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**Corresponding author:**
Maris Tõnso
maris.tonso@taltech.ee

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# On definition of difference field, associated to nonlinear control system: several options

Arvo Kaldmäe, Ülle Kotta and Maris Tõnso

Department of Software Science, Tallinn University of Technology, Akadeemia tee 15A, 12618 Tallinn, Estonia

ABSTRACT

A universal inversive difference field that always exists is constructed for a submersive discrete-time nonlinear control system. This field is unique up to permutation of the control variables. Using the unique field in proofs will simplify them significantly. The construction extends the one for state-invertible systems, being the subset of submersive systems. An algorithm is given for finding independent variables of this field. It is proven that the algorithm stops in at most $n + 1$ steps, where $n$ is the state dimension of the control system.

## 1. Introduction

Differential and difference algebraic methods, introduced in the 1980s to nonlinear control [3–5], have proven to be very useful. Though better suited for polynomial and rational systems, they motivated the development of the so-called constructive linear algebraic approach [2,6] that can handle also analytic and meromorphic systems. The key element in the linear algebraic approach is the differential or difference field in independent system variables, in the case of continuous- and discrete-time systems, respectively. This field gives a full algebraic description of the control system and plays a critical role in proofs and computations, since the dimensions, ranks, vector spaces, etc. are all defined and computed over this field.

In studying various control problems, it quickly became clear that one needs to extend the difference field to its inversive closure [1]. The difference field introduced in [1] can be viewed as an inversive closure of the field that was already defined in [6] under the assumption that the system is submersive. Note that though the construction of the inversive closure of the difference field is not unique, all field extensions are unique up to an isomorphism. In various control problems, non-uniqueness of the inversive closure does not affect the solvability conditions. In particular, if the solvability conditions are given in terms of the vector spaces (of one-forms or vector fields) over the field extension, then, by isomorphism, different field extensions simply result in different basis elements of the vector space.

However, in the case when the solvability conditions are given in terms of a single one-form or vector field, non-uniqueness may have a significant impact on the solvability conditions, since the shifts of one-forms and vector fields depend on the chosen extension. One such problem is the transformation of the state equations into the (generalized) observer forms [8]. If the state transition map is invertible with respect to the state, then there is the simple universal choice for the field extension. The existence of such a choice allows one to simplify the proofs of theorems, propositions and lemmas, and it has been used frequently.

In this paper, we will find the analogous fixed choice for non-invertible but submersive systems. The generalization to the nonsubmersive case is not trivial, since it does not correspond to the standard system extension equations. An algorithm is suggested that allows one to compute the key indices in the definition of this fixed field extension. A number of properties of the algorithm and the indices it computes are proven in order to guarantee that the indices are well-defined.

## 2. Standard definition of inversive difference field $\bar{\mathcal{K}}$

Consider a discrete-time nonlinear control system of the form

$$x^{\langle 1\rangle} = f(x,u), \tag{1}$$

where $x^{\langle 1\rangle} \equiv x(t+1)$, $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, $t \in \mathbb{Z}^{0+}$, $X, U$ are open subsets, and the state transition map $f : X \times U \to X$ is supposed to be real analytic. Since the map $f$ is not necessarily invertible with respect to $x$, we extend the equations (1) so that the extended state equations (2) below are valid for $t \in \mathbb{Z}$.

**Assumption 1.** Assume that the map $f$ can be extended to $F = (f^{\mathrm{T}}, \chi^{\mathrm{T}})^{\mathrm{T}} : X \times U \to X \times \mathbb{R}^m$ so that $F$ has a global real analytic inverse defined on its image $F(X \times U)$.

Introduce the additional set of variables $z = \chi(x,u)$. Then the inverse of the extended map

$$\begin{aligned} x^{\langle 1\rangle} &= f(x,u),\\ z &= \chi(x,u) \end{aligned} \tag{2}$$

can be described by

$$\begin{aligned} x &= \Lambda(x^{\langle 1\rangle}, z),\\ u &= \lambda(x^{\langle 1\rangle}, z). \end{aligned} \tag{3}$$

Under Assumption 1, there exists an inversive difference field $\bar{\mathcal{K}}$ of meromorphic functions in a finite number of independent variables from the set

$$\bar{\mathcal{C}} := \left\{x, u^{\langle i\rangle}, z^{\langle -j\rangle};\ i \geq 0, j > 0\right\},$$

where by $x$, $u$ and $z$ we mean the vectors with components $x_1, \ldots, x_n$, $u_1, \ldots, u_m$ and $z_1, \ldots, z_m$, respectively; $u^{\langle i\rangle}$ corresponds to the $i$-th order forward shift of the input $u$, and $z^{\langle -j\rangle}$ corresponds to the $j$-th order backward shift of the variable $z$ [9].

