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OPTICAL SOLITONS

Bright optical solitons with polynomial law of nonlinear refractive index by Adomian decomposition scheme

O. González-Gaxiola[a*], Anjan Biswas[b,c,d,e], Yakup Yildirim[f], Hashim M. Alshehri[c]

[a] Applied Mathematics and Systems Department, Universidad Autónoma Metropolitana-Cuajimalpa, Vasco de Quiroga 4871, 05348 Mexico City, Mexico
[b] Department of Physics, Chemistry and Mathematics, Alabama A&M University, Normal, AL 35762-4900, USA
[c] Mathematical Modeling and Applied Computation (MMAC) Research Group, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[d] Department of Applied Sciences, Cross-Border Faculty, Dunarea de Jos University of Galati, 111 Domneasca St., 800201 Galati, Romania
[e] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, Pretoria, South Africa
[f] Department of Mathematics, Faculty of Arts and Sciences, Near East University, 99138 Nicosia, Cyprus

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Abstract. This paper numerically addresses bright optical solitons with cubic-quintic-septic (polynomial) law of nonlinear refractive index. The adopted scheme is with Adomian decomposition. The surface and contour plots are presented along with negligibly small error count.

Keywords: nonlinear Schrödinger equation, polynomial law, bright optical solitons, Adomian decomposition method.

1. INTRODUCTION

One of the key features of optical soliton transmission is self-phase modulation (SPM). There are several forms of SPM that are known today which form the structure of the fiber optic material. The simplest, most renowned and commonly used is Kerr law nonlinearity that leads to cubic Schrödinger's equation as the governing model and is also referred to as nonlinear Schrödinger's equation (NLSE). A few other forms of SPM that are commonly studied are power law, parabolic law, log law, dual-power law, saturable law, quadratic-cubic law, anti-cubic law, and generalized anti-cubic law. Very recently, Kudryashov has proposed a variety of SPM structures that are gaining popularity in the field of quantum optics in spite of the fact that these theoretical forms of nonlinearity have not yet been reached in any laboratory [1–19]. This paper is the study of bright optical solitons from a numerical perspective for a specific form of non-Kerr law of nonlinear refractive index. It is called cubic-quintic-septic law of nonlinear refractive index that is occasionally referred to as polynomial law of nonlinearity. The Adomian decomposition method

* Corresponding author, ogonzalez@cua.uam.mx

(ADM) is the adopted scheme of this paper. The numerical simulations that are recovered by ADM are compared with the pre-existing results that have been obtained analytically. The results are astounding, and the measured error is significant. The results are all exhibited after a quick revisitation to the model and its known analytical results.

1.1. Description of the governing model

The NLSE with cubic-quintic-septic law of nonlinearity that models the problem to be solved, in its dimensionless form, is given by [20]:

$$iu_t + au_{xx} + \left(b_1|u|^2 + b_2|u|^4 + b_3|u|^6\right)u = 0, \quad a, b_1, b_2 \text{ and } b_3 \in \mathbb{R}. \tag{1}$$

Equation (1) models the evolution of pulses in an optical fiber where the complex field $u(x,t)$ represents the wave variable. Furthermore, in Eq. (1) the first term is temporal evolution, the second-order derivative with respect to the spatial variable x represents the dispersion, while in the nonlinear term, the coefficients b_1, b_2 and b_3 represent the SPM effect that stems from nonlinear refractive index of the fiber. This type of NLSE has been studied by few authors and the most important works have been reported in [20–22].

1.2. Bright solitons

The bright 1-soliton solution to Eq. (1) was recently established in [20,21], as

$$u(x,t) = A\,\mathrm{sech}^{1/3}[B(x-\nu t)] \times \exp\{i[-\kappa x + \omega t + \theta]\}, \tag{2}$$

where the amplitude and the velocity of the soliton are respectively

$$A = \sqrt[6]{\frac{4(\omega + a\kappa^2)}{b_3}}, \tag{3}$$

and

$$\nu = -2a\kappa, \tag{4}$$

whereas the inverse width of the soliton is

$$B = 3\sqrt{\frac{\omega + a\kappa^2}{a}}. \tag{5}$$

The mathematical restrictions for the existence of the soliton are as follows:

$$a\left(\omega + a\kappa^2\right) > 0, \quad b_3\left(\omega + a\kappa^2\right) > 0 \quad \text{and} \quad ab_3 > 0, \tag{6}$$

the last constraint in (6) is derived from relating A with B.

