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Colimits in the category Seg of Segal topological algebras

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Abstract. In this paper we find sufficient conditions for a direct system of Segal topological algebras to have a colimit in the category **Seg** of Segal topological algebras.

Keywords: Segal topological algebra, category, direct system, colimit.

1. INTRODUCTION AND MOTIVATION

It is a known result in the category theory that the category is cocomplete (i.e., all colimits exist) if and only if all coequalizers and all coproducts exist in this category. Usually this result is not proved in the category theory books. Instead, the authors prove a dual statement that the category is complete (i.e., all limits exist) if and only if all equalizers and all products exist in this category. The proof of this result is usually given quite schematically. If one takes a closer look at the proof of this result (or its dual result about limits, products and equalizers), then one can detect that only two particular coproducts and only one particular coequalizer is constructed in order to obtain the colimit of a particular direct system. Thus, in case one is interested in the existence or the description of only one particular colimit, then one does not have to demand that the category should be cocomplete. In this paper (see Theorem 1) we describe the sufficient conditions for the existence of a colimit of a particular direct system in the category and apply the obtained result in order to obtain some more complicated sufficient conditions for the existence of a colimit of a fixed direct system in the category **Seg** of Segal topological algebras.

The motivation for proving the category-theoretical result originates from the study of the category **Seg** of all Segal topological algebras (see below for the definition of a Segal topological algebra). For this category, it is known that all coequalizers exist (see [4]) but for the existence of the coproduct of a family of Segal topological algebras only sufficient conditions are known, which do not seem to guarantee the existence of coproducts of all families of Segal topological algebras (see [5]). Hence, the best we can currently hope to achieve in the case of the category **Seg** are the sufficient conditions for the existence of a colimit of a particular direct system of Segal topological algebras.

One of the aims of this paper is also to provide very detailed proofs of the main results involved (Theorems 1 and 4).

2. PRELIMINARY DEFINITIONS AND RESULTS

Let C be any category. Denote by $\mathrm{Ob}(C)$ the collection of all objects of C and by $\mathrm{Mor}(C)$ the collection of all morphisms of C. For particular fixed objects $A, B \in \mathrm{Ob}(C)$, denote by $\mathrm{Mor}(A, B)$ the collection of all morphisms from A to B. As usual, we will denote the morphisms in the diagrams by arrows. Thus, $A \xrightarrow{f} B$ will denote $f \in \mathrm{Mor}(A, B)$ in the diagram.

Let $(I, \leq)$ be **a partially ordered set**. This means that I is a set and $\leq$ is a homogeneous, reflexive, antisymmetric and transitive relation over I. Throughout the whole paper, we use [6] as the main source for the definitions in the general category theory.

A direct system (indexed by a partially ordered set I) in the category C is an ordered pair $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$, where the following three conditions are fulfilled:

1) $A_i \in \mathrm{Ob}(C)$ for all $i \in I$ and $\phi^i_j \in \mathrm{Mor}(A_i, A_j)$ for all $i, j \in I$ with $i \leq j$;

2) ϕ^i_i is the identity morphism on A_i for each $i \in I$;

3) $\phi^j_k \circ \phi^i_j = \phi^i_k$ for all $i, j, k \in I$ with $i \leq j \leq k$.

Let $(I, \leq)$ be a partially ordered set and consider the Cartesian product set

$$I \times I = \{(i, j) : i, j \in I\}.$$

On this set, we will consider **the product** (partial) **order** $\leq_P$, defined by

$$(i, j) \leq_P (k, l) \ \text{ if and only if } \ i \leq k \ \text{ and } \ j \leq l.$$

It is easy to check that $\leq_P$ is a partial order on the Cartesian product $I \times I$.

We will be using the subset $K = \{(i, j) \in I \times I : i \leq j\}$ of the Cartesian product $I \times I$. Considering the restriction $\leq_K = \leq_P|_K$ of the product order on K, the set $(K, \leq_K)$ becomes also a partially ordered set.

Suppose that we have a direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$. For all $(i, j), (k, l) \in K$ with $(i, j) \leq_K (k, l)$, define $A_{(i,j)} = A_i$ and $\phi^{(i,j)}_{(k,l)} = \phi^i_k$. Then we obtain another direct system $\left((A_{(i,j)})_{(i,j)\in K}, \left(\phi^{(i,j)}_{(k,l)}\right)_{(i,j)\leq_K(k,l)}\right)$, indexed by the partially ordered set $(K, \leq_K)$. Indeed:

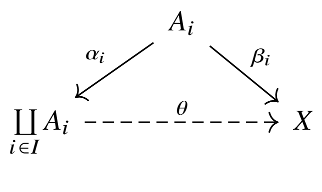
1) $A_{(i,j)} = A_i \in \mathrm{Ob}(C)$ and $\phi^{(i,j)}_{(k,l)} = \phi^i_k \in \mathrm{Mor}(A_{(i,j)}, A_{(k,l)})$ for all $(i, j), (k, l) \in K$ with $(i, j) \leq_k (k, l)$;

2) $\phi^{(i,j)}_{(i,j)} = \phi^i_i$ is the identity morphism on $A_{i,j} = A_i$ for each $(i, j) \in K$;

3) $\phi^{(k,l)}_{(m,n)} \circ \phi^{(i,j)}_{(k,l)} = \phi^k_m \circ \phi^i_k = \phi^i_m = \phi^{(i,j)}_{(m,n)}$ for all $(i, j), (k, l), (m, n) \in K$ with $(i, j) \leq_K (k, l) \leq_K (m, n)$.

