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On (m,n)-semiprime submodules

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Abstract. This paper aims to introduce a new class of submodules, called (m,n)-semiprime submodule, which is a generalization of semiprime submodule. Let M be a unital A-module and $m,n \in \mathbb{N}$. Then a proper submodule P of M is said to be an (m,n)-semiprime submodule if whenever $a^m x \in P$ for some $a \in A, x \in M$, then $a^n x \in P$. In addition to giving many characterizations and properties of this kind of submodules, we also use them to characterize von Neumann regular modules.

Key words: semiprime submodule, (m,n)-closed ideal, (m,n)-semiprime submodule, von Neumann regular module, descending chain condition.

1. INTRODUCTION

Throughout this article, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let A always represent such a ring and M represent such an A-module. Let P be a proper submodule of M, I be a nonempty subset of A and K be a nonempty subset of M. The residuals of P by I and K are defined as follows:

$$(P :_M I) = \{x \in M : Ix \subseteq P\}$$
$$(P : K) = \{a \in A : aK \subseteq P\}.$$

If $I = \{a\}$ and $K = \{x\}$ are the singletons, where $a \in A, x \in M$, we prefer $(P :_M a)$ and $(P : x)$ instead of $(P :_M \{a\})$ and $(P : \{x\})$, respectively.

Prime ideals/submodules and their generalizations have a distinguished place in commutative algebra since they are not only used in classifying rings/modules but they have also some applications in other areas such as algebraic geometry, graph theory, factorization theory, etc. For instance, reduced rings, a wide class of commutative rings including integral domains, von Neumann regular rings and the Cartesian product of integral domains are characterized in terms of semiprime ideals (i.e. an ideal which is equal to its radical). Furthermore, in 2007 Badawi defined the concept of 2-absorbing ideals, which is a generalization of prime ideals, and he used them to characterize Dedekind domains [7]. Recently, several other generalizations of prime ideals and 2-absorbing ideals have been introduced and they have been used to characterize some

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important class of rings such as valuation domains, von Neumann regular rings, divided domains, etc. See, for example, [2,9,16].

The extension of reduced rings to modules was first presented by Lee and Zhou in [19]. An A-module M is said to be a *reduced module* if for each $a \in A, x \in M$ with $ax = 0$, $aM \cap Ax = 0$, or equivalently, $a^2x = 0$ implies that $ax = 0$ [19]. Afterwards, Saraç extended the notion of semiprime ideals to modules as follows: a proper submodule P of M is said to be a *semiprime submodule* if $a^2x \in P$ implies that $ax \in P$ for each $a \in A, x \in M$ [21]. Note that an A-module M is a reduced module if and only if the zero submodule is a semiprime submodule. The concept of semiprime submodules has been widely studied in many papers (see, for example, [11,18]). Our aim in this research is to study a generalization of semiprime submodules and to use them to characterize a certain class of modules such as von Neumann regular modules and a subclass of Artinian modules. Let P be a proper submodule of M and $m,n \in \mathbb{N}$. P is said to be an (m,n)-*semiprime submodule* if $a^m x \in P$ implies that $a^n x \in P$ for each $a \in A, x \in M$. Among other results presented in this paper, we show that every semiprime submodule is an (m,n)-semiprime submodule for each positive integer m,n but the converse is not true in general (see Example 1 and Example 2). We also investigate the stability of (m,n)-semiprime submodules under homomorphisms, in factor modules, in quotient modules, in the Cartesian product of modules, in trivial extension $A \ltimes M$ of an A-module M, in the tensor product of modules (see Corollary 2 and Theorems 2–6). Finally, we characterize the descending chain condition of a certain type of submodules and von Neumann regular modules in terms of (m,n)-semiprime submodules (see Theorem 7 and Theorem 8).

2. CHARACTERIZATION OF (m,n)-SEMIPRIME SUBMODULES

Definition 1. *Let P be a proper submodule of M and $m,n \in \mathbb{N}$. P is said to be an (m,n)-semiprime submodule if $a^m x \in P$ implies that $a^n x \in P$ for each $a \in A, x \in M$.*

Recall from [2] that a proper ideal P of A is said to be an (m,n)-*closed ideal* if $x^m \in P$ implies that $x^n \in P$ for each $x \in A$.

