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TOPOLOGICAL ALGEBRAS

Initial objects, terminal objects, zero objects, and equalizers in the category Seg of Segal topological algebras

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Abstract. In this paper the initial, terminal, and zero objects in the category **Seg** of Segal topological algebras are described and some sufficient conditions under which the equalizers in **Seg** exist are found.

Key words: Segal topological algebras, category, initial object, terminal object, zero object, equalizer.

1. HISTORICAL OVERVIEW

The term ‘Segal algebra’ dates back for more than 55 years. It was introduced by Hans Reiter (see [13]). The notion of a Segal algebra was first used in the context of subalgebras of $L^1(G)$, where G was a locally compact group. However, soon it was understood that this concept could be generalized to the case of subalgebras of an arbitrary Banach algebra.

The algebra was called the Segal algebra in honour of Irwin Ezra Segal, who presented in [15] a set of axioms (known as Segal’s axioms) which was used later in the definition. Segal himself tried to describe a general algebraic structure underlying Wiener’s algebra, but he did not exploit this algebraic structure in a systematic fashion. A good reference about the history of Segal algebras is [12].

The usual context of Segal algebras during the last 25 years has been the setting of Banach algebras. About 5 years ago appeared some new generalizations to the case of locally multiplicatively convex (shortly, *lmc*) Fréchet algebras by Abtahi, Rahnama, and Rejali (see [10] and [11]), who called their generalization *Segal Fréchet algebra*, and to the case of complete lmc algebras by Yousofzadeh (see [16,17]).

In [1] (see also [2,4], etc.) it was demonstrated that one could actually take a much more general approach by considering any topological algebras (over $\mathbb{R}$ or $\mathbb{C}$) instead of limiting the study with Banach or lmc (Fréchet or just complete) algebras. For some examples of Segal topological algebras one could check the examples provided in [1].

In [1] the author’s motivation to study such general Segal topological algebras was to obtain results also in the context of topological algebras whose topology can not be described with any family of seminorms. For that, it was necessary to construct proofs without using the properties of seminorms.

After describing some properties of general Segal topological algebras, the categories $\mathscr{S}(B)$and **Seg** were introduced in [4] (the author of the present paper has not seen yet any other paper where the categories of Segal topological algebras, even in Banach case, are studied). Since then, several categorical properties of the category $\mathscr{S}(B)$ have already been studied in several papers (see [2,3,5–9]). The present paper is the first attempt to study also the categorical properties of the category **Seg**.

2. INTRODUCTION

By a *topological algebra* we will mean a topological linear space over the field $\mathbb{K}$ (here $\mathbb{K}$ could be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers) in which is defined a separately continuous (associative, but not necessarily commutative) multiplication. For (topological) algebras A,B we will denote by $1_A : A \to A$ the identity map, i.e. $1_A(a) = a$ for every $a \in A$, and by $\theta_{(A,B)} : A \to B$, a zero map which is defined by $\theta_{(A,B)}(a) = \theta_B$ for every $a \in A$, where θ_B is the zero element of B.

The simplest topological algebra is the algebra Θ, which consists of just one element θ, namely the zero element of Θ. The topology on Θ consists of sets $\Theta = \{\theta\}$ and $\emptyset$. This algebra is topologically isomorphic to any other topological algebra which consists of only one element.

We will start with recalling the definition of a (general) Segal topological algebra, first published in [1]. A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if
(1) $\mathrm{cl}_B(f(A)) = B$;
(2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$;
(3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In short, we will denote a Segal topological algebra by a triple (A, f, B). Condition (2) in the definition of a Segal topological algebra is equivalent to the condition that f is continuous.

For any category $\mathscr{C}$, we denote by $\mathrm{Ob}(\mathscr{C})$ the set of all objects of $\mathscr{C}$. For any $K,L \in \mathrm{Ob}(\mathscr{C})$, we denote by $\mathrm{Mor}(K,L)$ the set of all morphisms from K to L.

As everything will work similarly for left, right, or two-sided Segal topological algebras, we will not mention the sideness in the paper. For better understanding, the reader can think about the left Segal topological algebras, right Segal topological algebras, or two-sided Segal topological algebras, depending on which class of ideals seems to be more familiar.

