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FUNCTIONAL EQUATIONS

# About a function that allows calculation of all symmetric homogeneous bivariate means

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**Abstract.** In this paper we define a function that allows us to calculate all symmetric homogeneous bivariate means. We also provide examples for this function in case of 17 means.

**Key words:** bivariate means, functional equation.

## 1. INTRODUCTION

Means and bivariate means have been studied by several mathematicians during the last 20 years (see [1,2,7]). Different means appear in case of 17 means in mathematical analysis, statistics, elementary mathematics, etc. Therefore, it is good to have a single formula that would describe simultaneously many means in order to give a possibility of proving something for a whole class of different means with just a single proof. The search for this kind of a unique descriptive function has led mathematicians to different formulas, using integrals (see [6] or [5]) or integrals and measure (see [2]). However, there are actually more elementary formulas for describing the class of all symmetric homogeneous bivariate means, as we will show in the present paper.

In [6], the bivariate map $m_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ was defined for every function $f : \mathbb{R}^+ \to \mathbb{R}$ by the formula

$$m_f(a,b) = \begin{cases} \frac{2(a-b)}{f(\frac{a}{b})-f(\frac{b}{a})}, & a \neq b \\ a, & a = b \end{cases}.$$

The authors of [6] found also some necessary and sufficient conditions under which the map $m_f$ satisfies the conditions

$$\min(a,b) \leqslant m_f(a,b) \leqslant \max(a,b)$$

of a bivariate mean (see [6], Theorem 3.1, pp. 242–243).

In the same paper, some functions $f$ described are such that $m_f(a,b)$ presents some symmetric homogeneous bivariate means. The process of finding $f$ such that $m_f$ will coincide with some given mean is given

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through the selection of some suitable function $u \in C([0,\infty))$ such that $e^{-x} \leqslant u(x) \leqslant e^{x}$ for every $x \in [0,\infty)$, integrating it and composing with the logarithmic function (for details, see [6], Corollary 3.2 and Example 3.1, pp. 244–245). The authors start with $u$ (without any explanation why this $u$ was chosen) and show that after integrating and composing, the resulting function $f$ will force $m_f$ to be some known mean. However, it is not very clearly explained how to find the function $f$ when one wants to present some specific mean in a form $m_f$.

While working on his thesis and trying to complete the proofs, the author of [3] found a simpler method, which does not include any integration or composing and which is suitable for finding $f$ for any symmetric homogeneous bivariate mean $M$ such that $m_f = M$. This method was used in [3] succesfully for finding $f$ for 17 different symmetric homogeneous bivariate means. The present paper gives first an overview of this method with some necessary results. As an application, the functions $f$ for the 17 different means are provided.

In addition to the context of [3] and [6], we show that the function $m_f$ could be used for describing not just some but all symmetric homogeneous bivariate means and for no other means. Actually, for a symmetric homogeneous bivariate mean $M$ there are infinitely many different maps $f:\mathbb{R}^+ \to \mathbb{R}$ such that $m_f = M$. We finish the paper by giving a complete description of all maps $f$ that satisfy the condition $m_f = M$ for any fixed symmetric homogeneous bivariate mean $M$.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

In this paper we deal only with maps that are defined on the set $\mathbb{R}^+$ of positive real numbers or on the set $\mathbb{R}^+ \times \mathbb{R}^+$ of pairs of positive real numbers but can take values in the whole set $\mathbb{R}$ of real numbers.

First, we remind that a map $M:\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be
(a) *bivariate mean* if $\min(a,b) \leqslant M(a,b) \leqslant \max(a,b)$ for all $a,b \in \mathbb{R}^+$;
(b) *symmetric* if $M(a,b) = M(b,a)$ for all $a,b \in \mathbb{R}^+$;
(c) *homogeneous* if $M(ca,cb) = cM(a,b)$ for all $a,b,c \in \mathbb{R}^+$.