Remark 1. (i) In the previous definition, if $m \leq n$, then every proper submodule of an A-module M is an (m,n)-semiprime submodule. Thus, we always assume that $m > n$ if we mention (m,n)-semiprime submodule of a given module.

(ii) If we consider the ring A as a module over itself, then an (m,n)-semiprime submodule P of A is an (m,n)-closed ideal of A.

Example 1. Every semiprime submodule of M is an (m,n)-semiprime submodule for each $m > n$. To see this, take a semiprime submodule P of M and assume that $a^m x \in P$ for some $a \in A, x \in M$. Then note that $a^2(a^{m-2}x) \in P$. Since P is semiprime submodule, we conclude that $a^{m-1}x \in P$. By continuing in this manner, we have $a^n x \in P$.

Example 2. (An (m,n)-semiprime submodule that is not semiprime) Let us consider $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, where p is a prime number and $n > 2$. Then note that $P = (\overline{0})$ is not a semiprime submodule since $p^2(\overline{p}^{n-2}) = \overline{0}$ and $p(\overline{p}^{n-2}) = \overline{p}^{n-1} \neq \overline{0}$. On the other hand, let $a^m \overline{x} = \overline{0}$ for some $a, x \in \mathbb{Z}$. Then we have $p^n | a^m x$, which yields that $p^n | a^n x$. Thus, we have $a^n \overline{x} = \overline{0}$. Therefore, $P = (\overline{0})$ is an (m,n)-semiprime submodule.

Proposition 1. *Let P be a proper submodule of an A-module M and $m,n \in \mathbb{N}$ with $m > n$. The following statements are satisfied*:

(i) If P is an (m,n)-semiprime submodule, then P is a (k,n)-semiprime submodule for each $k \geq m$.

(ii) If P is an (m,n)-semiprime submodule, then P is an (m,k)-semiprime submodule for each $k \geq n$.

(iii) If P is an (m,n)-semiprime submodule, then P is a (k,k')-semiprime submodule for each $k \geq m$ and $k' \geq n$.

Proof. (i): Suppose that P is an (m,n)-semiprime submodule and $k \geq m$. Now, we will show that P is a (k,n)-semiprime submodule. Let $a^k x \in P$ for some $a \in A, x \in M$. Since P is an (m,n)-semiprime submodule and $a^m(a^{k-m}x) \in P$, we conclude that $a^{k+n-m}x \in P$. Note that $k+n-m \leq k-1$. Assume that $k+n-m \leq m$. Then we have $a^m x \in P$, which yields that $a^n x \in P$. Therefore, assume that $k+n-m > m$. Since $a^{k+n-m}x = a^m(a^{k+n-2m}x) \in P$, we have $a^{k+2n-2m}x \in P$. By continuing in this manner, we can obtain $a^t x \in P$ for some $t \leq m$ and thus we have $a^m x \in P$. Since P is an (m,n)-semiprime submodule, we get $a^n x \in P$, as required.

(ii): Suppose that P is an (m,n)-semiprime submodule and $a^m x \in P$ for some $a \in A, x \in M$. Then we have $a^n x \in P$. Since $k \geq n$, we conclude that $a^k x \in P$. Therefore, P is an (m,k)-semiprime submodule.

(iii): Follows from (i) and (ii). □

Proposition 2. *(i) Let P_i be an (m,n)-semiprime submodule for each $i \in \Delta$. Then $\bigcap_{i\in\Delta} P_i$ is an (m,n)-semiprime submodule.*

(ii) Let P_i be an (m_i,n_i)-semiprime submodule for each $i \in \Delta$, where $m_i > n_i$. Suppose that $\sup\{m_i : i \in \Delta\} < \infty$. *Then $\bigcap_{i\in\Delta} P_i$ is an (m,n)-semiprime submodule, where $m = \sup\{m_i : i \in \Delta\}$ and $n = \sup\{n_i : i \in \Delta\}$.*

(iii) Let P_i be an (m_i,n_i)-semiprime submodule for each $i = 1,2,\dots,k$. Then $\bigcap_{i=1}^{k} P_i$ is an (m,n)-semiprime submodule, where $m = m_1 + m_2 + \cdots + m_k$ and $n = n_1 + n_2 + \cdots + n_k$.

