(1-\varsigma_n)d^2(s_n,\mathscr{T}_1 s_n)]\\ & \leq (1-\zeta_n)d^2(s_n,\rho)+\zeta_n[(1-\eta_n)d^2(s_n,\rho)+\eta_n d^2(s_n,\rho)+\eta_n\varsigma_n(1-\varsigma_n)d^2(s_n,\mathscr{T}_1 s_n)]\\ & \leq d^2(s_n,\rho)-\zeta_n\eta_n\varsigma_n(1-\varsigma_n)d^2(s_n,\mathscr{T}_1 s_n) \end{array}$$

and

$$\begin{array}{rcl} d^2(z_n,\rho) & = & d^2((1-\zeta_n)\mathscr{T}_2 s_n \oplus \zeta_n \mathscr{T}_1 y_n, \rho) \\ & \leq & (1-\zeta_n)d^2(\mathscr{T}_2 s_n,\rho) + \zeta_n d^2(\mathscr{T}_1 y_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n) \\ & \leq & (1-\zeta_n)d^2(s_n,\rho) + \zeta_n d^2(y_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n) \\ & \leq & d^2(s_n,\rho) - \zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n). \end{array}$$

This implies that

$$\zeta_n \eta_n \varsigma_n (1-\varsigma_n) d^2(s_n, \mathscr{T}_1 s_n) \leq d^2(s_n,\rho) - d^2(z_n,\rho)$$

and

$$\zeta_n(1-\zeta_n)d^2(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n) \leq d^2(s_n,\rho) - d^2(z_n,\rho).$$

From our assumption and (3.26), we obtain

$$\lim_{n\to\infty} d(\mathscr{T}_1 s_n, s_n) = 0 \tag{3.27}$$

and

$$\lim_{n\to\infty} d(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n) = 0. \tag{3.28}$$

It follows from (3.27) that

$$\begin{array}{rcl} d(x_n,s_n) & \leq & d((1-\varsigma_n)t_n \oplus \varsigma_n \mathscr{T}_1 s_n, s_n) \\ & \leq & \varsigma_n d(\mathscr{T}_1 s_n, s_n) \\ & \to & 0 \ (as\ n\to\infty). \end{array} \tag{3.29}$$

It follows from (3.27) and (3.29) that

$$\begin{array}{rcl} d(y_n,s_n) & \leq & d((1-\eta_n)\mathscr{T}_1 s_n \oplus \eta_n \mathscr{T}_1 x_n, s_n) \\ & \leq & (1-\eta_n)d(\mathscr{T}_1 s_n, s_n) + \eta_n d(\mathscr{T}_1 x_n, s_n) \\ & \leq & (1-\eta_n)d(\mathscr{T}_1 s_n, s_n) + \eta_n d(\mathscr{T}_1 x_n, \mathscr{T}_1 s_n) + \eta_n d(\mathscr{T}_1 s_n, s_n) \\ & \leq & (1-\eta_n)d(\mathscr{T}_1 s_n, s_n) + \eta_n d(x_n, s_n) + \eta_n d(\mathscr{T}_1 s_n, s_n) \\ & \to & 0 \ (as\ n\to\infty). \end{array} \tag{3.30}$$

From (3.27), (3.28) and (3.30), we have

$$\begin{array}{rcl} d(\mathscr{T}_2 s_n, s_n) & \leq & d(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n) + d(\mathscr{T}_1 y_n, \mathscr{T}_1 s_n) + d(\mathscr{T}_1 s_n, s_n) \\ & \leq & d(\mathscr{T}_2 s_n, \mathscr{T}_1 y_n) + d(y_n, s_n) + d(\mathscr{T}_1 s_n, s_n) \\ & \to & 0 \ (as\ n\to\infty). \end{array} \tag{3.31}$$

From (3.27), (3.31) and Theorem 3.1, we get $w \in \mathscr{F}$.
Next, we show that $w = v = \mathscr{P}_{\mathscr{F}} s_1$. Since $s_n = \mathscr{P}_{\mathscr{C}_n} s_1$, we have

$$\langle \overrightarrow{s_1 \mathscr{P}_{\mathscr{C}_n} s_1}, \overrightarrow{\mathscr{P}_{\mathscr{C}_n} s_1 \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n,$$

$$\langle \overrightarrow{s_1 s_n}, \overrightarrow{s_n \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n. \tag{3.32}$$