The forward shift operator $\delta : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ is defined by

$$\begin{aligned} \delta x &:= f(x,u), & \delta u^{\langle i\rangle} &:= u^{\langle i+1\rangle},\ i \geq 0,\\ \delta z^{\langle -1\rangle} &:= \chi(x,u), & \delta z^{\langle -j\rangle} &:= z^{\langle -j+1\rangle},\ j \geq 2, \end{aligned} \tag{4}$$

where $f$ and $\chi$ are determined by the equations (2). The backward shift operator $\delta^{-1} : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ is defined by

$$\begin{aligned} \delta^{-1} x &:= \Lambda\left(x, z^{\langle -1\rangle}\right), & \delta^{-1} u^{\langle i\rangle} &:= u^{\langle i-1\rangle}, i \geq 1,\\ \delta^{-1} u &:= \lambda\left(x, z^{\langle -1\rangle}\right), & \delta^{-1} z^{\langle -j\rangle} &:= z^{\langle -j-1\rangle}, j \geq 1, \end{aligned} \tag{5}$$

where $\Lambda$ and $\lambda$ are determined by the equations (3). The forward and backward shift operators of the function $\varphi \in \bar{\mathcal{K}}$ are defined by applying the respective operator to the arguments of $\varphi$, see [9]. For brevity, the notations $\delta\varphi \equiv \varphi^{\langle 1\rangle}$ and $\delta^{-1}\varphi \equiv \varphi^{\langle -1\rangle}$, $\varphi \in \bar{\mathcal{K}}$, are used below.

Next, we introduce the infinite set of symbols $\mathrm{d}\bar{\mathcal{C}} := \left\{\mathrm{d}x, \mathrm{d}u^{\langle i\rangle}, \mathrm{d}z^{\langle -j\rangle};\ i \geq 0, j > 0\right\}$ and denote by $\mathcal{E}$ the vector space spanned over $\bar{\mathcal{K}}$ by the elements of $\mathrm{d}\mathcal{C}$, i.e. $\mathcal{E} := \mathrm{span}_{\bar{\mathcal{K}}}\{\mathrm{d}\mathcal{C}\}$. The elements of the vector space $\mathcal{E}$ are called one-forms. The one-form $\omega \in \mathcal{E}$ is said to be an *exact* one-form if $\omega = \mathrm{d}\varphi$ for some $\varphi \in \bar{\mathcal{K}}$. A dual space to $\mathcal{E}$ is the vector space $\mathcal{E}^*$ of vector fields.

## 3. Universal inversive difference field $\mathcal{K}$

The field $\bar{\mathcal{K}}$ was defined above as the field of meromorphic functions in variables from the set $\bar{\mathcal{C}}$. The selection of the variable $z = \chi(x,u)$ is not unique and, as such, affects the expressions of the forward and backward shifts of the functions in $\bar{\mathcal{K}}$ (see (4) and (5)) and, consequently, the shifts of one-forms and vector fields [9]. The goal of this section is to redefine the field $\bar{\mathcal{K}}$ in such a way that the shifts of functions in the new field $\mathcal{K}$ are independent of the selection of the $z$ variable.

**Definition 1.** *Define $\mathcal{K}$ as a difference field of meromorphic functions in a finite number of independent variables from the set*

$$C := \left\{x, u^{\langle i\rangle}, u_k^{\langle -j_k\rangle};\ i \geq 0, j_k > q_k, k = 1, \ldots, m\right\},$$

*where the indices $q_k \geq 0$, $k = 1, \ldots, m$, are defined as minimal integers such that $u_k^{\langle -q_k-1\rangle}$ cannot be expressed as a function of $x$ and $u_j^{\langle -q_j-1\rangle}, \ldots, u_j^{\langle -q_k-1\rangle}$, $j = 1, \ldots, m$, $j \neq k$.*

Definition 1 is not suitable for finding the indices $q_k$, $k = 1, \ldots, m$, in practice. The aim of the following algorithm is to provide such a tool.