In the present paper, the following bivariate means will be used:
(1) *minimum*, which is defined as

$$\min(a,b) = \begin{cases} a, & a \leqslant b \\ b, & a > b \end{cases};$$

(2) *maximum*, which is defined as

$$\max(a,b) = \begin{cases} a, & a \geqslant b \\ b, & a < b \end{cases};$$

(3) *arithmetic mean*, which is defined as $A(a,b) = \frac{a+b}{2}$;
(4) *harmonic mean*, which is defined as $H(a,b) = \frac{2ab}{a+b}$;
(5) *geometric mean*, which is defined as $G(a,b) = \sqrt{ab}$;
(6) *root mean square*, which is defined as $RMS(a,b) = \sqrt{\frac{a^2+b^2}{2}}$;
(7) *logarithmic mean*, which is defined as

$$L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b}, & a \neq b \\ a, & a = b \end{cases};$$

(8) *first Seiffert mean*, which is defined as

$$SF(a,b) = \begin{cases} \frac{a-b}{2\arcsin\frac{a-b}{a+b}}, & a \neq b \\ a, & a = b \end{cases};$$

(9) *second Seiffert mean*, which is defined as

$$SS(a,b)=\begin{cases}\frac{a-b}{2\arctan\frac{a-b}{a+b}}, & a\neq b\\ a, & a=b\end{cases};$$

(10) *Neuman–Sándor mean*, which is defined as

$$NS(a,b)=\begin{cases}\frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}}, & a\neq b\\ a, & a=b\end{cases};$$

(11) *identric mean*, which is defined as

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, & a\neq b\\ a, & a=b\end{cases};$$

(12) *centroidal mean*, which is defined as $C(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)}$;
(13) *contraharmonic mean*, which is defined as $CH(a,b)=\frac{a^2+b^2}{a+b}$;
(14) *Heron mean*, which is defined as $HE(a,b)=\frac{a+\sqrt{ab}+b}{3}$;
(15) *Lehmer mean*, which is defined as $LE(a,b)=\frac{a^p+b^p}{a^{p-1}+b^{p-1}}$ for some fixed $p\in\mathbb{R}$;
(16) *Hölder mean* (for some $p\in\mathbb{R}\setminus\{0\}$), which is defined as

$$H\ddot{O}(a,b)=\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}};$$

(17) *Stolarsky mean* (for some $p,q\in\mathbb{R}\setminus\{0\}$ with $p\neq q$), which is defined as

$$S(a,b)=\begin{cases}\left(\frac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}}, & a\neq b\\ a, & a=b\end{cases};$$

(18) *weighted arithmetic mean* (for some $p,q\in\mathbb{R}^+$), which is defined as $WA(a,b)=\frac{pa+qb}{p+q}$;
(19) *weighted harmonic mean* (for some $p,q\in\mathbb{R}^+$), which is defined as $WH(a,b)=\frac{p+q}{\frac{p}{a}+\frac{q}{b}}$;
(20) *weighted root mean square* (for some $p,q\in\mathbb{R}^+$), which is defined as $WRMS(a,b)=\sqrt{\frac{pa^2+qb^2}{p+q}}$.

It is straightforward to check that the first 17 means are symmetric homogeneous bivariate means and that the last 3 means are not symmetric if $p\neq q$. Examples of bivariate means that are not homogeneous are given in [4], Theorem 1, p. 158.

Following the notation of [6], we denote

$$F_-=\{g=\mu\circ\ln;\ \mu:\mathbb{R}\to\mathbb{R}\ \text{ is an odd function}\},\quad F_+=\{g=\nu\circ\ln;\ \nu:\mathbb{R}\to\mathbb{R}\ \text{ is an even function}\}.$$

In [6], the following two results (we present them here in a slightly different form) are given with short proofs or hints for a proof (the complete proofs of these results are given in [3]):

**Lemma 1** ([6], Lemma 2.1, p. 240; [3], Lemma 2.1, p. 5.)**.** *Let $g:\mathbb{R}^+\to\mathbb{R}$. Then the following two assertions are equivalent:*

(i) *For all* $x > 0$ *we have* $g(\frac{1}{x}) = -g(x)$ *(respectively,* $g(\frac{1}{x}) = g(x)$*).*
(ii) *There exists an odd (respectively, an even) function* $\mu : \mathbb{R} \to \mathbb{R}$ *such that* $g(x) = (\mu \circ \ln)(x)$ *for every* $x \in \mathbb{R}^+$.

**Lemma 2** ([6], Lemma 2.3, p. 242; [3], Lemma 2.4, p. 9.)**.** *Let* $g \in F_-$ *and* $f : \mathbb{R}^+ \to \mathbb{R}$*. Then the following two assertions are equivalent:*
(i) $f$ *is a solution of the functional equation*

$$f(x) - f\left(\frac{1}{x}\right) = 2g(x) \text{ for every } x \in \mathbb{R}^+;$$

(ii) *there exists* $h \in F_+$ *such that* $f = g + h$*.*

We will use these results in the following section.