2. SOLUTION ALGORITHM

We know that u is a complex-valued function, then, considering its real part and its imaginary part we can decompose it as $u(x,t) = p(x,t) + iq(x,t)$, then we have the following nonlinear Cauchy problem:

$$\begin{cases} -L_t q + aRp + N_1(p,q) = 0, \\ L_t p + aRq + N_2(p,q) = 0, \\ u(x,0) = (p(x,0), q(x,0)). \end{cases} \tag{7}$$

In the above decomposition we have considered the following nomenclature for the operators involved:

$$L_t = \frac{\partial}{\partial t}, \quad R = \frac{\partial^2}{\partial x^2}.$$

In addition, the nonlinear components N_1 and N_2 in terms of p and q are, respectively, given as

$$N_1(p,q) = p[b_1(p^2+q^2) + b_2(p^2+q^2)^2 + b_3(p^2+q^2)^3] \tag{8}$$

and

$$N_2(p,q) = q[b_1(p^2+q^2) + b_2(p^2+q^2)^2 + b_3(p^2+q^2)^3]. \tag{9}$$

Assuming that the operator L is invertible and its inverse is $L_t^{-1} = \int_0^t (\cdot)ds$, we apply the inverse to Eq. (7) and considering $p(x,0)$ and $q(x,0)$, we have

$$p(x,t) = p(x,0) - \int_0^t (aRp(x,s) + N_1(p(x,s),q(x,s)))ds, \tag{10}$$

$$q(x,t) = q(x,0) + \int_0^t (aRq(x,s) + N_2(p(x,s),q(x,s)))ds. \tag{11}$$

The Adomian decomposition method essentially allows us to assume that the p and q components of the solution can be expressed as the series [23]

$$p(x,t) = \sum_{n=0}^{\infty} p_n(x,t) \quad \text{and} \quad q(x,t) = \sum_{n=0}^{\infty} q_n(x,t), \tag{12}$$

where each of the summands p_n, q_n will be found recursively as will be established below. While the nonlinear operators N_1 and N_2 can be decomposed as

$$N_1(p,q) = \sum_{n=0}^{\infty} A_n(p_0,p_1,\ldots,p_n;q_0,q_1,\ldots,q_n) \tag{13}$$

and

$$N_2(p,q) = \sum_{n=0}^{\infty} B_n(p_0,p_1,\ldots,p_n;q_0,q_1,\ldots,q_n), \tag{14}$$

where each of the summands A_n, B_n are known as Adomian polynomials in two variables, which will be calculated by means of the following integral formulas [24]:

$$A_n(p_0,\ldots,p_n;q_0,\ldots,q_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} N_1\big(\sum_{k=0}^{n} p_k e^{ik\omega}, \sum_{k=0}^{n} q_k e^{ik\omega}\big)e^{-in\omega}d\omega, \quad n \geq 1, \tag{15}$$

$$B_n(p_0,\ldots,p_n;q_0,\ldots,q_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} N_2\big(\sum_{k=0}^{n} p_k e^{ik\omega}, \sum_{k=0}^{n} q_k e^{ik\omega}\big)e^{-in\omega}d\omega, \quad n \geq 1, \tag{16}$$

in the case where $n = 0$, we have $A_0(p_0) = N_1(p_0)$ and $B_0(q_0) = N_2(q_0)$.
Substituting Eqs (12), (13) and (14) in Eqs. (10) and (11), we obtain the following equalities:

$$\sum_{n=0}^{\infty} p_n(x,t) = p_0(x,0) - \int_0^t \Big(a\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p_n(x,s) + \sum_{n=0}^{\infty} A_n(p_0,\ldots,p_n;q_0,\ldots,q_n)\Big)ds, \tag{17}$$

$$\sum_{n=0}^{\infty} q_n(x,t) = q_0(x,0) + \int_0^t \Big(a\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} q_n(x,s) + \sum_{n=0}^{\infty} B_n(p_0,\ldots,p_n;q_0,\ldots,q_n)\Big)ds. \tag{18}$$