Taking limit in (3.32), we obtain

$$\langle \overrightarrow{s_1 w}, \overrightarrow{w \rho} \rangle \geq 0, \forall \rho \in \mathscr{C}_n. \tag{3.33}$$

Since $\mathscr{F} \subset \mathscr{C}_n$, so $w = \mathscr{P}_{\mathscr{F}} s_1$. □

4. NUMERICAL EXAMPLE

This section includes an example which assists our main theorems.
Example. Let $\mathscr{X} = \mathbb{R}^3$ and $\mathscr{C} = \mathbb{R} \times \mathbb{R} \times [0,2]$. Let $(x,y) \in E(\mathscr{G})$ iff $x_1, y_1 \leq 1.5$, $-1 \leq x_2, y_2 \leq 1$ and $.25 \leq x_3, y_3 \leq 1.75$ or $x = y$ for all $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in \mathscr{C}$. Let mappings $\mathscr{T}_1, \mathscr{T}_2 : \mathscr{C} \to \mathscr{C}$ be defined by

$$\mathscr{T}_1 x = \left(\frac{10^{x_1-1}+9}{10}, 0, \frac{\sin^{-1}(x_2-1)}{2} + 1 \right)$$

and

$$\mathscr{T}_2 x = \left(1, \frac{\tan x_2}{4}, 1 \right)$$

for all $x = (x_1,x_2,x_3) \in \mathscr{C}$.

Clearly, $\mathscr{T}_1$ and $\mathscr{T}_2$ are two mappings such that $\mathscr{F}(\mathscr{T}) = \{(1,0,1)\}$. On the flip side, by taking $x = (2,1,0.18)$ and $y = (1.44,1,0.06)$, we see that $\mathscr{T}_1$ is not nonexpansive, $\|\mathscr{T}_1 x - \mathscr{T}_1 y\| > 0.7 > \|x-y\|$ and $\mathscr{T}_2$ is not nonexpansive by taking $x = (3,1.11,2)$ and $y = (3,1.03,2)$ as $\|\mathscr{T}_2 x - \mathscr{T}_2 y\| > 0.084 > \|x-y\|$.

It is clear that $\rho = (1,0,1)$ is a point of $\mathscr{F}$. Choosing the values of $\zeta_n, \eta_n, \varsigma_n = \frac{3n}{4n+1}$ and stopping criteria as $\|x_{n+1} - x_n\| < 10^{-9}$. By reckoning, we get the sequences $\{x_n\}$ generated by Theorems 3.2–3.4 converge to the point $(1,0,1)$ of $\mathscr{F}$ for Choice 1: $x_1 = (-1,1,1.95)$, Choice 2: $x_1 = (-10,1,.95)$ and Choice 3: $x_1 = (-9,-0.9,1.95)$. In Tables 1 and 2, comparison of the convergence rate of three modified iterative schemes has been shown. We see that $\rho = (1,0,1)$ is a point of $\mathscr{F}$. By reckoning, we get the sequences $\{x_n\}$ generated by Theorems 3.2–3.4 converge to the point $(1,0,1)$ of $\mathscr{F}$.

Remark.
(1) From figures and tables, it is clear that modified method (iii) (defined in Theorem 3.4) has better convergence rate and needs lower number of iterations than the other two modified methods (i) and (ii) for three different choices.
(2) The sequences $\zeta_n, \eta_n, \varsigma_n$ are not the optimized parameters, they are fixed for three algorithms *(i)–(iii)* for comparison.
(3) The results in the Tables 1–2 depend on mappings $\mathscr{T}_1, \mathscr{T}_2$ and the initialization x_1.

Table 1. For Choices 1 and 2, comparison for the sequences defined in Theorems 3.2–3.4

Iter.	Choice 1: No. of iter.	(–1, 1, 1.95) Time taken by CPU (sec)	Choice 2: No. of iter.	(–10, 1, .95) CPU Time (sec)
(i)	187	.0089	202	.0089
(ii)	63	.0029	69	.0031
(iii)	32	.0080	12	.0024

Table 2. For Choice 3, comparison for the sequences defined in Theorems 3.2–3.4

Iter.	Choice 3: No. of iter.	(–9, –0.9, 1.95) Time taken by CPU (sec)
(i)	200	.0101
(ii)	97	.0048
(iii)	11	.0027