## 3. THE FUNCTION $M_F$ AND SYMMETRIC HOMOGENEOUS BIVARIATE MEANS

All bivariate means $M$ that were expressed via the map $m_f$ (defined as $m_f(a,b) = \frac{2(a-b)}{f(\frac{a}{b})-f(\frac{b}{a})}$ for all $a,b \in \mathbb{R}^+$, with $f : \mathbb{R}^+ \to \mathbb{R}$) in [6] were symmetric and homogeneous. Although the authors of [6] do not give any reason for considering only symmetric homogeneous bivariate means, Proposition 1 will show that other types of bivariate means $M$ can not be expressed in the form $M = m_f$ for any $f : \mathbb{R}^+ \to \mathbb{R}$. Moreover, the authors of [6] do not mention either that if $M = m_f$, then the map $f : \mathbb{R}^+ \to \mathbb{R}$ will have to satisfy the condition $f(x) > f(\frac{1}{x})$ for every $x > 1$.

**Proposition 1.** *Let* $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a bivariate mean. If there exists* $f : \mathbb{R}^+ \to \mathbb{R}$ *such that* $M = m_f$*, then* $f(x) > f(\frac{1}{x})$ *for every* $x > 1$ *and* $M$ *is both symmetric and homogeneous.*

*Proof.* Suppose that the function $m_f$, defined in the introductory part, coincides with some bivariate mean $M$. Then $0 < \min(a,b) \leqslant m_f(a,b) \leqslant \max(a,b)$ for every $a,b \in \mathbb{R}^+$. Hence, $m_f(a,b) > 0$ for all $a,b \in \mathbb{R}^+$. Take any $x > 1$ and choose $a = x, b = 1$. Then $a,b \in \mathbb{R}^+$ with $a > b$ and $m_f(a,b) = \frac{2(x-1)}{f(x)-f(\frac{1}{x})}$. We see that from $M(x,1) = m_f(x,1) > 0$ it follows that $f(x) > f(\frac{1}{x})$. As $x > 1$ was chosen arbitrarily, we must have $f(x) > f(\frac{1}{x})$ for every $x > 1$.

Take any $a,b,c \in \mathbb{R}^+$. Notice that since $M = m_f$, then

$$M(a,b) = m_f(a,b) = \frac{2(a-b)}{f(\frac{a}{b}) - f(\frac{b}{a})} = \frac{2(b-a)}{f(\frac{b}{a}) - f(\frac{a}{b})} = m_f(b,a) = M(b,a)$$

and

$$M(ca,cb) = m_f(ca,cb) = \frac{2(ca-cb)}{f(\frac{ca}{cb}) - f(\frac{cb}{ca})} = c\frac{2(a-b)}{f(\frac{a}{b}) - f(\frac{b}{a})} = cm_f(a,b) = cM(a,b).$$

Hence, $M$ is both symmetric and homogeneous. □

Therefore, in what follows, we need to study only these functions $f : \mathbb{R}^+ \to \mathbb{R}$ for which $f(x) > f(\frac{1}{x})$ for every $x > 1$.

In [3], the following modification of Theorem 3.1 from pp. 242–243 in [6] was proved:

**Theorem 1** ([3], Theorem 3.2, pp. 11–12)**.**
(i) *Let* $f : \mathbb{R}^+ \to \mathbb{R}$ *be such that* $f(x) > f(\frac{1}{x})$ *for every* $x > 1$*. If* $m_f$ *coincides with some bivariate mean, then there exist* $g \in F_-$ *and* $h \in F_+$ *such that* $f = g + h$ *and* $1 - \frac{1}{x} \leqslant g(x) \leqslant x - 1$ *for every* $x > 1$*.*
(ii) *Let* $h \in F_+$ *and* $g \in F_-$ *be such that* $1 - \frac{1}{x} \leqslant g(x) \leqslant x - 1$ *for every* $x > 1$*. Then* $f = g + h$ *is such that* $f(x) > f(\frac{1}{x})$ *for every* $x > 1$ *and* $m_f$ *coincides with some bivariate mean.*