From the above equalities (17) and (18), by matching terms of the same order, we obtain the following algorithm to find each of the components of the solution to our model:

$$p_m(x,t) = -\int_0^t \Big(a\frac{\partial^2}{\partial x^2}p_{m-1}(x,s) + A_{m-1}(p_0,\ldots,p_{m-1};q_0,\ldots,q_{m-1})\Big)ds,\ m \geq 1, \tag{19}$$

$$q_m(x,t) = \int_0^t \Big(a\frac{\partial^2}{\partial x^2}q_{m-1}(x,s) + B_{m-1}(p_0,\ldots,p_{m-1};q_0,\ldots,q_{m-1})\Big)ds,\ m \geq 1. \tag{20}$$

Once each of the components has been obtained, the solution according to the series given in Eq. (12) is

$$u(x,t) = p(x,t) + iq(x,t) = \sum_{n=0}^{\infty} p_n(x,t) + i\sum_{n=0}^{\infty} q_n(x,t). \tag{21}$$

For practical cases it is sufficient to obtain an approximation to the solution, which we acquire through summations up to order N, i.e., for our simulations we will consider $u_N(x,t)$ given by:

$$u_N(x,t) = p_N(x,t) + iq_N(x,t) = \sum_{n=0}^{N}\Big(p_n(x,t) + iq_n(x,t)\Big). \tag{22}$$

The decomposition method that we are proposing in the present study is a very useful and powerful tool for solving nonlinear partial differential equations and in particular nonlinear Schrödinger type equations; for further details see [25,26] and their references.
Next, we will illustrate the application of the proposed method by solving Eq. (1) in some particular examples. We will also compare our results with those previously obtained for the case of singular solitons in [27].

3. NUMERICAL EXAMPLES

In this section, we will apply the ADM-based algorithm described in the previous section to solve several particular cases of Eq. (1). The results obtained will be shown by means of graphs in which the profiles of the solutions as well as the absolute errors resulting from the N level of approximation can be observed.

Example 1. For this example we will consider the parameters given in Table 1.

Therefore the initial condition is given by the following function:

$$u(x,t) = 1.53\,\mathrm{sech}^{1/3}[4.54x] \times \exp\{i[-2.03x + 2.05]\}. \tag{23}$$

The results obtained from the simulation and the absolute error with N iteration steps are shown in Fig. 1.

Table 1. Parameters for model (1) corresponding to example 1

a	b_1	b_2	b_3	ω	κ	A	B	ν	θ	N	Max Error
1.52	2.03	3.00	1.08	1.22	2.03	1.53	4.54	−6.17	2.05	15	4.0×10^{-8}

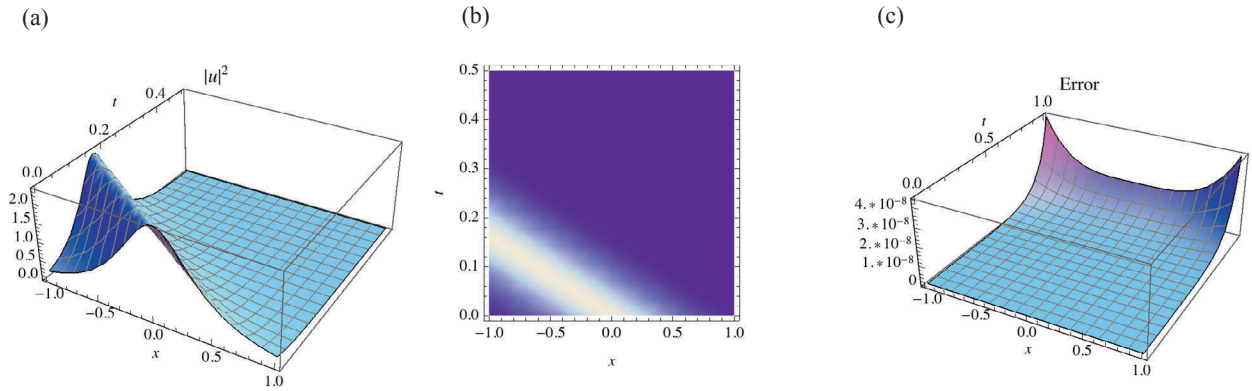

Fig. 1. Graphical representation of the results of example 1: (a) 3D bright soliton, (b) 2D density contour and (c) Absolute error with $N = 15$ iteration steps.