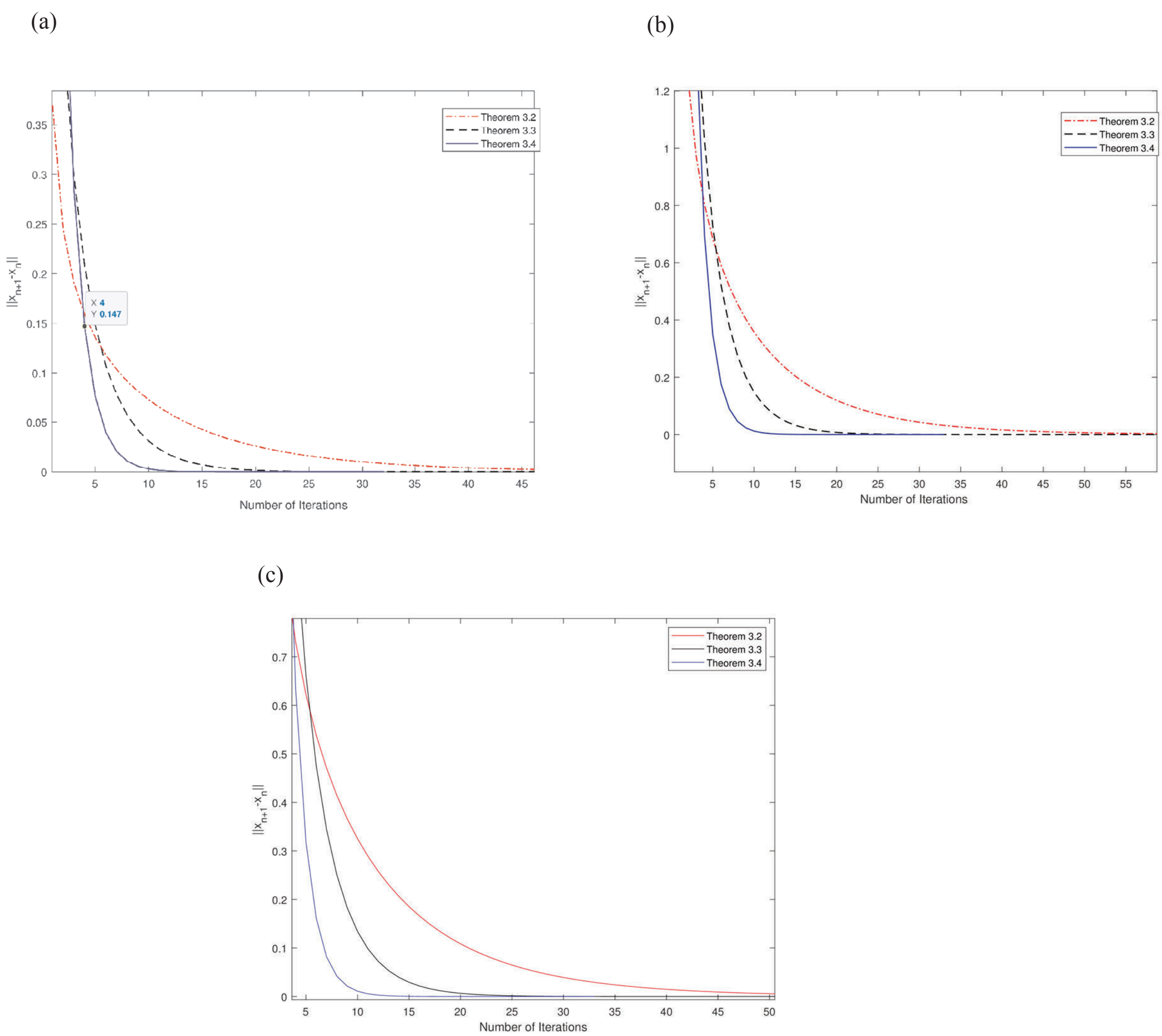

Fig. 1. Error plots for first (a), second (b) and third (c) choice.

5. CONCLUSIONS

We introduced three iterative schemes by modifying shrinking projection method involving $\mathscr{G}$-nonexpansive mappings. We proved convergence theorems to obtain common fixed points of $\mathscr{G}$-nonexpansive mappings and constructed a numerical example which supports our main results and comparison of among three iterative schemes has been shown.