**Remark 1.** In (i) in [6], it is demanded that $f$ should be a monotonic function on $(0,\infty)$ and claimed that then $1-\frac{1}{x}\leqslant g(x)\leqslant x-1$ for every $x\geqslant 1$. Actually, the monotonic function could be also decreasing, which does not give us the necessary condition $f(x)>f(\frac{1}{x})$. In [3], the monotonicity is replaced by the condition $f(x)>f(\frac{1}{x})$ for every $x>1$, which guarantees the condition $m_f(a,b)>0$ for all $a,b\in\mathbb{R}^+$. The condition $1-\frac{1}{x}\leqslant g(x)\leqslant x-1$ for $x=1$ gives us $g(1)=0$, which is automatically fulfilled if one defines $g(x)=\frac{1}{2}(f(x)-f(\frac{1}{x}))$ for every $x>0$, as it is actually done in the proof both in [6] and [3].

In the proof of part (ii) of Theorem 1, one just has to notice that since $g\in F_-$ and $h\in F_+$, then, by Lemma 1, $g(x)=-g(\frac{1}{x})$ and $h(x)=h(\frac{1}{x})$, which gives us $f(x)-f(\frac{1}{x})=2g(x)\geqslant 2(1-\frac{1}{x})>0$ for any $x>1$. Hence, $f(x)>f(\frac{1}{x})$ for every $x>1$, as desired.

The following result gives a formula how to find for a given symmetric homogeneous bivariate mean $M$ a map $f$ such that $m_f=M$. This map $f$ is certainly not the unique map for which $M=m_f$, but all other suitable maps $f'$, with $m_{f'}=M$, are of the form $f'=f+h$, where $h\in F_+$ (as will be shown later on in Theorem 2).

**Proposition 2** ([3], Lemma 4.1, p. 14.)**.** *Let $M:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be a symmetric homogeneous bivariate mean and let $f:\mathbb{R}^+\to\mathbb{R}$ be a map defined by $f(x)=\frac{x-1}{M(1,x)}$ for every $x\in\mathbb{R}^+$. Then $f\in F_-$, $m_f=M$ and $f(x)>f(\frac{1}{x})$ for every $x>1$.*

*Proof.* As $M$ is homogeneous and symmetric, then $M(1,x)=xM(\frac{1}{x},1)=xM(1,\frac{1}{x})$ for every $x\in\mathbb{R}^+$. Thus,

$$f\Big(\frac{1}{x}\Big)=\frac{\frac{1}{x}-1}{M(1,\frac{1}{x})}=\frac{1-x}{xM(1,\frac{1}{x})}=-\frac{x-1}{M(1,x)}=-f(x)$$

for every $x\in\mathbb{R}^+$. By Lemma 1, $f\in F_-$.

If $a=b>0$, then, by the definitions of $m_f$ and the bivariate mean, we see that $m_f(a,b)=a=M(a,b)$.

Let $a,b\in\mathbb{R}^+$ be such numbers that $a\neq b$. As $M$ is homogeneous and symmetric, then $bM(1,\frac{a}{b})=M(b,a)=M(a,b)=aM(1,\frac{b}{a})$. Therefore,

$$m_f(a,b)=\frac{2(a-b)}{f(\frac{a}{b})-f(\frac{b}{a})}=\frac{2(a-b)}{\frac{\frac{a}{b}-1}{M(1,\frac{a}{b})}-\frac{\frac{b}{a}-1}{M(1,\frac{b}{a})}}=\frac{2(a-b)}{\frac{a-b}{bM(1,\frac{a}{b})}+\frac{a-b}{aM(1,\frac{b}{a})}}=M(a,b).$$

Hence, $m_f(a,b)=M(a,b)$ for all $a,b\in\mathbb{R}^+$. Thich means that $m_f=M$.

By Proposition 1, $f(x)>f(\frac{1}{x})$ for every $x>1$. □

From Propositions 1 and 2 we obtain the following result.

**Corollary 1.** *Let $M:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be an arbitrary bivariate mean. Then the following are equivalent:*
(i) *there exists $f:\mathbb{R}^+\to\mathbb{R}$ such that $m_f=M$;*
(ii) *$M$ is homogeneous and symmetric.*
*Moreover, in both cases the map $f$, mentioned in* (i)*, satisfies condition*
(iii) *$f(x)>f(\frac{1}{x})$ for every $x>1$.*

*Proof.* Proposition 1 tells us that if there exists $f:\mathbb{R}^+\to\mathbb{R}$ such that $m_f=M$, then $f(x)>f(\frac{1}{x})$ for every $x>1$ and $M$ is both symmetric and homogeneous. Hence, (ii) follows from (i) and condition (iii) is also fulfilled.