We will continue with recalling the definition of the category **Seg** of all Segal topological algebras, which was first introduced in [4] together with the definition of a category $\mathscr{S}(B)$ of Segal topological algebras. The category $\mathscr{S}(B)$ is more thoroughly studied in several papers (e.g. [2]). The category **Seg** has all Segal topological algebras as its objects. For any $(A, f, B), (C, g, D)$, the set $\mathrm{Mor}((A, f, B), (C, g, D))$ of morphisms from (A, f, B) to (C, g, D) consists of all such pairs (α, β) of continuous algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to D$ for which $g \circ \alpha = \beta \circ f$. Hence, in case $(A, f, B), (C, g, D) \in \mathrm{Ob}(\mathbf{Seg})$ and $(\alpha, \beta) \in \mathrm{Mor}((A, f, B), (C, g, D))$, we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ C & \xrightarrow{g} & D \end{array}.$$

The composition of morphisms of **Seg** is defined componentwise as follows: for any $(A, f, B), (C, g, D), (E, h, F) \in \mathrm{Ob}(\mathbf{Seg})$ and arbitrary morphisms $(\alpha, \beta) : (A, f, B) \to (C, g, D)$, $(\gamma, \delta) : (C, g, D) \to (E, h, F)$, the composition of (γ, δ) and (α, β) is $(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta)$.

In [4], pp. 2–4, it is shown that this composition of morphisms is correctly defined and associative. Moreover, it is shown that the identity morphism for an object (A, f, B) of **Seg** is a pair $(1_A, 1_B)$ of identity maps.

Fix a topological algebra B. The category $\mathscr{S}(B)$ of Segal topological algebras is a subcategory of **Seg** having as objects all Segal topological algebras in the form (A, f, B), where A and f vary, but B is fixed. The morphisms between $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ are pairs $(\alpha, 1_B)$, where $\alpha : A \to C$ denotes such continuous algebra homomorphism for which $f = 1_B \circ f = g \circ \alpha$. Since here all the objects have one fixed topological algebra (in our case B) and one fixed algebra homomorphism (in our case 1_B), it was easier to study the properties of the category $\mathscr{S}(B)$ first. In the present paper we begin the study of the properties of a more complex category **Seg**.

3. INITIAL, TERMINAL, AND ZERO OBJECTS IN THE CATEGORY SEG

Let us recall from the category theory that an object K of a category $\mathscr{C}$ is
(a) *an initial object* (see for example [14], p. 216) of the category $\mathscr{C}$ if for any object $L \in \mathscr{C}$ the set $\mathrm{Mor}(K,L)$ consists of exactly one morphism;
(b) *a terminal object* (see for example [14], p. 218) of the category $\mathscr{C}$ if for any object $L \in \mathscr{C}$ the set $\mathrm{Mor}(L,K)$ consists of exactly one morphism;
(c) *a zero object* (see for example [14], p. 226) of the category $\mathscr{C}$ if K is both the initial and the terminal object of the category $\mathscr{C}$.

In [2], the initial, terminal, and zero objects of the category $\mathscr{S}(B)$ are described through the properties of the topological algebra B. As we will show below, the description of initial, terminal, and zero objects is more simple in the case of the category **Seg**.

Consider the objects $(A,f,B),(C,g,D)$ of **Seg** and the pair $(\theta_{(A,C)},\theta_{(B,D)})$ of zero maps $\theta_{(A,C)}: A \to C$, $\theta_{(B,D)}: B \to D$. Obviously $\theta_{(A,C)}$ and $\theta_{(B,D)}$ are continuous algebra homomorphisms. Moreover,

$$g \circ \theta_{(A,C)} = \theta_{(A,D)} = \theta_{(B,D)} \circ f.$$

Hence, $(\theta_{(A,C)},\theta_{(B,D)}) \in \mathrm{Mor}((A,f,B),(C,g,D))$ and $\mathrm{Mor}((A,f,B),(C,g,D)) \neq \emptyset$ for any pair $((A,f,B),(C,g,D))$ of objects of **Seg**.

Proposition 1. *In the category* **Seg***, the following claims hold:*
(a) *every initial object of* **Seg** *is topologically isomorphic to* $(\Theta,1_\Theta,\Theta)$*;*
(b) *every terminal object of* **Seg** *is topologically isomorphic to* $(\Theta,1_\Theta,\Theta)$*;*
(c) *every zero object of* **Seg** *is topologically isomorphic to* $(\Theta,1_\Theta,\Theta)$.