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REFERENCES

1. Alfuraidan, M. R. Fixed points of monotone nonexpansive mappings with a graph. *Fixed Point Theory Appl*., 2015, **49**. https://doi.org/10.1186/s13663-015-0299-0
2. Alfuraidan, M. R. and Khamsi, M. A. Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph. *Fixed Point Theory Appl*., 2015, **44**. https://doi.org/10.1186/s13663-015-0294-5
3. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux quations intgrales. *Fundam. Math*., 1922, **3**(1), 133–181.
4. Berg, I. D. and Nikolaev, I. G. Quasilinearization and curvature of Aleksandrov spaces. *Geom. Dedicata*, 2008, **133**, 195–218. https://doi.org/10.1007/s10711-008-9243-3
5. Bridson, M. R. and Haefliger, A. *Metric Spaces of Non-positive Curvature*. Springer–Verlag, Berlin 1999.
6. Dehghan, H. and Rooin, J. Metric projection and convergence theorems for nonexpansive mappings in Hadamard spaces. *J. Nonlinear Convex Anal*., 2013. https://doi.org/10.48550/arXiv.1410.1137
7. Hammad, H. A., Cholamjiak, W. and Yambangwai, D. A Modified shrinking projection method for numerical reckoning fixed points of G-nonexpansive mappings in Hilbert spaces with graphs. *Miskolc Math. Notes*, 2019, **20**(2), 941–956. https://doi.org/10.18514/MMN.2019.2954
8. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math., Soc*. 2008, **136**(4), 1359–1373.
9. Khatibzadeh, H. and Ranjbar, S. Monotone operators and the proximal point algorithm in complete CAT(0) metric spaces. *J. Aust. Math. Soc*., 2017, **103**(1), 70–90
10. Kirk, W. A. Geodesic geometry and fixed point theory. In *Proceedings of the Seminar of Mathematical Analysis, Malaga/Seville, 2002/2003*. Seville University Publications, 2003, **64**, 195–225.
11. Kirk, W. A. and Panyanak, B. A concept of convergence in geodesic spaces. *Nonlinear Anal*., 2008, **68**(12), 3689–3696.
12. Lim, T. C. Remarks on some fixed point theorems. *Proc. Am. Math. Soc*., 1976, **60**, 179–182.
13. Suparatulatorn, R., Cholamjiak, W. and Suantai, S. A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs. *Numer. Algorithms*, 2018, **77**, 479–490.
14. Takahashi, W., Takeuchi, Y. and Kubota, R. Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl*., 2008, **341**(1), 276–286.
15. Thianwan, T. and Yambangwai, D. Convergence analysis for a new two-step iteration process for G-nonexpansive mappings with directed graphs. *J. Fixed Point Theory Appl*., 2019, **21**(44). https://doi.org/10.1007/s11784-019-0681-3
16. Tiammee, J., Kaewkhao, A. and Suantai, S. On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs. *Fixed Point Theory Appl*., 2015, **187**. https://doi.org/10.1186/s13663-015-0436-9
17. Tripak, O. Common fixed points of G-nonexpansive mappings on Banach spaces with a graph. *Fixed Point Theory Appl*., 2016, **87**. https://doi.org/10.1186/s13663-016-0578-4
18. Yambangwai, D. and Thianwan, T. $\triangle$-Convergence and Strong Convergence for Asymptotically Nonexpansive Mappings on a CAT(0) Space. *Thai J. Math*., 2021, **19**(3), 813–826.
19. Yambangwai, D. and Thianwan, T. Convergence point of G-nonexpansive mappings in Banach spaces endowed with graphs applicable in image deblurring and signal recovering problems. *Ric. di mat*., 2021. https://doi.org/10.1007/s11587-021-00631-y
20. Yambangwai, D. and Thianwan, T. An efficient iterative algorithm for common solutions of three G-nonexpansive mappings in Banach spaces involving a graph with applications to signal and image restoration problems. *AIMS Mathematics*, 2022, **7**(1), 1366–1398. https://doi.org/10.3934/math.2022081
21. Yambangwai, D., Aunruean, S. and Thianwan, T. A new modified three-step iteration method for G-nonexpansive mappings in Banach spaces with a graph. *Numer. Algorithms*, 2019, **84**(2), 437–565. https://doi.org/10.1007/s11075-019-00768-w