In what follows, we will refer to the direct system $\left((A_{(i,j)})_{(i,j)\in K}, \left(\phi^{(i,j)}_{(k,l)}\right)_{(i,j)\leq_K(k,l)}\right)$ by the name **the direct system of the domains** of the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$.

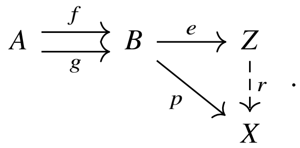
Let I be any set. Recall that **the coproduct of the family** $(A_i)_{i\in I}$ of objects of C is an ordered pair $\left(\coprod_{i\in I} A_i, (\alpha_i)_{i\in I}\right)$, where $\coprod_{i\in I} A_i \in \mathrm{Ob}(C)$ and $\alpha_i \in \mathrm{Mor}\left(A_i, \coprod_{i\in I} A_i\right)$ for each $i \in I$ such that for every $X \in \mathrm{Ob}(C)$ and every family of morphisms $(\beta_i \in \mathrm{Ob}(A_i, X))_{i\in I}$, there exists a unique morphism $\theta \in \mathrm{Mor}\left(\coprod_{i\in I} A_i, X\right)$, making the diagram



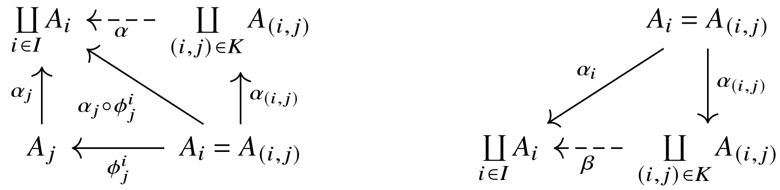
commutative for every $i \in I$.

Take any $A, B \in \mathrm{Ob}(C)$ and $f, g \in \mathrm{Mor}(A, B)$. Recall that **the coequalizer of the morphisms** f **and** g is the ordered pair (Z, e), where $Z \in \mathrm{Ob}(C)$ and $e \in \mathrm{Mor}(B, Z)$ are such that $ef = eg$ and for each $X \in \mathrm{Ob}(C)$

and $p \in \mathrm{Mor}(B,X)$ with $pf = pg$, there exists a unique $r \in \mathrm{Mor}(Z,X)$ such that $p = r \circ e$, i.e., the following diagram is commutative:



Suppose that C is such a category that the coproducts $\left(\coprod_{i\in I} A_i, (\alpha_i)_{i\in I}\right)$ and $\left(\coprod_{(i,j)\in K} A_{(i,j)}, (\alpha_{(i,j)})_{(i,j)\in K}\right)$ of a direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\le j})$ and its direct system of domains exist. Then we obtain the following diagrams



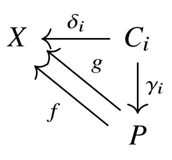
for all pairs $(i,j) \in K$. As $\left(\coprod_{(i,j)\in K} A_{i,j}, (\alpha_{i,j})_{(i,j)\in K}\right)$ is the coproduct, then there exists a unique morphism $\alpha \in \mathrm{Mor}\left(\coprod_{(i,j)\in K} A_{(i,j)}, \coprod_{i\in I} A_i\right)$ such that the first diagram becomes commutative. For similar reasons, there exists a unique morphism $\beta \in \mathrm{Mor}\left(\coprod_{(i,j)\in K} A_{(i,j)}, \coprod_{i\in I} A_i\right)$ such that the second diagram becomes commutative.

It is clear that if we have any direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\le j})$ in category C, then we will automatically obtain its direct system of domains. Moreover, if the coproducts of the direct system and its direct system of domains exist, then we will automatically obtain morphisms α and β. Thus, whenever the coproducts of the direct system and its direct system of domains exist, the morphisms α and β are uniquelly determined by the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\le j})$. Therefore, in order to shorten the text that will follow, we will call these morphisms α and β **morphisms induced by the direct system** $((A_i)_{i\in I}, (\phi^i_j)_{i\le j})$.

In what follows, we will need two results, the duals of which could be found with proofs in the books on the category theory. Since we need them in the "co"-situation and the books do not seem to have the proofs for this case, we provide hereby those two results with their complete proofs.

Lemma 1. *Let C be any category. If I is any set, $(C_i)_{i\in I}$ is a family of objects of the category C, $(P,(\gamma_i)_{i\in I})$ is the coproduct of the family $(C_i)_{i\in I}$, $X \in \mathrm{Ob}(C)$ and $f,g \in \mathrm{Mor}(P,X)$ such that $f\circ\gamma_i = g\circ\gamma_i$ for each $i \in I$, then $f = g$.*

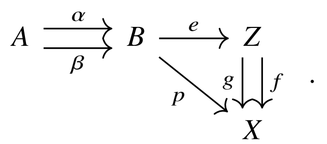
Proof. Denote $\delta_i = f\circ\gamma_i = g\circ\gamma_i$ for each $i \in I$. Notice that then we will obtain a commutative diagram



for each $i \in I$. By the definition of the coproduct, there exists a unique morphism $\theta \in \mathrm{Mor}(P,X)$ such that $\theta\circ\gamma_i = \delta_i$ for each $i \in I$. As h and k satisfy the same condition, then we must have $h = \theta = k$. □

Lemma 2. *Let C be any category, $A,B \in \mathrm{Ob}(C)$ and $\alpha,\beta \in \mathrm{Mor}(A,B)$. If (Z,e) is the coequalizer of the morphisms α and β, $X \in \mathrm{Ob}(C)$ and $f,g \in \mathrm{Mor}(Z,X)$ are such that $f\circ e = g\circ e$, then $f = g$.*