Proof. Notice that $(\Theta,1_\Theta,\Theta) \in \mathrm{Ob}(\mathbf{Seg})$. Obviously, $(\Theta,1_\Theta,\Theta)$ is an initial object of **Seg** because for any $(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$ the set $\mathrm{Mor}((\Theta,1_\Theta,\Theta),(C,g,D))$ consists of exactly one pair $(\theta_{(\Theta,C)},\theta_{(\Theta,D)})$.

Similarily, $(\Theta,1_\Theta,\Theta)$ is a terminal object of **Seg**, because for any $(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$, the set $\mathrm{Mor}((C,g,D),(\Theta,1_\Theta,\Theta))$ consists of exactly one pair $(\theta_{(C,\Theta)},\theta_{(D,\Theta)})$. With this we have also shown that $(\Theta,1_\Theta,\Theta)$ is a zero object of **Seg**.

Take any $(A,f,B) \in \mathrm{Ob}(\mathbf{Seg})$ and suppose that (A,f,B) is either an initial object of **Seg** or a terminal object of **Seg**. It means that there exists exactly one morphism $(\alpha,\beta) \in \mathrm{Mor}((A,f,B),(A,f,B))$. From what was noticed before, we know that $(1_A,1_B),(\theta_{(A,A)},\theta_{(B,B)}) \in \mathrm{Mor}((A,f,B),(A,f,B))$. As there could be only one element in $\mathrm{Mor}((A,f,B),(A,f,B))$, we must have $1_A = \theta_{(A,A)}$ and $1_B = \theta_{(B,B)}$, which means that $A = 1_A(A) = \theta_{(A,A)}(A) = \theta_A$ and $B = 1_B(B) = \theta_{(B,B)}(B) = \theta_B$. Hence, both A and B are topologically isomorphic to Θ and (A,f,B) is topologically isomorphic to $(\Theta,1_\Theta,\Theta)$. Therefore every initial or terminal object of **Seg** is topologically isomorphic to $(\Theta,1_\Theta,\Theta)$.

Suppose that $(A,f,B) \in \mathrm{Ob}(\mathbf{Seg})$ is a zero object of **Seg**. Then (A,f,B) is also an initial object of **Seg**, which means that (A,f,B) is topologically isomorphic to $(\Theta,1_\Theta,\Theta)$. With this we have proved that the category **Seg** has (within topological isomorphism) exactly one initial, exactly one terminal, and exactly one zero object, all of which are in the form $(\Theta,1_\Theta,\Theta)$. □

4. EQUALIZERS IN THE CATEGORY SEG

In a category $\mathscr{C}$ with $K,L \in \mathrm{Ob}(\mathscr{C})$ and $\alpha_1,\alpha_2 \in \mathrm{Mor}(K,L)$, the equalizer (see for example [14], p. 225) of morphisms α_1 and α_2 is the pair (M,δ), where $M \in \mathrm{Ob}(\mathscr{C})$ and $\delta \in \mathrm{Mor}(M,K)$ are such that the following two conditions hold:
(1) $\alpha_1 \circ \delta = \alpha_2 \circ \delta$;
(2) for any pair (N,ε) with $N \in \mathrm{Ob}(\mathscr{C})$ and $\varepsilon \in \mathrm{Mor}(N,K)$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$, there exists a unique morphism $\lambda : N \to M$ such that $\varepsilon = \gamma \circ \lambda$.

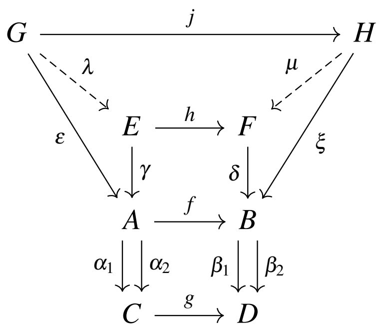
It is well known in category theory that the equalizer of two morphisms $\alpha_1, \alpha_2 \in \mathrm{Mor}(K,L)$ in any universal algebraic category (including the categories Sets of all sets, Ring of rings and Alg of algebras over the field $\mathbb{K}$) is the pair $(K_0, 1_{K_0})$, where $K_0 = \{k \in K : \alpha_1(k) = \alpha_2(k)\}$ and 1_{K_0} is just the inclusion map. In the category TopAlg of all topological algebras, the only difference with the category Alg is that every object is a topological algebra and the morphisms are continuous algebra homomorphisms. The equalizer of $\alpha_1, \alpha_2 \in \mathrm{Mor}(K,L)$ in TopAlg is still the same pair $(K_0, 1_{K_0})$ because the (topological) inclusion is always continuous.