Proposition 2 gives us the existence of a map $f$ with $f(x)>f(\frac{1}{x})$ for every $x>1$ such that $m_f=M$. Hence, from (ii) follows (i) and condition (iii) is also fulfilled.

Thus, claims (i) and (ii) are really equivalent and condition (iii) follows from any of conditions (i) and (ii). □

With Proposition 2, we have obtained a scheme for finding the maps $f$ for given $M$ such that $m_f$ coincides with $M$:

Step 0: Check that $M$ is a symmetric homogeneous bivariate mean.

Step 1: Define $f(x) = \frac{x-1}{M(1,x)}$ for every $x \in \mathbb{R}^+$.

Step 2: Using the technique given in the proof of Proposition 1, check that $m_f(a,b) = M(a,b)$ for all $a,b \in \mathbb{R}^+$.

Using these steps, the maps $f$ for 17 different symmetric homogeneous means were obtained in [3]. The results are given in Table 1. The easy calculations (which are similar to the calculations made in the proof of Proposition 1 and could be found in [3]) are left for the reader.

**Table 1.** Maps $f$ for different means

| Mean | Value of $M(a, b)$ | Value of $f(x)$ |
|---|---|---|
| Minimum | $\min(a,b)$ | $\frac{x-1}{\min(1,x)}$ |
| Maximum | $\max(a,b)$ | $\frac{x-1}{\max(1,x)}$ |
| Arithmetic mean | $\frac{a+b}{2}$ | $\frac{2(x-1)}{x+1}$ |
| Harmonic mean | $\frac{2ab}{a+b}$ | $\frac{x^2-1}{2x}$ |
| Geometric mean | $\sqrt{ab}$ | $\frac{x-1}{\sqrt{x}}$ |
| Root mean square | $\sqrt{\frac{a^2+b^2}{2}}$ | $\frac{\sqrt{2}(x-1)}{\sqrt{1+x^2}}$ |
| Logarithmic mean | $\begin{cases} \frac{a-b}{\ln a - \ln b}, & a \neq b \\ a, & a = b \end{cases}$ | $\ln x$ |
| First Seiffert mean | $\begin{cases} \frac{a-b}{2\arcsin\frac{a-b}{a+b}}, & a \neq b \\ a, & a = b \end{cases}$ | $2\arcsin\frac{x-1}{x+1}$ |
| Second Seiffert mean | $\begin{cases} \frac{a-b}{2\arctan\frac{a-b}{a+b}}, & a \neq b \\ a, & a = b \end{cases}$ | $2\arctan\frac{x-1}{x+1}$ |
| Neuman–Sándor mean | $\begin{cases} \frac{a-b}{2\,\mathrm{arcsinh}\frac{a-b}{a+b}}, & a \neq b \\ a, & a = b \end{cases}$ | $2\,\mathrm{arcsinh}\frac{x-1}{x+1}$ |
| Identric mean | $\begin{cases} \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, & a \neq b \\ a, & a = b \end{cases}$ | $\begin{cases} e(x-1)x^{\frac{x}{x-1}}, & x \neq 1 \\ 0, & x = 1 \end{cases}$ |
| Centroidal mean | $\frac{2(a^2+ab+b^2)}{3(a+b)}$ | $\frac{3(x^2-1)}{2(1+x+x^2)}$ |
| Contraharmonic mean | $\frac{a^2+b^2}{a+b}$ | $\frac{x^2-1}{x^2+1}$ |
| Heron mean | $\frac{a+\sqrt{ab}+b}{3}$ | $\frac{3(x-1)}{1+\sqrt{x}+x}$ |
| Lehmer mean | $\frac{a^p+b^p}{a^{p-1}+b^{p-1}}$ | $\frac{(x-1)(1+x^{p-1})}{1+x^p}$ |
| Hölder mean | $\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$ | $\frac{(x-1)2^{\frac{1}{p}}}{(1+x^p)^{\frac{1}{p}}}$ |
| Stolarsky mean | $\begin{cases} \left(\frac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}}, & a \neq b \\ a, & a = b \end{cases}$, | $\begin{cases} \frac{x-1}{\left(\frac{q(1-x^p)}{p(1-x^q)}\right)^{\frac{1}{p-q}}}, & x \neq 1 \\ 0, & x = 1 \end{cases}$ |