Proof. (i): It is clear.

(ii): First note that $\sup\{n_i : i \in \Delta\} \leq \sup\{m_i : i \in \Delta\}$. Suppose that $\sup\{m_i : i \in \Delta\} = m$, $\sup\{n_i : i \in \Delta\} = n$. Without loss of generality, we may assume that $m \neq n$. Since P_i is an (m_i,n_i)-semiprime submodule, by Proposition 1, we have that P_i is an (m,n)-semiprime submodule. Then, by (i), we have that $\bigcap_{i\in\Delta} P_i$ is an (m,n)-semiprime submodule.

(iii): It is an analogue of (ii). □

Now, we give a characterization of (m,n)-semiprime submodules in terms of (m,n)-closed ideals.

Theorem 1. *Let P be a proper submodule of M and $m > n$ be two integers. The following statements are equivalent*:

(i) P is an (m,n)-semiprime submodule.

(ii) $(P : x)$ is an (m,n)-closed ideal for each $x \in M - P$.

(iii) $(P :_M a^m) = (P :_M a^n)$ for each $a \in A$.

(iv) For each $a \in A$ and each submodule K of M, $a^m K \subseteq P$ implies that $a^n K \subseteq P$.

(v) For each submodule K of M with $K \nsubseteq P$, $(P : K)$ is an (m,n)-closed ideal.

Proof. $(i) \Rightarrow (ii)$: Suppose that P is an (m,n)-semiprime submodule and $x \in M - P$. Let $a^m \in (P : x)$. Then we have $a^m x \in P$. Since P is an (m,n)-semiprime submodule, we have $a^n x \in P$, which yields that $a^n \in (P : x)$. Therefore, $(P : x)$ is an (m,n)-closed ideal.

$(ii) \Rightarrow (iii)$: Since $m > n$, we always have $(P :_M a^n) \subseteq (P :_M a^m)$. Let $x \in (P :_M a^m)$. Then we have $a^m \in (P : x)$. If $x \in P$, then clearly, we have $x \in (P : a^n)$. So assume that $x \in M - P$. Then, by (ii), $(P : x)$ is an (m,n)-closed ideal. Since $a^m \in (P : x)$, we conclude that $a^n \in (P : x)$, which yields that $x \in (P :_M a^n)$. Therefore, $(P :_M a^m) \subseteq (P :_M a^n)$.

$(iii) \Rightarrow (iv)$: Suppose that $a^m K \subseteq P$ for some $a \in A$ and a submodule K of M. Thus, we have $K \subseteq (P :_M a^m)$. Then, by (iv), we conclude that $K \subseteq (P :_M a^n)$, and this implies that $a^n K \subseteq P$.

$(iv) \Rightarrow (v)$: It is clear.

$(v) \Rightarrow (i)$: Let $a^m x \in P$ for some $a \in A, x \in M$. If $x \in P$, then we have $a^n x \in P$. Therefore, assume that $x \notin P$. Now, put $K = Ax$. Since $a^m \in (P : K)$ and $(P : K)$ is an (m,n)-closed ideal, we get $a^n \in (P : K)$, which implies that $a^n x \in P$. □

As a result of Theorem 1, we give the following explicit result.

Corollary 1. *Suppose that P is an (m,n)-semiprime submodule of M. Then $(P : M)$ is an (m,n)-closed ideal of A.*

We now give the following example to illustrate that the converse of the previous corollary is not true in general.