Proof. Denote $p = f \circ e = g \circ e$. Notice that we will obtain a commutative diagram



As (Z, e) is the coequalizer of the morphisms α and β, then $e \circ \alpha = e \circ \beta$. But then also

$$p \circ \alpha = (f \circ e) \circ \alpha = f \circ (e \circ \alpha) = f \circ (e \circ \beta) = (f \circ e) \circ \beta = p \circ \beta.$$

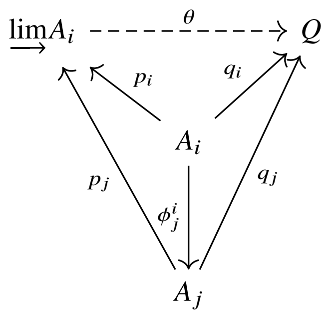
By the definition of the coequalizer, we know that in such a situation there exists exactly one morphism $r \in \mathrm{Mor}(Z, X)$ such that $r \circ e = p$. As f and g satisfy the same condition, then we must have $f = r = g$. □

Let us conclude this section by recalling the definition of a colimit of a direct system.

The **colimit** of a direct system $((A_i)_{i\in I}; (\phi^j_k)_{j\leq k})$ in C is the pair $(\varinjlim A_i; (p_i)_{i\in I})$, where $\varinjlim A_i$ is an object of C and $(p_j : A_j \to \varinjlim A_i)_{j\in I}$ is a collection of morphisms in C such that

(i) $p_j \circ \phi^i_j = p_i$ whenever $i \leq j$;

(ii) for every $Q \in \mathrm{Ob}(C)$ and morphisms $(q_i : A_i \to Q)_{i\in I}$, which satisfy $q_j \circ \phi^i_j = q_i$ whenever $i \leq j$, there exists a unique morphism $\theta : \varinjlim A_i \to Q$, making the diagram



commutative.

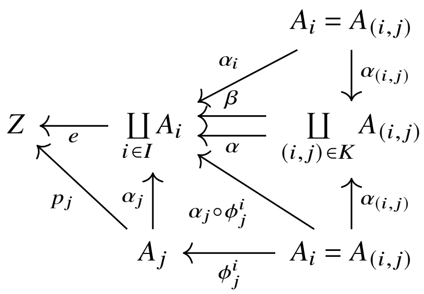
3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE COLIMIT OF A DIRECT SYSTEM

Using the notation and lemmas from the previous section, we are ready to prove the following result.

Theorem 1. *Let C be any category and $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ a direct system in the category C. If the coproducts of the system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ and the direct system of the domains of $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ exist and the coequalizer of the morphisms α and β, induced by the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$, exist in C, then the colimit of the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ also exists.*

Proof. Suppose that the coproducts $\left(\coprod_{i\in I} A_i, (\alpha_i)_{i\in I}\right)$ and $\left(\coprod_{(i,j)\in K} A_{(i,j)}, (\alpha_{(i,j)})_{(i,j)\in K}\right)$ of the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ and its direct system of domains $\left((A_{(i,j)})_{(i,j)\in K}, \left(\phi^{(i,j)}_{(k,l)}\right)_{(i,j)\leq_K (k,l)}\right)$ exist. As we showed before, it follows that the morphisms $\alpha \in \mathrm{Mor}\left(\coprod_{(i,j)\in K} A_{(i,j)}, \coprod_{i\in I} A_i\right)$ and $\beta \in \mathrm{Mor}\left(\coprod_{(i,j)\in K} A_{(i,j)}, \coprod_{i\in I} A_i\right)$, induced by the direct system $((A_i)_{i\in I},(\phi^i_j)_{i\le j})$, also exist and that $\alpha\circ\alpha_{(i,j)}=\alpha_j\circ\phi^i_j$ and $\beta\circ\alpha_{(i,j)}=\alpha_i$ for all $(i,j)\in K=\{(i,j)\in I\times I : i\le j\}$.

By the assumption the coequalizer of α and β exists. Let (Z,e) be the coequalizer of the morphisms α and β. Then $e\circ\alpha=e\circ\beta$. Define morphisms $p_i:=e\circ\alpha_i$ for each $i\in I$. Then $p_i\in \mathrm{Mor}(A_i,Z)$ for each $i\in I$ and we obtain the following commutative diagram



for each $(i,j)\in K$.

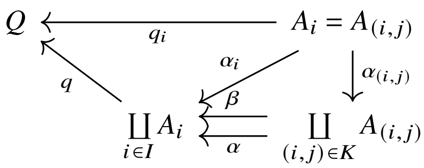
Let us show that $(Z,(p_i)_{i\in I})$ is the colimit of the direct system $((A_i)_{i\in I},(\phi^i_j)_{i\le j})$. For it, take any $(i,j)\in K$. Then

$$p_i=e\circ\alpha_i=e\circ(\beta\circ\alpha_{(i,j)})=(e\circ\beta)\circ\alpha_{(i,j)}=(e\circ\alpha)\circ\alpha_{(i,j)}$$

$$=e\circ(\alpha\circ\alpha_{(i,j)})=e\circ(\alpha_j\circ\phi^i_j)=(e\circ\alpha_j)\circ\phi^i_j=p_j\circ\phi^i_j.$$

Hence, the condition (i) of the definition of the colimit is fulfilled.