In [4], p. 5, the definition for an equalizer in the category $\mathscr{S}(B)$ is given. Hereby, we generalize this definition for the category **Seg** as follows.

Definition 1. *Let* $(A,f,B),(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$. *The* **equalizer** *of morphisms*

$$(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \mathrm{Mor}((A,f,B),(C,g,D))$$

is a pair $((E,h,F);(\gamma,\delta))$ *such that*
(1) $(E,h,F) \in \mathrm{Ob}(\mathbf{Seg})$ *and* $(\gamma,\delta) \in \mathrm{Mor}((E,h,F),(A,f,B))$ *with* $\alpha_1 \circ \gamma = \alpha_2 \circ \gamma$ *and* $\beta_1 \circ \delta = \beta_2 \circ \delta$*;*
(2) *for any pair* $((G,j,H);(\varepsilon,\xi))$ *with* $(G,j,H) \in \mathrm{Ob}(\mathbf{Seg})$ *and* $(\varepsilon,\xi) \in \mathrm{Mor}((G,j,H),(A,f,B))$ *with* $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$ *and* $\beta_1 \circ \xi = \beta_2 \circ \xi$*, there exists unique* $(\lambda,\mu) \in \mathrm{Mor}((G,j,H),(E,h,F))$ *with* $\varepsilon = \gamma \circ \lambda$ *and* $\xi = \delta \circ \mu$*:*



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We show that if the equalizer $((E,h,F);(\gamma,\delta))$ of morphisms $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \mathrm{Mor}((A,f,B),(C,g,D))$ exists, then $(E;\delta)$ has to be the equalizer of morphisms $\alpha_1, \alpha_2 \in \mathrm{Mor}(A,C)$ in the category TopAlg of topological algebras. Hence, E has to be topologically isomorphic to $A_0 = \{a \in A : \alpha_1(a) = \alpha_2(a)\}$, equipped with the subspace topology inherited from the topology of A.

Lemma 1. *Let* $(A,f,B),(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$, $A_0 = \{a \in A : \alpha_1(a) = \alpha_2(a)\}$ *and* $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \mathrm{Mor}((A,f,B),(C,g,D))$. *If there exist such* $(E,h,F) \in \mathrm{Ob}(\mathbf{Seg})$ *and* $(\gamma,\delta) \in \mathrm{Mor}((E,h,F),(A,f,B))$ *that* $((E,h,F);(\gamma,\delta))$ *is the equalizer of morphisms* (α_1,β_1) *and* (α_2,β_w)*, then* E *is topologically isomorphic to* A_0 *through some topological isomoprhism* $\sigma : E \to A_0$ *and* $\delta = 1_{A_0} \circ \sigma$*.*

Proof. Suppose that $((E,h,F);(\gamma,\delta))$ is the equalizer of morphisms $(\alpha_1,\beta_1),(\alpha_2,\beta_2)$ in the category **Seg** of Segal topological algebras. Take any topological algebra G and a continuous algebra homomorphism $\varepsilon : G \to A$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$. Then $(G,1_G,G) \in \mathrm{Ob}(\mathbf{Seg})$ and there exists a continuous algebra homomorphism $\xi = f \circ \varepsilon : G \to B$ such that

$$\begin{aligned}\beta_1 \circ \xi &= \beta_1 \circ (f \circ \varepsilon) = (\beta_1 \circ f) \circ \varepsilon = (g \circ \alpha_1) \circ \varepsilon = g \circ (\alpha_1 \circ \varepsilon) \\ &= g \circ (\alpha_2 \circ \varepsilon) = (g \circ \alpha_2) \circ \varepsilon = (\beta_2 \circ f) \circ \varepsilon = \beta_2 \circ (f \circ \varepsilon) = \beta_2 \circ \xi.\end{aligned}$$