Example 2. For this example we will consider the parameters given in Table 2.

Therefore the initial condition is given by the following function:

$$u(x,t) = 1.21\,\mathrm{sech}^{1/3}[2.32x] \times \exp\{i[-1.05x + 1.25]\}. \tag{24}$$

The results obtained from the simulation and the absolute error with N iteration steps are shown in Fig. 2.

Table 2. Parameters for model (1) corresponding to example 2

a	b_1	b_2	b_3	ω	κ	A	B	ν	θ	N	Max Error
2.11	2.15	−2.51	1.58	1.05	1.05	1.21	2.32	−4.43	1.25	14	2.0×10^{-7}

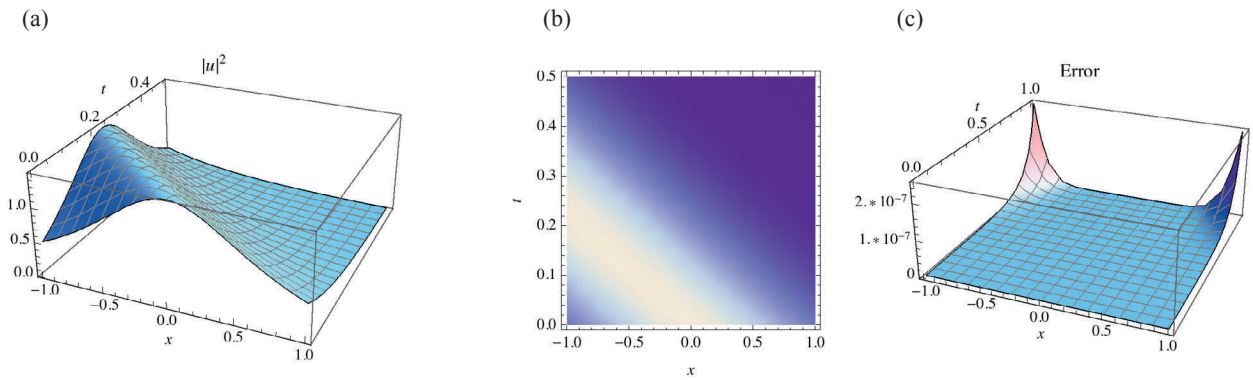

Fig. 2. Graphical representation of the results of example 2: (a) 3D bright soliton, (b) 2D density contour and (c) Absolute error with $N = 14$ iteration steps.

Table 3. Parameters for model (1) corresponding to example 3

a	b_1	b_2	b_3	ω	κ	A	B	ν	θ	N	Max Error
2.05	1.28	2.05	3.10	4.25	−0.50	1.35	4.57	2.05	−0.55	14	1.0×10^{-7}

Example 3. For this example we will consider the parameters given in Table 3.
Therefore the initial condition is given by the following function:

$$u(x,t) = 1.35\,\mathrm{sech}^{1/3}[4.57x]\times\exp\{i[0.5x-0.55]\}. \tag{25}$$

The results obtained from the simulation and the absolute error with N iteration steps are shown in Fig. 3.
Example 4. For this example we will consider the parameters given in Table 4.

Therefore the initial condition is given by the following function:

$$u(x,t) = 1.45\,\mathrm{sech}^{1/3}[4.06x]\times\exp\{i[0.2x-1.12]\}. \tag{26}$$

The results obtained from the simulation and the absolute error with N iteration steps are shown in Fig. 4.