Suppose that there exists $Q\in\mathrm{Ob}(C)$ and morphisms $(q_i:A_i\to Q)_{i\in I}$, which satisfy $q_j\circ\phi^i_j=q_i$ whenever $i\le j$. As $\left(\coprod_{i\in I}A_i,(\alpha_i)_{i\in I}\right)$ is the coproduct of the family $((A_i)_{i\in I},(\phi^i_j)_{i\le j})$, then there exists a map $q\in\mathrm{Mor}\left(\coprod_{i\in I}A_i,Q\right)$ such that $q_i=q\circ\alpha_i$ for each $i\in I$. Hence, we obtain the following commutative diagram



for each $(i,j)\in K$.

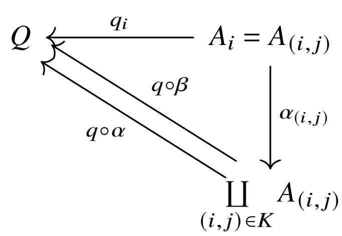
Notice that

$$(q\circ\beta)\circ\alpha_{(i,j)}=q\circ(\beta\circ\alpha_{(i,j)})=q\circ\alpha_i=q_i=q_j\circ\phi^i_j$$

$$=(q\circ\alpha_j)\circ\phi^i_j=q\circ(\alpha_j\circ\phi^i_j)=q\circ(\alpha\circ\alpha_{(i,j)})=(q\circ\alpha)\circ\alpha_{(i,j)}$$

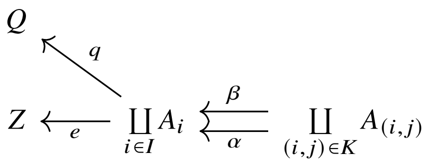
for each $(i,j)\in K$.

Hence, we obtain a commutative diagram of the form



for each $(i,j) \in K$. Now we are in the situation of Lemma 1, taking $C_i = A_i = A_{(i,j)}$, $P = \coprod_{(i,j)\in K} A_{(i,j)}$, $X = Q, \delta_i = q_i, \gamma_i = \alpha_{(i,j)}, f = q \circ \alpha$ and $g = q \circ \beta$ for each $(i,j) \in K$. Therefore, by Lemma 1, we have $q \circ \alpha = q \circ \beta$.

This means that we get a diagram of the form



As (Z,e) is the coequalizer of α and β, then there exists a unique morphism $m \in \mathrm{Mor}(Z,Q)$ such that $m \circ e = q$. Notice that then we also have

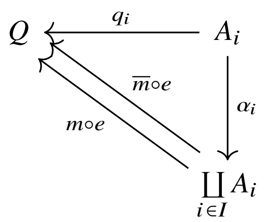
$$m \circ p_i = m \circ (e \circ \alpha_i) = (m \circ e) \circ \alpha_i = q \circ \alpha_i = q_i$$

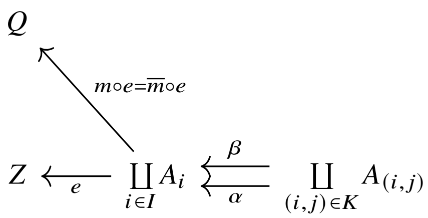
for each $i \in I$.

Suppose that there exists a morphism $\overline{m} \in \mathrm{Mor}(Z,Q)$ such that $\overline{m} \circ p_i = q_i$ for each $i \in I$. Then

$$(m \circ e) \circ \alpha_i = q \circ \alpha_i = q_i = \overline{m} \circ p_i = \overline{m} \circ (e \circ \alpha_i) = (\overline{m} \circ e) \circ \alpha_i$$

for each $i \in I$. Hence, we obtain the following two commutative diagrams for each $i \in I$:





Using Lemma 1 with $X = Q, C_i = A_i, P = \coprod_{i\in I} A_i, \delta_i = q_i, \gamma_i = \alpha_i, f = m \circ e$ and $g = \overline{m} \circ e$, we obtain that $m \circ e = \overline{m} \circ e$ in the second diagram.

Using Lemma 2 with $A = \coprod_{(i,j)\in K} A_{(i,j)}, B = \coprod_{i\in I} A_i, X = Q, p = m \circ e = \overline{m} \circ e, f = m$ and $g = \overline{m}$, we obtain that $m = \overline{m}$.

Hence, $m \in \mathrm{Mor}(Z,Q)$ is the unique morphism with $m \circ e = q$ and the last condition of the colimit is also fulfilled.

With that we have shown that the colimit of the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ exists and is of the form $(Z, (p_i)_{i\in I})$. □

4. PRELIMINARY RESULTS KNOWN FOR THE CATEGORY SEG OF SEGAL TOPOLOGICAL ALGEBRAS

In this paper, **a topological algebra** (A, τ_A) is a topological linear space (A, τ_A) over the field $\mathbb{K}$ of real or complex numbers, in which a separately continuous associative multiplication is defined. Hence, A is an algebra, τ_A is a topology on A, the addition and multiplication by scalars are continuous and the multiplication is separately continuous in the topology τ_A. Notice that we do not demand that the algebra A should be unital.

Let us recall the notions introduced in [1].

A topological algebra (A,τ_A) is a left (right or two-sided) **Segal topological algebra** in a topological algebra (B,τ_B) via an algebra homomorphism $f: A \to B$, if

1) $\mathrm{cl}_B(f(A)) = B$;

2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$, i.e., f is continuous;

3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a left (right or two-sided) Segal topological algebra will be denoted shortly by a triple (A,f,B). The reader might think of either left, right or two-sided Segal topological algebras, depending on which "sideness" is more familiar. Everything will actually work similarily in all of those three cases. Therefore, we will omit the words indicating "sideness" and will only use the phrase "Segal topological algebras". Only in cases where the sideness is more important, we will write "left (right or two-sided) Segal topological algebras" explicitly in the text.