Thus, $((G,1_G,G);(\varepsilon,\xi))$ satisfies the assumptions of condition (2) of the equalizer of (α_1,β_1) and (α_2,β_2). Hence, by Definition 1, there exists unique $(\lambda,\mu) \in \mathrm{Mor}((G,1_G,G),(E,h,F))$ such that $\varepsilon = \gamma \circ \lambda$ and $\xi = \delta \circ \mu$. This means also that there exists unique $\lambda \in \mathrm{Mor}(G,E)$ such that $\varepsilon = \gamma \circ \lambda$. As this holds for every topological algebra G and a continuous algebra homomorphism $\varepsilon : G \to A$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$, then (E,δ) is the equalizer of α_1 and α_2 in the category TopAlg of topological algebras. As the equalizer is unique up to isomorphism, then E is topologically isomorphic to A_0 through some topological isomorphism $\sigma : E \to A_0$ and $\delta = 1_{A_0} \circ \sigma$. □

By Lemma 1, the equalizer of $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \mathrm{Mor}((A,f,B),(C,g,D))$, if it exists, has the form (A_0,h,F). So, we have to work only with the Segal topological algebras in the form (A_0,h,F), where F is some topological algebra.

We will state the following proposition in the case of left Segal topological algebras. In the case of right or two-sided Segal topological algebras one should demand that $f(A_0)F \subseteq f(A_0)$ or that both $Ff(A_0), f(A_0)F \subseteq f(A_0)$.

Proposition 2. *Suppose that* $(A,f,B),(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$ *and* $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \mathrm{Mor}((A,f,B),(C,g,D))$. *Set* $A_0 = \{a \in A : \alpha_1(a) = \alpha_2(a)\}$, $B_0 = \{b \in B : \beta_1(b) = \beta_2(b)\}$, $F = \mathrm{cl}_B(f(A_0)) \cap B_0$ *and* $F_1 = \{b \in F : bf(A_0) \subseteq f(A_0)\}$. *Then* A_0 *is a subalgebra of* A *and, equipped with the subspace topology, is a topological algebra. Also* B_0, F, *and* F_1 *are subalgebras of* B *and, equipped with the subspace topology, are topological algebras.*

(a) *If* $\mathrm{cl}_B(F_1) = F_1$, *then the equalizer of morphisms* $(\alpha_1,\beta_1),(\alpha_2,\beta_2)$ *exists and is of the form* $((A_0,h,F_1);(\gamma,\delta))$, *where* $h = f|_{A_0}, \gamma = 1_{A_0}$ *and* $\delta = 1_{F_1}$.

(b) *If* $Ff(A_0) \subseteq f(A_0)$, *then the equalizer of morphisms* $(\alpha_1,\beta_1),(\alpha_2,\beta_2)$ *exists and is of the form* $((A_0,h,F);(\gamma,\delta))$, *where* $h = f|_{A_0}, \gamma = 1_{A_0}$ *and* $\delta = 1_F$.

Proof. It is easy to check that A_0 and B_0 are subalgebras of A and B, respectively, because the maps $\alpha_1,\alpha_2,\beta_1$, and β_2 are algebra homomorphisms. As f is an algebra homomorphism, then $f(A_0)$ is a subalgebra of B. As f is also continuous, then $\mathrm{cl}_B(f(A_0))$ is a closed subalgebra of B. Hence, F, as the intersection of two sublagebras of B, is also a subalgebra of B. Because $f(A_0)$ is a subalgebra of B, the set F_1 is also a subalgebra of B. Equipping a subalgebra of a topological algebra with the subspace topology results in a topological algebra.

(a) Take any $a_1,a_2 \in A_0$ and $\lambda \in \mathbb{K}$. Then

$$\alpha_1(a_1+a_2) = \alpha_1(a_1)+\alpha_1(a_2) = \alpha_2(a_1)+\alpha_2(a_2) = \alpha_2(a_1+a_2)$$

and

$$\alpha_1(\lambda a_1) = \lambda\alpha_1(a_1) = \lambda\alpha_2(a_1) = \alpha_2(\lambda a_1).$$