We would like to remark here that for the identric mean and the Stolarsky mean we have

$$\lim_{x \to 1} f(x) = 0.$$

Comparison of our table with the table given in [6] (see *Table 1.1 Fitted standard means*, p. 240) shows that the maps $f_R$ (we use the subindex $R$ to distinguish the maps given in [6] from the maps $f$ given in Table 1 in the present paper) for constructing $m_f$ are in some cases different from the maps $f$ in the present paper. More precisely, in [6] they offer for the harmonic mean $f_R(x) = x$ or $f_R(x) = -\frac{1}{x}$, for the arithmetic mean $f_R(x) = -\frac{4}{x+1}$, for the geometric mean $f_R(x) = 2\sqrt{x}$, for the first Seiffert mean $f_R(x) = 4\arctan\sqrt{x}$, and for the second Seiffert mean $f_R(x) = 2\arctan x$. For the logarithmic mean and the Neuman–Sándor mean the values of $f_R(x)$ given in [6] coincide with the ones presented in the table above.

Notice that the difference $d = f - f_R$ of the map $f$ given in the present paper and $f_R$ given in [6] is a map from $F_+$, which, by Lemma 1, is equivalent to the fact that $d(x) = d(\frac{1}{x})$ for every $x \in \mathbb{R}^+$. We will show that $d \in F_-$ by checking that $d(x) = d(\frac{1}{x})$ for every $x \in \mathbb{R}^+$.

For the harmonic mean, we have two possible differences with

$$d_1(x) = \frac{x^2-1}{2x} - x = -\frac{1}{2}\left(x + \frac{1}{x}\right) = -\frac{1}{2}\left(\frac{1}{x} + x\right) = \frac{\left(\frac{1}{x}\right)^2 - 1}{2 \cdot \frac{1}{x}} - \frac{1}{x} = d_1\left(\frac{1}{x}\right)$$

and

$$d_2(x) = \frac{x^2-1}{2x} - \left(-\frac{1}{x}\right) = \frac{1}{2}\left(x + \frac{1}{x}\right) = \frac{1}{2}\left(\frac{1}{x} + x\right) = \frac{\left(\frac{1}{x}\right)^2 - 1}{2 \cdot \frac{1}{x}} - \left(-\frac{1}{\frac{1}{x}}\right) = d_2\left(\frac{1}{x}\right)$$

for every $x \in \mathbb{R}^+$.

For the arithmetic mean, we have

$$d(x) = \frac{2(x-1)}{x+1} - \left(-\frac{4}{x+1}\right) = \frac{2x+2}{x+1} = 2$$

for every $x \in \mathbb{R}^+$. Hence, $d(x) = 2 = d(\frac{1}{x})$ for every $x \in \mathbb{R}^+$.

For the geometric mean, we have

$$d(x) = \frac{x-1}{\sqrt{x}} - 2\sqrt{x} = -\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = -\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right) = \frac{\frac{1}{x} - 1}{\sqrt{\frac{1}{x}}} - 2\sqrt{\frac{1}{x}} = d\left(\frac{1}{x}\right)$$

for every $x \in \mathbb{R}^+$.

Let us recall some trigonometric identities that hold for all $w \in [-1, 1]$ and for all $u, v \in \mathbb{R}$:

$$\arcsin w = 2\arctan\frac{w}{1+\sqrt{1-w^2}}; \qquad \arctan u - \arctan v = \arctan\frac{u-v}{1+uv}.$$

We will use these identities in the calculations for the first Seiffert mean and the second Seiffert mean.

For the first Seiffert mean, we obtain

$$d(x) = 2\arcsin\frac{x-1}{x+1} - 4\arctan\sqrt{x} = 4\arctan\frac{\frac{x-1}{x+1}}{1+\sqrt{1-(\frac{x-1}{x+1})^2}} - 4\arctan\sqrt{x}$$

$$= 4\left(\arctan\frac{x-1}{(\sqrt{x}+1)^2} - \arctan\sqrt{x}\right) = 4\arctan\frac{\frac{x-1}{(\sqrt{x}+1)^2} - \sqrt{x}}{1+\frac{x-1}{(\sqrt{x}+1)^2}\sqrt{x}} = 4\arctan(-1) = 4\left(-\frac{\pi}{4}\right) = -\pi$$

for every $x \in \mathbb{R}^+$. Hence, $d(x) = -\pi = d(\frac{1}{x})$ for every $x \in \mathbb{R}^+$.