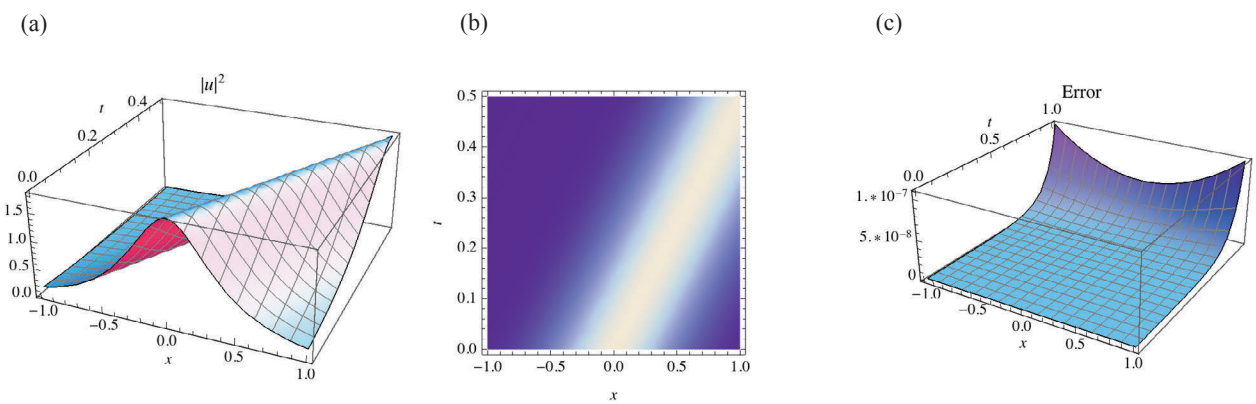

Fig. 3. Graphical representation of the results of example 3: (a) 3D bright soliton, (b) 2D density contour and (c) Absolute error with $N = 14$ iteration steps.

Table 4. Parameters for model (1) corresponding to example 4

a	b_1	b_2	b_3	ω	κ	A	B	ν	θ	N	Max Error
1.95	3.21	2.35	1.52	3.50	−0.20	1.45	4.06	0.78	−1.12	14	6.0×10^{-7}

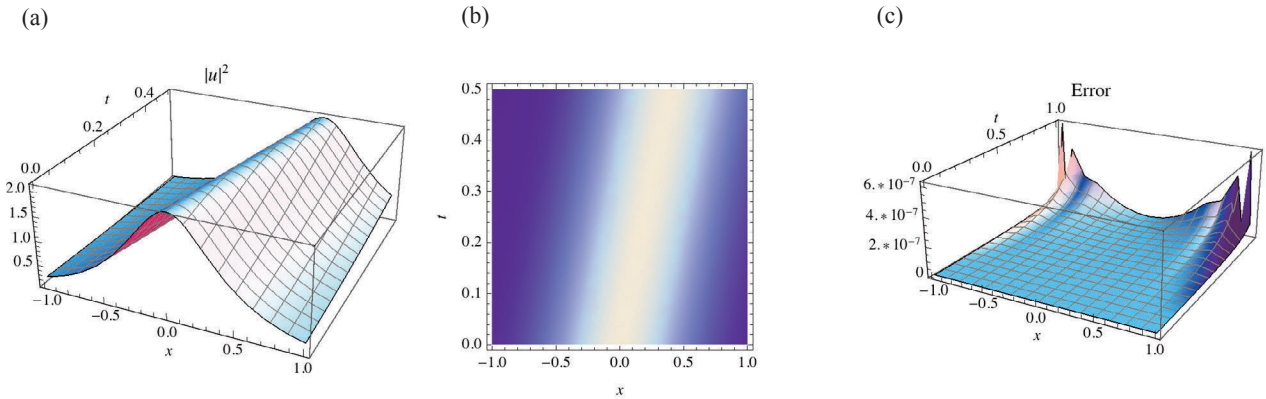

Fig. 4. Graphical representation of the results of example 4: (a) 3D bright soliton, (b) 2D density contour and (c) Absolute error with $N = 14$ iteration steps.

4. CONCLUSIONS

In this paper we applied ADM to numerically address optical solitons with polynomial law of nonlinearity. The scheme compares the analytical results with the numerical surface plots as well as the corresponding contour plots, and the error measure, as seen, is truly remarkable. The results appear to encourage venturing further along in this direction. The ADM scheme as well as the Laplace ADM (LADM) will be later implemented to numerically address SPM in birefringent fibers as well as in dispersion-flattened fibers. Additional forms of SPM that will be later handled are anti-cubic law, generalized anti-cubic law, saturable law, Kudryashov's law and many others. The results of such research activities will be sequentially reported and their exhibits will be gradually visible.

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