Example 3. (A non-(m,n)-semiprime submodule whose residual is (m,n)-closed ideal) Consider $\mathbb{Z}$-module $M = \mathbb{Z}\times\mathbb{Z}$ and submodule $P = (0)\times p^{m+1}\mathbb{Z}$, where p is a prime number and $m > 2$. Then P is not an (m,n)-semiprime submodule since $p^m(0,p) = (0,p^{m+1}) \in P$ and $p^n(0,p) = (0,p^{n+1}) \notin P$ for each $n < m$. On the other hand, $(P : M) = (0)$ is a prime ideal, thus is an (m,n)-closed ideal.

Theorem 2. *Let $f: M \to L$ be an A-homomorphism. The following statements are satisfied*:

(i) If P is an (m,n)-semiprime submodule of M containing $Ker(f)$ and f is surjective, then $f(P)$ is an (m,n)-semiprime submodule of L.

(ii) If K is an (m,n)-semiprime submodule of L such that $f^{-1}(K) \neq M$, then $f^{-1}(K)$ is an (m,n)-semiprime submodule of M.

Proof. (i): Let $a^m y \in f(P)$ for some $a \in A, y \in L$. Since f is surjective, we can write $y = f(x)$ for some $x \in M$. Thus, we have $a^m y = a^m f(x) = f(a^m x) \in f(P)$. As $Ker(f) \subseteq P$, we obtain $a^m x \in P$. Since P is an (m,n)-semiprime submodule of M, we have $a^n x \in P$, so that we have $f(a^n x) = a^n y \in f(P)$. Hence, $f(P)$ is an (m,n)-semiprime submodule of L.

(ii): Let $a^m x \in f^{-1}(K)$ for some $a \in A, x \in M$. Then we have $f(a^m x) = a^m f(x) \in K$. Since K is an (m,n)-semiprime submodule of L, we conclude that $a^n f(x) = f(a^n x) \in K$, which implies that $a^n x \in f^{-1}(K)$. Therefore, $f^{-1}(K)$ is an (m,n)-semiprime submodule of M. $\square$

As an immediate consequence of the previous theorem, we give the following results.

Corollary 2. *Let P be a proper submodule of M. The following statements are satisfied*:

(i) Suppose that K is a submodule of M with $K \subseteq P$. Then P/K is an (m,n)-semiprime submodule of M/K if and only if P is an (m,n)-semiprime submodule of M.

(ii) If P is an (m,n)-semiprime submodule of M and K is a submodule of M with $K \nsubseteq P$, then $P\cap K$ is an (m,n)-semiprime submodule of K.

Let M be an A-module. Then an element $a \in A$ is said to be a *zero divisor* on M if there exists $0 \neq x \in M$ such that $ax = 0$. The set of all zero divisors on M is denoted by $z(M)$.

Theorem 3. *Let M be an A-module and $S \subseteq A$ be a multiplicatively closed set of A. The following statements are satisfied*:

(i) If P is an (m,n)-semiprime submodule of M with $(P : M)\cap S = \emptyset$, then $S^{-1}P$ is an (m,n)-semiprime submodule of $S^{-1}M$.

(ii) If $S^{-1}P$ is an (m,n)-semiprime submodule of $S^{-1}M$ such that $z(M/P)\cap S = \emptyset$, then P is an (m,n)-semiprime submodule of M.

Proof. (i): Suppose that P is an (m,n)-semiprime submodule of M and $\left(\frac{a}{s}\right)^m \frac{x}{t} \in S^{-1}P$ for some $a \in A, x \in M$; $s,t \in S$. This implies that $ua^m x \in P$ and thus $a^m(ux) \in P$ for some u $\in$ S. Since P is an (m,n)-semiprime submodule, we conclude that $a^n(ux) \in P$. This implies that $\left(\frac{a}{s}\right)^n \frac{x}{t} = \frac{a^n(ux)}{s^n ut} \in S^{-1}P$. Therefore, $S^{-1}P$ is an (m,n)-semiprime submodule of $S^{-1}M$.