Hence, $a_1+a_2, \lambda a_1 \in A_0$. Now, take any $b_1,b_2 \in h(A_0)$ and $\lambda \in \mathbb{K}$. Then there exist $a_1,a_2 \in A_0$ such that $b_1 = h(a_1)$ and $b_2 = h(a_2)$. As f is an algebra homomorphism, then $h = f|_{A_0}$ is also an algebra homomorphism and

$$b_1+b_2 = h(a_1)+h(a_2) = h(a_1+a_2) \in h(A_0),\ \ \lambda b_1 = \lambda h(a_1) = h(\lambda a_1) \in h(A_0).$$

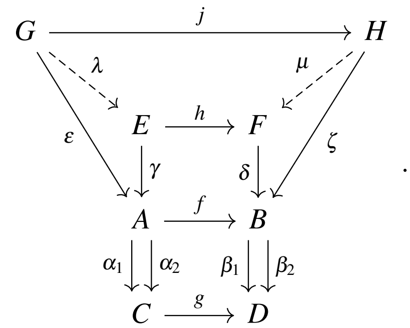
By the definition of F_1,

$$F_1h(A_0) = F_1f|_{A_0}(A_0) = F_1f(A_0) \subseteq f(A_0) = f|_{A_0}(A_0) = h(A_0).$$

Thus, $h(A_0)$ is a left ideal of F_1. Notice that $F_1 \subseteq \mathrm{cl}_B(f(A_0))$. Hence, $\mathrm{cl}_{F_1}(f(A_0)) = \mathrm{cl}_B(f(A_0)) \cap F_1 = F_1$, which means that $f(A_0)$ is dense in F_1. Moreover, as f is a continuous map, then $h = f|_{A_0}$ is also continuous. Therefore, $h(A_0)$ is a dense left ideal of F_1 and $(A_0,h,F_1) \in \mathrm{Ob}(\mathbf{Seg})$.

Notice that $(\alpha_1 \circ \gamma)(a) = \alpha_1(a) = \alpha_2(a) = (\alpha_2 \circ \gamma)(a)$ for every $a \in A_0$ and $(\beta_1 \circ \delta)(b) = \beta_1(b) = \beta_2(b) = (\beta_2 \circ \delta)(b)$ for every $b \in F_1$. Hence, $\alpha_1 \circ \gamma = \alpha_2 \circ \gamma$ and $\beta_1 \circ \delta = \beta_2 \circ \delta$.

Suppose that there are $(G,j,H) \in \mathrm{Ob}(\mathbf{Seg})$ and $(\varepsilon,\zeta) \in \mathrm{Mor}((G,j,H),(A,f,B))$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$ and $\beta_1 \circ \zeta = \beta_2 \circ \zeta$. Take any $g \in G$. Then $\alpha_1(\varepsilon(g)) = (\alpha_1 \circ \varepsilon)(g) = (\alpha_2 \circ \varepsilon)(g) = \alpha_2(\varepsilon(g))$, which means that $\varepsilon(g) \in A_0$. Thus, $\varepsilon(G) \subseteq A_0$ and there exists exactly one map $\lambda = \varepsilon : G \to A_0$ such that $\varepsilon = \gamma \circ \lambda$



Notice that as $f(A_0) \subseteq F_1$, then $\zeta(j(g)) = f(\varepsilon(g)) \in f(A_0) \subseteq F_1$ for every $g \in G$. Hence, $\zeta(j(G)) \subseteq F_1$. As the map ζ is continuous, then, using the assumption $\mathrm{cl}_B(F_1) = F_1$, we obtain that

$$\zeta(H) = \zeta(\mathrm{cl}_H(j(G))) \subseteq \mathrm{cl}_B(\zeta(j(G))) \subseteq \mathrm{cl}_B(F_1) = F_1.$$

Hence, there exists exactly one map $\mu = \zeta : H \to F_1$ such that $\zeta = 1_{F_1} \circ \mu = \delta \circ \mu$.

With this, we have shown that there exists unique morphism $(\lambda, \mu) = (\varepsilon, \zeta) \in \mathrm{Mor}((G, j, H), (A_0, h, F_1))$ such that $\varepsilon = \gamma \circ \lambda$ and $\zeta = \delta \circ \mu$. Hence, $((A_0, 1_{A_0}, F_1); (1_{A_0}, 1_{F_1}))$ is the equalizer of (α_1, β_1) and (α_2, β_2).

(b) As $Ff(A_0) \subseteq f(A_0)$, then $F_1 = F$. Set $h = f|_{A_0}$, $\delta = 1_{A_0}$ and $\delta = 1_F$. Exactly as in the part (a) of the proof, we can show that $(A_0, h, F) \in \mathrm{Ob}(\mathbf{Seg})$.