For second Seiffert mean, we have

$$d(x)=2\arctan\frac{x-1}{x+1}-2\arctan x=2\arctan\frac{\frac{x-1}{x+1}-x}{1+\frac{x-1}{x+1}x}=2\arctan(-1)=2\left(-\frac{\pi}{4}\right)=-\frac{\pi}{2}$$

for every $x\in\mathbb{R}^+$. Hence, $d(x)=-\frac{\pi}{2}=d(\frac{1}{x})$ for every $x\in\mathbb{R}^+$.

To sum this all up, we have the following theorem, which describes all functions $f$ for a given symmetric homogeneous bivariate mean $M$ for which $M=m_f$.

**Theorem 2.** *Let $M$ be a symmetric homogeneous bivariate mean and $f':\mathbb{R}^+\to\mathbb{R}$. Then $m_{f'}=M$ if and only if there exists $h\in F_+$ such that $f'=f+h$, where $f(x)=\frac{x-1}{M(1,x)}$ for every $x\in\mathbb{R}^+$.*

*Proof.* Suppose that $m_{f'}=M$. By Proposition 2, we already know that $M=m_f$. Hence, for all $a,b\in\mathbb{R}^+$,

$$\frac{2(a-b)}{f'(\frac{a}{b})-f'(\frac{b}{a})}=m_{f'}=m_f=\frac{2(a-b)}{f(\frac{a}{b})-f(\frac{b}{a})}.$$

From this we obtain that $f'(\frac{a}{b})-f'(\frac{b}{a})=f(\frac{a}{b})-f(\frac{b}{a})$, which gives us $f'(\frac{a}{b})-f(\frac{a}{b})=f'(\frac{b}{a})-f(\frac{b}{a})$ for all $a,b\in\mathbb{R}^+$. Set $h=f'-f$ and take any $x\in\mathbb{R}^+$. Put $a=x,b=1$. Then

$$h(x)=(f'-f)(x)=f'(x)-f(x)=f'\left(\frac{a}{b}\right)-f\left(\frac{a}{b}\right)=f'\left(\frac{b}{a}\right)-f\left(\frac{b}{a}\right)$$
$$=f'\left(\frac{1}{x}\right)-f\left(\frac{1}{x}\right)=(f'-f)\left(\frac{1}{x}\right)=h\left(\frac{1}{x}\right).$$

Since this holds for every $x\in\mathbb{R}^+$, then $h\in F_+$. Hence, there exists $h\in F_+$ such that $f'=f+h$.

To prove the converse implication, take any $h\in F_+$ and set $f'=f+h$. Then, by Lemma 1, $h(x)=h(\frac{1}{x})$ for every $x\in\mathbb{R}^+$. Hence, $h(\frac{a}{b})=h(\frac{b}{a})$ for every $a,b\in\mathbb{R}^+$. By Proposition 2, we know that $M=m_f$. Therefore, we obtain

$$m_{f'}(a,b)=\frac{2(a-b)}{(f+h)(\frac{a}{b})-(f+h)(\frac{b}{a})}=\frac{2(a-b)}{(f(\frac{a}{b})-f(\frac{b}{a}))+(h(\frac{a}{b})-h(\frac{b}{a}))}=\frac{2(a-b)}{f(\frac{a}{b})-f(\frac{b}{a})}=m_f(a,b)=M(a,b)$$

for every $a,b\in\mathbb{R}^+$. Hence, for all $m_{f'}=M$. □

## 4. CONCLUSION

In the present paper we found a simple formula that allows us to represent all symmetric homogeneous bivariate means in a similar form without too much use of technical calculations.

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## Funktsioonist, mis võimaldab arvutada kõiki sümmeetrilisi homogeenseid kahe muutuja keskmisi

Mart Abel ja Raido Marmor

On sisse toodud üsna lihtsal kujul olev funktsioon, mis võimaldab samaaegselt kirjeldada kõiki sümmeetrilisi homogeenseid kahe muutuja keskmisi. On uuritud ja tõestatud selle funktsiooni mõningaid omadusi ning toodud näiteid selle funktsiooni konkreetsest kujust 17 erineva keskmise korral.