Let us recall the definition of the category **Seg** of Segal topological algebras, introduced in [2]. The objects of the category **Seg** are all Segal topological algebras, i.e., all triples in the form $(A,f,B),(C,g,D),\ldots$. The morphisms between Segal topological algebras (A,f,B) and (C,g,D) are all ordered pairs (α,β) of continuous algebra homomorphisms $\alpha: A \to C, \beta: B \to D$, satisfying $(g\circ\alpha)(a) = (\beta\circ f)(a)$ for each $a \in A$, i.e., making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow{g} & D \end{array}$$

commutative.

In [5] (Theorem 1, p. 231) we found some sufficient conditions for the existence of the coproduct of a family of Segal topological algebras in the category **Seg**. For that we needed the notion of a tensor algebra, which we will introduce shortly below.

For a set Λ and a collection $(A_\lambda)_{\lambda\in\Lambda}$ of algebras, their **tensor algebra** is an algebra

$$T = \Big(\bigoplus_{\lambda\in\Lambda} A_\lambda\Big) \oplus \Big(\bigoplus_{\lambda,\mu\in\Lambda} (A_\lambda\otimes A_\mu)\Big) \oplus \Big(\bigoplus_{\lambda,\mu,\nu\in\Lambda} (A_\lambda\otimes A_\mu\otimes A_\nu)\Big) \oplus \ldots$$

and every element $t \in T$ is in the form

$$t = \bigoplus_{l=1}^{k}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} t_{q,m,1}\otimes\ldots\otimes t_{q,m,i_l}\Big)\Big)$$

for some $k, p_l, r_{m,l} \in \mathbb{Z}^+$ and $t_{q,m,1},\ldots,t_{q,m,i_l} \in \bigcup_{\lambda\in\Lambda} A_\lambda$.

In [3], pp. 203–205, we defined the algebraic operations in T as follows. If $\rho \in \mathbb{K}$,

$$t = \bigoplus_{l=1}^{k_t}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} t_{q,m,1}\otimes\ldots\otimes t_{q,m,i_l}\Big)\Big) \in T$$

and

$$s = \bigoplus_{f=1}^{k_s}\Big(\bigoplus_{g=1}^{u_f}\Big(\sum_{h=1}^{v_{g,f}} s_{h,g,1}\otimes\ldots\otimes s_{h,g,j_f}\Big)\Big) \in T,$$

then

$$\rho t = \bigoplus_{l=1}^{k_t}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} (\rho t_{q,m,1})\otimes\ldots\otimes t_{q,m,i_l}\Big)\Big),$$

$$t+s=\bigoplus_{l=1}^{k_t+k_s}\left(\bigoplus_{m=1}^{w_l}\left(\sum_{q=1}^{x_{m,l}} z_{q,m,1}\otimes\ldots\otimes z_{q,m,L_l}\right)\right),$$

where

$$L_l=\begin{cases} i_l, & \text{if } 1\leqslant l\leqslant k_t\\ j_{l-k_t}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases},\quad w_l=\begin{cases} p_l, & \text{if } 1\leqslant l\leqslant k_t\\ u_{l-k_t}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases}, \tag{1}$$

$$x_{m,l}=\begin{cases} r_{m,l}, & \text{if } 1\leqslant l\leqslant k_t\\ v_{m,l-k_t}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases}\quad \text{and}\quad z_{q,m,d}=\begin{cases} t_{q,m,d}, & \text{if } 1\leqslant l\leqslant k_t\\ s_{q,m,d}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases}. \tag{2}$$

The multiplication of elements had to satisfy the rule

$$t\cdot s=\bigoplus_{\epsilon=1}^{k_tk_s}\bigoplus_{\delta=1}^{p_{X_1}u_{X_2}}\sum_{y=1}^{r_{X_3,X_1}v_{X_4,X_2}}\left(\bigotimes_{u=1}^{i_{X_1}} t_{X_5,X_3,u}\otimes\bigotimes_{d=1}^{j_{X_2}} s_{X_6,X_4,d}\right),$$

where

$$X_1=\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1,\quad X_2=\epsilon-X_1k_s=\epsilon-\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor k_s,$$

$$X_3=\left\lfloor\frac{\delta-1}{p_{X_1}}\right\rfloor+1=\left\lfloor\frac{\delta-1}{p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1}}\right\rfloor+1,\quad X_4=\delta-X_3p_{X_1}=\delta-\left\lfloor\frac{\delta-1}{p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1}}\right\rfloor p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1},$$

$$X_5=\left\lfloor\frac{y-1}{v_{X_4,X_2}}\right\rfloor+1=\left\lfloor\frac{y-1}{v_{\delta-\left\lfloor\frac{\delta-1}{p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1}}\right\rfloor p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1},\,\epsilon-\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor k_s}}\right\rfloor+1$$

and

$$X_6=y-(X_5-1)v_{X_4,X_2}=y-\left\lfloor\frac{y-1}{v_{X_4,X_2}}\right\rfloor+1$$

$$=y-\left\lfloor\frac{y-1}{v_{\delta-\left\lfloor\frac{\delta-1}{p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1}}\right\rfloor p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1},\,\epsilon-\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor k_s}}\right\rfloor v_{\delta-\left\lfloor\frac{\delta-1}{p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1}}\right\rfloor p_{\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor+1},\,\epsilon-\left\lfloor\frac{\epsilon-1}{k_s}\right\rfloor k_s}.$$