(ii): Suppose that $S^{-1}P$ is an (m,n)-semiprime submodule of $S^{-1}M$ such that $z(M/P)\cap S = \emptyset$. Let $a^m x \in P$ for some $a \in A, x \in M$. Then we have $\left(\frac{a}{1}\right)^m \frac{x}{1} \in S^{-1}P$. Since $S^{-1}P$ is an (m,n)-semiprime submodule, we get $\left(\frac{a}{1}\right)^n \frac{x}{1} \in S^{-1}P$, which yields that $ua^n x \in P$. If $a^n x \notin P$, we have $u \in S\cap z(M/P)$, which is a contradiction. Hence, we have $a^n x \in P$ and thus P is an (m,n)-semiprime submodule of M. $\square$

Let M be an A-module. The trivial extension or idealization $A \ltimes M = A \oplus M$ of M is a commutative ring with the componentwise addition and the multiplication defined by $(a,x)(b,y) = (ab, ay+bx)$ for each $a,b \in A$; $x,y \in M$ [20]. If I is an ideal of A and P is a submodule of M, then $I \ltimes P$ is an ideal of $A \ltimes M$ if and only if $IM \subseteq P$ [4,13]. In that case, $I \ltimes P$ is said to be a *homogeneous ideal* of $A \ltimes M$. Now, we are ready to determine homogeneous (m,n)-semiprime ideals of the trivial extension $A \ltimes M$.

Theorem 4. *Let I be an ideal of A and P is a proper submodule of M such that $IM \subseteq P$. The following statements are satisfied*:

(i) If $I \ltimes P$ is an (m,n)-semiprime ideal of $A \ltimes M$, then I is an (m,n)-semiprime ideal of A and P is an (m,n)-semiprime submodule of M.

(ii) If I is an (m,n)-semiprime ideal of A and P is an (m,n)-semiprime submodule of M, then $I \ltimes P$ is an $(m,n+1)$-semiprime ideal of $A \ltimes M$.

Proof. (i): Suppose that $I \ltimes P$ is an (m,n)-semiprime ideal of $A \ltimes M$. First, we will show that I is an (m,n)-semiprime ideal of A. To prove this, choose $a,b \in A$ such that $a^m b \in I$. Then we have $(a,0)^m(b,0) = (a^m b,0) \in I \ltimes P$. Since $I \ltimes P$ is an (m,n)-semiprime ideal of $A \ltimes M$, we have $(a,0)^n(b,0) = (a^n b,0) \in I \ltimes P$, which implies that $a^n b \in I$. Thus, I is an (m,n)-semiprime ideal of A. Let $a^m x \in P$ for some $a \in A, x \in M$. Then we have $(a,0)^m(0,x) = (0,a^m x) \in I \ltimes P$, which implies that $(a,0)^n(0,x) = (0,a^n x) \in I \ltimes P$. Thus, we obtain $a^n x \in P$. Therefore, P is an (m,n)-semiprime submodule of M.

(ii): Suppose that I is an (m,n)-semiprime ideal of A and P is an (m,n)-semiprime submodule of M. Let $(a,x)^m(b,y) = (a^m b, ma^{m-1}bx + a^m y) \in I \ltimes P$ for some $a,b \in A$; $x,y \in M$. Then we have $a^m b \in I$ and also $ma^{m-1}bx + a^m y \in P$. Since I is an (m,n)-semiprime ideal of A, we get $a^n b \in I$. Since $m > n$, we have $ma^{m-1}bx \in IM \subseteq P$ and thus we get $a^m y \in P$. As P is an (m,n)-semiprime submodule of M, we have $a^n y \in P$. Therefore, we conclude that $(a,x)^{n+1}(b,y) = (a^{n+1}b, (n+1)a^n bx + a^{n+1}y) \in I \ltimes P$. Hence, $I \ltimes P$ is an $(m,n+1)$-semiprime ideal of $A \ltimes M$. □

Proposition 3. *Let M_1 and M_2 be two A-modules and $M = M_1 \times M_2$. Suppose that P_i is a proper submodule of M_i for each $i = 1,2$. The following statements are equivalent*:

(i) $P = P_1 \times P_2$ is an (m,n)-semiprime submodule of M.