Suppose that there are $(G, j, H) \in \mathrm{Ob}(\mathbf{Seg})$ and $(\varepsilon, \zeta) \in \mathrm{Mor}((G, j, H), (A, f, B))$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$ and $\beta_1 \circ \zeta = \beta_2 \circ \zeta$. Exactly as in the part (a) of the proof, we can show that there exists exactly one map $\lambda = \varepsilon : G \to A_0$ such that $\varepsilon = \gamma \circ \lambda$.

Take any $x \in H$. Then one has $\beta_1(\zeta(x)) = (\beta_1 \circ \zeta)(x) = (\beta_2 \circ \zeta)(x) = \beta_2(\zeta(x))$, which means that $\zeta(x) \in B_0$. On the other hand, as $H = \mathrm{cl}_H(j(G))$, there exists a family $(g_\kappa)_{\kappa \in \mathscr{K}}$ such that the family $j(g_\kappa)$ converges to x. Now it follows from the continuity of ζ that the family $(f \circ \varepsilon)(g_\kappa) = (\zeta \circ j)(g_\kappa)$ converges to $\zeta(x)$. As $(f \circ \varepsilon)(g_\kappa) \in f(\varepsilon(G)) \subseteq f(A_0)$ for every $\kappa \in \mathscr{K}$, then $\zeta(x) \in \mathrm{cl}_B(f(A_0))$. Hence, $\zeta(H) \subseteq F$ and there exists exactly one map $\mu = \zeta : H \to F$ such that $\zeta = \delta \circ \mu$.

With this we have shown that there exists unique morphism $(\lambda, \mu) = (\varepsilon, \zeta) \in \mathrm{Mor}((G, j, H), (A_0, h, F))$ such that $\varepsilon = \gamma \circ \lambda$ and $\zeta = \delta \circ \mu$. Thus, $((A_0, 1_{A_0}, F); (1_{A_0}, 1_F))$ is the equalizer of (α_1, β_1) and (α_2, β_2). □

Remark 1. Notice that if the assumptions of parts (a) and (b) of Theorem 1 are fulfilled at the same time, then $F_1 = F$.

Corollary 1. *Suppose that* $(A, f, B), (C, g, D) \in \mathrm{Ob}(\mathbf{Seg})$ *and* $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathrm{Mor}((A, f, B), (C, g, D))$. *Set* $A_0 = \{a \in A : \alpha_1(a) = \alpha_2(a)\}$, $B_0 = \{b \in B : \beta_1(b) = \beta_2(b)\}$, *and* $F = \mathrm{cl}_B(f(A_0)) \cap B_0$. *If* $B_0 f(A_0) \subseteq f(A_0)$, *then the equalizer of morphisms* $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ *exists and is of the form* $((A_0, h, F); (\gamma, \delta))$, *where* $h = f|_{A_0}, \gamma = 1_{A_0}$ *and* $\delta = 1_F$.

Proof. As $F \subseteq B_0$, then $Ff(A_0) \subseteq B_0 f(A_0) \subseteq f(A_0)$. Hence, the claim follows from the part (b) of Proposition 2. □

Open questions. (1) We have seen that the conditions $\mathrm{cl}_B(F_1) = F_1$ and $Ff(A_0) \subseteq f(A_0)$ of Proposition 2 are both sufficient for the existence of an equalizer. Is any of these conditions actually a necessary condition for the existence of an equalizer?

(2) Which are the necessary and sufficient conditions for the existence of equalizers in **Seg**?

5. CONCLUSIONS

In the present paper we started the study of the category **Seg** of Segal topological algebras by describing the initial, terminal, and zero objects in this category and finding some sufficient conditions under which the equalizers exist in this category.

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Alg-, lõpp- ja nullobjektidest ning võrdsustajatest Segali topoloogiliste algebrate kategoorias Seg

Mart Abel

On alustatud Segali topoloogiliste algebrate kategooria **Seg** kategoorsete omaduste uurimist. Sel teel on antud selle kategooria kõigi alg-, lõpp- ja nullobjektide kirjeldused ning leitud mõned piisavad tingimused selleks, et selles kategoorias eksisteeriksid võrdsustajad.