Take any family $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ of Segal topological algebras in the category **Seg**. Define a map $\widetilde{h_T}:\bigcup\limits_{\lambda\in\Lambda}A_\lambda\to\bigcup\limits_{\lambda\in\Lambda}B_\lambda$ by $\widetilde{h_T}(a)=f_{\lambda_a}(a)$. Next, define a map $h_T:T\to S$ by setting

$$h_T(t)=\bigoplus_{l=1}^{k}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}}\widetilde{h_T}(t_{q,m,1})\otimes\cdots\otimes\widetilde{h_T}(t_{q,m,i_l})$$

for every element

$$t=\bigoplus_{l=1}^{k}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1}\otimes\ldots\otimes t_{q,m,i_l}\right)\right)$$

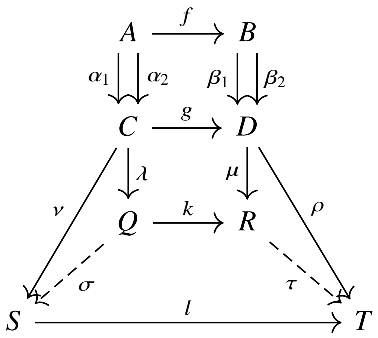
of T. In [5] we showed that the map h_T is an algebra homomorphism. In what follows, the map h_T will be called **the tensor map for Segal topological algebras** $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$.

Let us recall some notions from the category theory for the category **Seg** of Segal topological algebras.

Let (A,f,B) and (C,g,D) be objects of the category **Seg**. The **coequalizer of morphisms** $(\alpha_1,\beta_1),(\alpha_2,\beta_2 \in \mathrm{Mor}((A,f,B),(C,g,D))$ is a pair $((Q,k,R);(\lambda,\mu))$ such that

1) $(Q,k,R) \in \mathrm{Ob}(\mathbf{Seg})$ and $(\lambda,\mu) \in \mathrm{Mor}((C,g,D),(Q,k,R))$ with $\lambda\circ\alpha_1 = \lambda\circ\alpha_2$ and $\mu\circ\beta_1 = \mu\circ\beta_2$;

2) for any pair $((S,l,T);(\nu,\rho))$ with $(S,l,T) \in \mathrm{Ob}(\mathbf{Seg})$ and $(\nu,\rho) \in \mathrm{Mor}((C,g,D),(S,l,T))$ with $\nu\circ\alpha_1 = \nu\circ\alpha_2$ and $\rho\circ\beta_1 = \rho\circ\beta_2$, there exists a unique $(\sigma,\tau) \in \mathrm{Mor}((Q,k,R),(S,l,T))$ with $\nu = \sigma\circ\lambda$ and $\rho = \tau\circ\mu$:



.

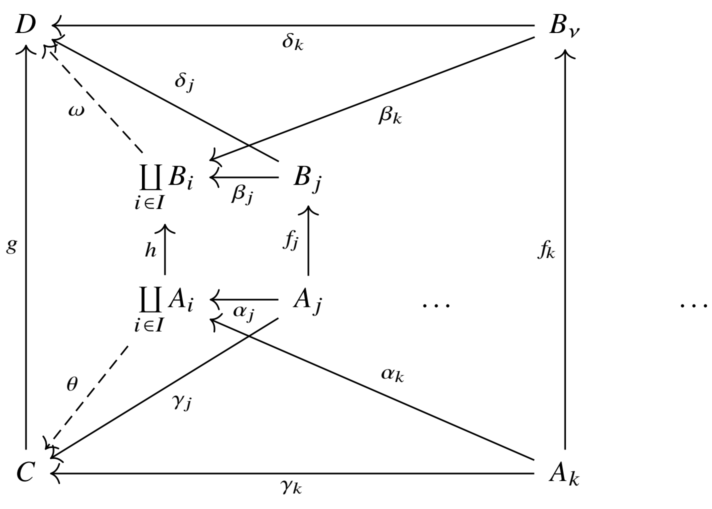
The coproduct of the family $(A_i,f_i,B_i)_{i\in I}$ **of Segal topological algebras** in the category **Seg** is an ordered pair $\Big((\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i),((\alpha_j,\beta_j))_{j\in I}\Big)$, consisting of a Segal topological algebra $\Big(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\Big)$ and a family $\Big((\alpha_j,\beta_j):(A_j,f_j,B_j)\to\Big(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\Big)\Big)_{j\in I}$ of morphisms in **Seg** such that for any object (C,g,D) of **Seg** and every family

$$((\gamma_j,\delta_j):(A_j,f_j,B_j)\to(C,g,D))_{j\in I}$$

of morphisms in **Seg**, there exists a unique morphism

$$(\theta,\omega):\Big(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\Big)\to(C,g,D)$$

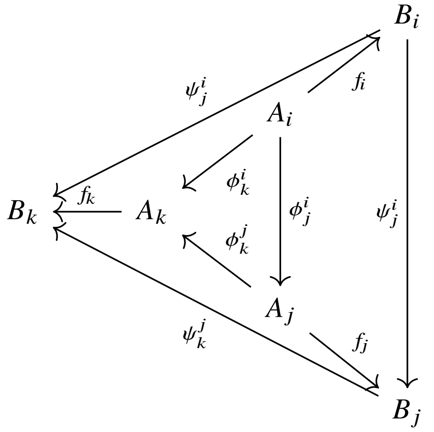
in **Seg** such that the diagram



commutes.

Now, we are ready to formulate the definitions of the direct system and the colimit in the context of the category **Seg**.