(ii) P_i is an (m,n)-semiprime submodule of M_i for each $i = 1,2$.

Proof. $(i) \Rightarrow (ii)$: Suppose that P is an (m,n)-semiprime submodule of M. Let $a^m x \in P_1$ for some $a \in A, x \in M_1$. Then we have $a^m(x,0) \in P$. Since P is an (m,n)-semiprime submodule of M, we conclude that $a^n(x,0) = (a^n x,0) \in P$, which implies that $a^n x \in P_1$. Thus, P_1 is an (m,n)-semiprime submodule of M_1. A similar argument shows that P_2 is an (m,n)-semiprime submodule of M_2.

$(ii) \Rightarrow (i)$: Suppose that P_i is an (m,n)-semiprime submodule of M_i for each $i = 1,2$. Let $a^m(x_1,x_2) = (a^m x_1, a^m x_2) \in P$ for some $a \in A, x_i \in M_i$. Then we have $a^m x_i \in P_i$. Since P_i is an (m,n)-semiprime submodule, we get $a^n x_i \in P_i$, which yields that $a^n(x_1,x_2) = (a^n x_1, a^n x_2) \in P$. Hence, P is an (m,n)-semiprime submodule of M. □

Let M_i be an A_i-module for each $i = 1,2,\ldots,k$. Suppose that $M = M_1 \times M_2 \times \cdots \times M_k$ and $A = A_1 \times A_2 \times \cdots \times A_k$. Then M is an A-module and each submodule P of M has the form $P = P_1 \times P_2 \times \cdots \times P_k$, where P_i is a submodule of M_i. Now, we characterize (m,n)-semiprime submodules of the Cartesian product of modules.

Theorem 5. *Let M_i be an A_i-module for each $i = 1,2,\ldots,k$, $M = M_1 \times M_2 \times \cdots \times M_k$ and $A = A_1 \times A_2 \times \cdots \times A_k$. Suppose that P_i is a proper submodule of M_i and $P = P_1 \times P_2 \times \cdots \times P_k$. The following statements are equivalent*:

(i) P is an (m,n)-semiprime submodule of M.

(ii) P_i is an (m,n)-semiprime submodule of M_i for each $i = 1,2,\ldots,k$.

Proof. $(i) \Rightarrow (ii)$: Suppose that P is an (m,n)-semiprime submodule of M. Now, choose $t \in \{1,2,\ldots,k\}$ and we will show that P_t is an (m,n)-semiprime submodule of M_t. To see this, take $a_t \in A_t, x_t \in M_t$ such that $a_t^m x_t \in P_t$. Now, put $a = (0,0,\ldots,0,a_t,0,\ldots,0)$ and $x = (0,0,\ldots,0,x_t,0,\ldots,0)$. Then note that $a^m x \in P$. As P is an (m,n)-semiprime submodule, we conclude that $a^n x \in P$, which implies that $a_t^n x_t \in P_t$ and this shows that P_t is an (m,n)-semiprime submodule of M_t.

$(ii) \Rightarrow (i)$: Suppose that P_i is an (m,n)-semiprime submodule of M_i for each $i = 1,2,\ldots,k$. Let $a = (a_1,a_2,\ldots,a_k) \in A$ and $x = (x_1,x_2,\ldots,x_k) \in M$ such that $a^m x \in P$. This implies that $a_i^m x_i \in P_i$. Since P_i is an (m,n)-semiprime submodule of M_i, we have $a_i^n x_i \in P_i$, which yields that $a^n x \in P$. Therefore, P is an (m,n)-semiprime submodule of M. $\square$

Theorem 6. *Let M be an A-module. The following statements are satisfied*:

(i) Suppose that F is a flat A-module and P is an (m,n)-semiprime submodule of M such that $F \otimes P \neq F \otimes M$. Then $F \otimes P$ is an (m,n)-semiprime submodule of $F \otimes M$.