Definition 1. *Given a partially ordered set $(I,\leq)$, a* **direct system in Seg** *is an ordered pair $(((A_i, f_i, B_i))_{i\in I};\ (\phi^j_k,\ \psi^j_k)_{j\leq k})$, where $((A_i, f_i, B_i))_{i\in I}$ is an indexed family of objects of* **Seg** *and $(\phi^j_k : A_j \to A_k)_{j\leq k}, (\psi^j_k : B_j \to B_k)_{j\leq k}$ are indexed families of morphisms in* **Seg** *such that $\phi^j_j = 1_{A_j}, \psi^j_j = 1_{B_j}$ for each $j \in I$ and $\phi^i_k = \phi^j_k \circ \phi^i_j, \psi^i_k = \psi^j_k \circ \psi^i_j$ for each $i, j, k \in I$ with $i \leq j \leq k$. The last condition means that the diagram*

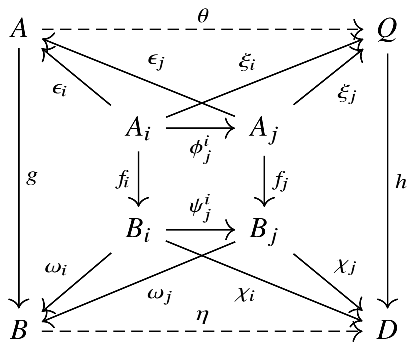


commutes.

Definition 2. *The* **colimit** *of a direct system $(((A_i, f_i, B_i))_{i\in I};\ (\phi^j_k, \psi^j_k)_{j\leq k})$ in* **Seg** *is the pair $((A, g, B);\ (\epsilon_i, \omega_i)_{i\in I})$, where $(A, g, B) \in$ Ob(**Seg**) and $((\epsilon_i, \omega_i) \in \mathrm{Mor}((A_i, f_i, B_i), (A, g, B)))_{i\in I}$ is a collection of morphisms in* **Seg** *such that*

(i) $(\epsilon_j, \omega_j) \circ (\phi^i_j, \psi^i_j) = (\epsilon_i, \omega_i)$ *whenever $i \leq j$;*

(ii) *for each $(C, h, D) \in$ Ob(**Seg**) and $((\xi_i, \chi_i) \in \mathrm{Mor}((A_i, f_i, B_i), (C, h, D))_{i\in I}$, which satisfy $(\xi_j, \chi_j) \circ (\phi^i_j, \psi^i_j) = (\xi_i, \chi_i)$ whenever $i \leq j$, there exists a unique morphism $(\theta, \eta) : (A, g, B) \to (C, h, D)$, making the diagram*



commutative.

In [4] we proved the following theorem (see [4], Theorem 1, p. 157).

Theorem 2. *Let $(A, f, B), (C, g, B) \in$ Ob(**Seg**) and $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathrm{Mor}((A, f, B), (C, g, D))$. Let I be the smallest two-sided ideal of C, generated by the set $M = \{\alpha_1(a) - \alpha_2(a) : a \in A\}$, and J be the smallest two-sided ideal of D, generated by the set $N = \{\beta_1(b) - \beta_2(b) : b \in B\}$. Then the coequalizer of morphisms $(\alpha_1,\ \beta_1)$, $(\alpha_2,\ \beta_2)$ always exists and is the pair $((C/I, \tilde{g},\ D/J);\ (p, q))$, where $p : C \to C/I$, $q : D \to D/J$ are the canonical projections, C/I, D/J are equipped with the quotient topologies $\tau_{C/I} = \{V \subseteq C/I : p^{-1}(V) \in \tau_C\}$, $\tau_{D/J} = \{W \subseteq D/J : q^{-1}(W) \in \tau_D\}$, respectively, and $\tilde{g} : C/I \to D/J$ is defined by $\tilde{g}([c]) = [g(c)] = q(g(c))$ for each $[c] \in C/I$.*

In [5] we proved the following theorem (see [5], Theorem 1, p. 231), which we present here in a slightly modified form.

Theorem 3. *Let I be any set, $(A_i, f_i, B_i)_{i\in I}$ be a family of left (right or two-sided) Segal topological algebras, T the tensor algebra of algebras $(A_i)_{i\in I}$, S the tensor algebra of algebras $(B_i)_{i\in I}$, J and L two-sided ideals of T and S, generated by the sets*

$$\{x\otimes y - xy : x, y \in A_i, i \in I\} \quad \text{and} \quad \{z\otimes w - zw : z, w \in B_i, i \in I\},$$

respectively, and $h_T : T \to S$ the tensor map for Segal topological algebras $(A_i, f_i, B_i)_{i\in I}$. If $S\cdot h_T(T) \subseteq h_T(T)$ (respectively, $h_T(T)\cdot S \subseteq h_T(T)$ or $S\cdot h_T(T)\cdot S \subseteq h_T(T)$), then the coproduct of the family $(A_i, f_i, B_i)_{i\in I}$ exists and is of the form $\left(\left(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\right), ((\alpha_j, \beta_j))_{j\in I}\right)$, where $\coprod_{i\in I} A_i = T/J$, $\coprod_{i\in I} B_i = S/L$, $\alpha_j = \kappa_J \circ \lambda_j$, $\beta_j = \kappa_L \circ \mu_j$ and $\lambda_j : A_j \to T, \mu_j : B_j \to S$ are the inclusion maps for each $j \in I$.

5. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE COLIMIT OF A DIRECT SYSTEM OF SEGAL TOPOLOGICAL ALGEBRAS

Let $(I, \leq)$ be a partially ordered set. Take any direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$ in **Seg**. Set $K = \{(i, j) \in I\times I : i \leq j\}$ and let $\left(((A_{(i,j)}, f_{(i,j)}, B_{(i,j)})_{(i,j)\in K}, \left(\phi^{(i,j)}_{(k,l)}, \psi^{(i,j)}_{(k,l)}\right)_{(i,j)\leq_K (k,l)}\right)$ be the direct system of the domains of the direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$. Then, for each $(i, j) \in K$, we have $A_{(i,j)} = A_i, B_{(i,j)} = B_i, f_{(i,j)} = f_i$. From Section 2 we know that whenever the coproducts $\left(\left(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\right), ((\gamma_i, \delta_i))_{i\in I}\right)$ and $\left(\left(\coprod_{(i,j)\in K} A_{(i,j)}, \tilde{h}, \coprod_{(i,j)\in K} B_{(i,j)}\right), ((\gamma_{(i,j)}, \delta_{(i,j)}))_{(i,j)\in K}\right)$ exist in **Seg**, then there exist also the morphisms

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \operatorname{Mor}\left(\left(\coprod_{(i,j)\in K} A_{(i,j)}, \tilde{h}, \coprod_{(i,j)\in K} B_{(i,j)}\right), \left(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\right)\right),$$

induced by the direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$.

Now we are ready to state and prove the main result of this paper.

Theorem 4. *Let $(I, \leq)$ be a partially ordered set, $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$ a direct system of left (right or two-sided) Segal topological algebras in* **Seg**, *$K = \{(i, j) \in I\times I : i \leq j\}$ and*

$$\left(((A_{(i,j)}, f_{(i,j)}, B_{(i,j)})_{(i,j)\in K}, (\phi^{(i,j)}_{(k,l)}, \psi^{(i,j)}_{(k,l)})_{(i,j)\leq_K (k,l)}\right)$$

the direct system of the domains of the direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$.

Let T be the tensor algebra of the family $(A_i)_{i\in I}$, S the tensor algebra of the family $(B_i)_{i\in I}$, W the tensor algebra of the family $(A_{(i,j)})_{(i,j)\in K}$ and Z the tensor algebra of the family $(B_{(i,j)})_{(i,j)\in K}$. Let J, L, M and N the left (respectively, right or two-sided) ideals, generated, respectively, by the sets

$$\{x\otimes y - xy : x, y \in A_i, i \in I\},\ \{z\otimes w - zw : z, w \in B_i, i \in I\},$$

$$\{x\otimes y - xy : x, y \in A_{(i,j)}, (i, j) \in K\},\ \{z\otimes w - zw : z, w \in B_{(i,j)}, (i, j) \in K\}.$$

Let $h_T : T \to S$ and $h_W : W \to Z$ be the tensor maps for Segal topological algebras $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$ and $\left(((A_{(i,j)}, f_{(i,j)}, B_{(i,j)})_{(i,j)\in K}, \left(\phi^{(i,j)}_{(k,l)}, \psi^{(i,j)}_{(k,l)}\right)_{(i,j)\leq_K (k,l)}\right)$, respectively.

If $S\cdot h_T(T) \subseteq h_T(T)$ and $Z\cdot h_W(W) \subseteq h_W(W)$ (respectively, $h_T(T)\cdot S \subseteq h_T(T)$ and $h_W(W)\cdot Z \subseteq h_W(W)$ or $S\cdot h_T(T)\cdot S \subseteq h_T(T)$ and $Z\cdot h_W(W)\cdot Z \subseteq h_W(W)$), then the colimit of the direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$ exists.

Proof. By Theorem 3, we know that the coproducts $\left(\left(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\right), ((\gamma_i,\delta_i))_{i\in I}\right)$ and $\left(\left(\coprod_{(i,j)\in K} A_{(i,j)}, \tilde{h}, \coprod_{(i,j)\in K} B_{(i,j)}\right), ((\gamma_{(i,j)},\delta_{(i,j)}))_{(i,j)\in K}\right)$ of the families $((A_i, f_i, B_i))_{i\in I}$ and $((A_{(i,j)}, f_{(i,j)}, B_{(i,j)}))_{(i,j)\in K}$, respectively, exist in **Seg**. Hence, we also obtain the morphisms

$$(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \operatorname{Mor}\left(\left(\coprod_{(i,j)\in K} A_{(i,j)}, \tilde{h}, \coprod_{(i,j)\in K} B_{(i,j)}\right), \left(\coprod_{i\in I} A_i, h, \coprod_{i\in I} B_i\right)\right),$$

induced by the direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$. By Theorem 2, the coequalizer of the morphisms (α_1,β_1) and (α_2,β_2) exists in **Seg**. Therefore, by Theorem 1, the colimit of the direct system $(((A_i, f_i, B_i))_{i\in I}; (\phi^j_k, \psi^j_k)_{j\leq k})$ also exists. □

Open question. Let C be an arbitrary category and $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ be such a direct system in C, for which the colimit exists in C. Is it true then that also the coproduct of the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$, the coproduct of the direct system of the domains of $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$ and the coequalizer of the morphisms α and β, induced by the direct system $((A_i)_{i\in I}, (\phi^i_j)_{i\leq j})$, exist in C?

6. CONCLUSIONS

In this paper we found some sufficient conditions for the existence of the colimit of a direct system in an arbitrary category and applied this result in order to find sufficient conditions for the existense of the colimit of a direct system in the category **Seg** of Segal topological algebras. The publication costs of this article were covered by the Estonian Academy of Sciences.

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Kopiirid Segali topoloogiliste algebrate kategoorias Seg

Mart Abel

Käesolevas artiklis leitakse piisavad tingimused konkreetse otsesüsteemi kopiiri leidumiseks suvalises kategoorias ning rakendatakse saadud tulemust piisavate tingimuste leidmiseks konkreetse otsesüsteemi kopiiri leidumiseks Segali topoloogiliste algebrate kategoorias **Seg**.