(ii) Suppose that F is a faithfully flat A-module. Then P is an (m,n)-semiprime submodule of M if and only if $F \otimes P$ is an (m,n)-semiprime submodule of $F \otimes M$.

Proof. (i): Let F be a flat A-module and P be an (m,n)-semiprime submodule such that $F \otimes P \neq F \otimes M$. Let $a \in A$. Then, by Theorem 1, $(P :_M a^m) = (P :_M a^n)$. Also, by [6, Lemma 3.2], we have

$$\begin{aligned}(F \otimes P :_{F\otimes M} a^m) &= F \otimes (P :_M a^m)\\ &= F \otimes (P :_M a^n)\\ &= (F \otimes P :_{F\otimes M} a^n).\end{aligned}$$

Again, by Theorem 1, $F \otimes P$ is an (m,n)-semiprime submodule of $F \otimes M$.

(ii): Suppose that F is a faithfully flat A-module.

$(\Rightarrow)$ Let P be an (m,n)-semiprime submodule of M. Assume that $F \otimes P = F \otimes M$. Now, consider the exact sequence $0 \to F \otimes P \overset{\subseteq}{\to} F \otimes M \to 0$. Since F is faithfully flat, the sequence $0 \to P \overset{\subseteq}{\to} M \to 0$ is exact so that $P = M$, which is a contradiction. Thus, $F \otimes P \neq F \otimes M$. As P is an (m,n)-semiprime submodule of M, by (i), $F \otimes P$ is an (m,n)-semiprime submodule of $F \otimes M$.

$(\Leftarrow)$ Let $F \otimes P$ be an (m,n)-semiprime submodule of $F \otimes M$. Take $a \in A$. Then, by Theorem 1 and [6, Lemma 3.2], we have

$$\begin{aligned}F \otimes (P :_M a^m) &= (F \otimes P :_{F\otimes M} a^m)\\ &= (F \otimes P :_{F\otimes M} a^n)\\ &= F \otimes (P :_M a^n).\end{aligned}$$

Thus, we conclude that $F \otimes (P :_M a^m) = F \otimes (P :_M a^n)$. Now, consider the exact sequence $0 \to F \otimes (P :_M a^n) \overset{\subseteq}{\to} F \otimes (P :_M a^m) \to 0$. As F is faithfully flat, we get the exact sequence $0 \to (P :_M a^n) \overset{\subseteq}{\to} (P :_M a^m) \to 0$, which implies that $(P :_M a^n) = (P :_M a^m)$. Again, by Theorem 1, we have that P is an (m,n)-semiprime submodule of M. $\square$

Now, we investigate the conditions under which every submodule of a module is an (m,n)-semiprime.

Theorem 7. *Let M be an A-module and $m > n$ be two positive integers. The following statements are equivalent*:

(i) Every proper submodule P of M is an (m,n)-semiprime submodule.

(ii) For each submodule N of M and each element $a \in A$, the descending chain

$$aN \supseteq a^2N \supseteq a^3N \supseteq \cdots \supseteq a^mN \supseteq \cdots$$

of submodules of M terminates at the n^{th} step.

(iii) $a^nN = a^mN$ for each $a \in A$ and each submodule N of M.

Proof. $(i) \Rightarrow (ii)$: Suppose that every proper submodule P of M is an (m,n)-semiprime submodule of M. Let $a \in A$ and N be a submodule of M. Consider the descending chain $aN \supseteq a^2N \supseteq a^3N \supseteq \cdots \supseteq a^mN \supseteq \cdots$. If $a^nN = M$, then the descending chain terminates at the n^{th} step since $a^nN = a^{n+1}N = \cdots = M$. Therefore, assume that a^nN is a proper submodule of M. As $a^mN \subseteq a^mN$ and a^mN is an (m,n)-semiprime submodule of M, by Theorem 1, $a^nN \subseteq a^mN$ and this implies that $a^nN = a^mN$. This gives that the descending chain $aN \supseteq a^2N \supseteq a^3N \supseteq \cdots \supseteq a^mN \supseteq \cdots$ terminates at the n^{th} step.

$(ii) \Leftrightarrow (iii)$: It is straigtforward.

$(ii) \Rightarrow (i)$: Let P be a proper submodule of M. Now, we will show that P is an (m,n)-semiprime submodule of M. To see this, choose $a \in A$ and a submodule K of M such that $a^m K \subseteq P$. Then, by assumption, $a^m K = a^n K \subseteq P$. Again, by Theorem 1, P is an (m,n)-semiprime submodule of M. □

Recall from [8] that an A-module M is said to be a *multiplication module* if each submodule P of M has the form $P = IM$ for some ideal I of A. In this case, it is clear that $P = (P : M)M$, where $(P : M) = \{a \in A : aM \subseteq P\}$. For more details on multiplication modules, we refer the reader to [1] and [12].

The concept of von Neumann regular rings and its generalizations have aroused great interest, and have been widely studied in many papers (see, for example, [3,10,15,17]. Recall from [22] that a commutative ring A is said to be a *von Neumann regular ring* if for each $a \in A$, there exists $b \in A$ such that $a = a^2 b$. In this case, the principal ideal $(a) = (e)$ is generated by an idempotent element $e \in A$. Recently, Jayaram and Tekir [14] extended the concept of von Neumann regular ring to modules by defining *M-von Neumann regular elements* of a module. Let M be an A-module. Then an element $a \in A$ is said to be an *M-von Neumann regular* if $aM = a^2 M$ [14]. An A-module M is said to be a *von Neumann regular module* if for each $x \in M$, $Ax = aM = a^2 M$ for some $a \in A$. Now, we characterize von Neumann regular modules in terms of (m,n)-semiprime submodules.

Theorem 8. *Let M be a finitely generated A-module. The following statements are equivalent*:

(i) M is a von Neumann regular module.

(ii) M is a multiplication reduced module, in which every proper submodule is (m,n)-semiprime.

Proof. $(i) \Rightarrow (ii)$: Suppose that M is a finitely generated von Neumann regular module. Then clearly M is a multiplication module. According to [14, Lemma 10], M is a reduced module. Also, by [14, Theorem 1], for each $a \in A$, $aM = a^2 M$. Now, let P be a proper submodule of M. Choose an element $a \in A$ and a submodule K of M such that $a^m K \subseteq P$. Since M is multiplication module, we have $K = (K : M)M$, which implies that $a^m K = (K : M)a^m M$. As $aM = a^2 M$, we conclude that $a^m K = (K : M)a^m M = (K : M)a^n M = a^n (K : M)M = a^n K$. This gives that $a^m K = a^n K \subseteq P$. Then, by Thoerem 1, P is an (m,n)-semiprime submodule of M.

$(ii) \Rightarrow (i)$: Let M be a finitely generated reduced multiplication module, in which every proper submodule is (m,n)-semiprime. Let $a \in A$. Now, we will show that $aM = a^2 M$. If $a^m M = M$, then there is nothing to prove. Thus, assume that $a^m M$ is proper. Then, by assumption, $a^m M$ is an (m,n)-semiprime submodule of M. Since $a^m M \subseteq a^m M$, by Theorem 1, $a^n M = a^m M$, which implies that $a^n M = a^{n+1} M$. Since M is a finitely generated module, by [5, Corollary 4], $(1 - ab)a^n M = 0$ for some $b \in A$. Since M is reduced, we conclude that $(1 - ab)aM = 0$, which implies that $aM = a^2 bM \subseteq a^2 M \subseteq aM$. Thus, we have $aM = a^2 M$. Then, by [14, Theorem 1], M is a von Neumann regular module. □

3. CONCLUSIONS

In this article, we have discussed the concept of (m,n)-semiprime submodules, which is a generalization of semiprime submodules of modules over commutative rings. We have investigated many properties of (m,n)-semiprime submodules, as well as the relations between (m,n)-semiprime submodules and other classical ideals/submodules such as (m,n)-closed ideals and semiprime submodules. We have also characterized von Neumann regular modules and a subclass of Artinian modules in terms of this concept.

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