**Algorithm 1.** Computation of the indices $q_1, \ldots, q_m$ for the extended system of the form (2).

**Step 1.** (a) Compute the backward shift $u^{\langle -1\rangle}$.

(b) Let

$$\xi_1 := \operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial u^{\langle -1\rangle}}{\partial z^{\langle -1\rangle}}.$$

(c) Decompose $u^{\langle -1\rangle}$ into two parts, $u^{\langle -1\rangle} = \left(\tilde{u}_1^{\langle -1\rangle}, \hat{u}_1^{\langle -1\rangle}\right)$, such that $\dim_{\bar{\mathcal{K}}} \tilde{u}_1^{\langle -1\rangle} = \xi_1$ and

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial \tilde{u}_1^{\langle -1\rangle}}{\partial z^{\langle -1\rangle}} = \xi_1. \tag{6}$$

(d) The vector

$$\tilde{u}_1^{\langle -1\rangle} = \left(u_{\sigma(1)}^{\langle -1\rangle}, \ldots, u_{\sigma(\xi_1)}^{\langle -1\rangle}\right)^{\mathrm{T}}, \tag{7}$$

where the bijection $\sigma$ is a permutation of the set $S = \{1, \ldots, m\}$. Define the indices $q_{\sigma(1)} := \cdots := q_{\sigma(\xi_1)} := 0$. That is, for each $u_k$ such that $u_k^{\langle -1\rangle}$ is the element of $\tilde{u}_1^{\langle -1\rangle}$, define $q_k := 0$.

(e) If $\partial \hat{u}_1^{\langle -1\rangle}/\partial z^{\langle -1\rangle} \not\equiv 0$, then $\hat{u}_1^{\langle -1\rangle}$ can be expressed by (6) as a function of $x$ and $\tilde{u}_1^{\langle -1\rangle}$, i.e. $\hat{u}_1^{\langle -1\rangle} = \psi_1(x, \tilde{u}_1^{\langle -1\rangle})$.

**Step $i$ ($i \geq 2$).** (a) Compute the backward shift of $\hat{u}_{i-1}^{\langle 1-i\rangle}$, obtained at previous step, to get $\hat{u}_{i-1}^{\langle -i\rangle}$.

(b) Let

$$\xi_i := \operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial \left(\tilde{u}_1^{\langle -1\rangle}, \ldots, \tilde{u}_{i-1}^{\langle -i+1\rangle}, \hat{u}_{i-1}^{\langle -i\rangle}\right)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}}.$$

(c) Decompose $\hat{u}_{i-1}^{\langle -i\rangle}$ into two parts, $\hat{u}_{i-1}^{\langle -i\rangle} = \left(\tilde{u}_i^{\langle -i\rangle}, \hat{u}_i^{\langle -i\rangle}\right)$, such that $\dim_{\bar{\mathcal{K}}} \tilde{u}_i^{\langle -i\rangle} = \xi_i - \xi_{i-1}$ and

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial \left(\tilde{u}_1^{\langle -1\rangle}, \ldots, \tilde{u}_i^{\langle -i\rangle}\right)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}} = \xi_i.$$

(d) The vector

$$\tilde{u}_i^{\langle -i\rangle} = \left(u_{\sigma(\xi_{i-1}+1)}^{\langle -i\rangle}, \ldots, u_{\sigma(\xi_i)}^{\langle -i\rangle}\right)^{\mathrm{T}}.$$

Define the indices $q_{\sigma(\xi_{i-1}+1)} := \cdots := q_{\sigma(\xi_i)} := i - 1$. That is, for each $u_k$ such that $u_k^{\langle -i\rangle}$ is the element of $\tilde{u}_i^{\langle -i\rangle}$, define $q_k := i - 1$.

(e) If $\partial \hat{u}_i^{\langle -i\rangle}/\partial z^{\langle -1\rangle} \not\equiv 0$, then $\hat{u}_i^{\langle -i\rangle}$ can be expressed as a function of $x$ and $\tilde{u}_j^{\langle -j\rangle}, \ldots, \tilde{u}_j^{\langle -i\rangle}$, $j = 1, \ldots, i$, i.e. $\hat{u}_i^{\langle -i\rangle} = \psi_i(x, \tilde{u}_j^{\langle -j\rangle}, \ldots, \tilde{u}_j^{\langle -i\rangle}, j = 1, \ldots, i)$.

The algorithm ends at the step $i = \rho$, when the value of $\xi_i = m$.

At each step of Algorithm 1, we have found the vector $\tilde{u}_i^{\langle -i\rangle}$, $i = 1, \ldots, \rho$. For each $u_k^{\langle -i\rangle}$, being the element of $\tilde{u}_i^{\langle -i\rangle}$, denote

$$u_k^{\langle -i\rangle} = u_k^{\langle -q_k-1\rangle} =: \phi_k(x, z^{\langle -1\rangle}, u_j^{\langle -q_j-2\rangle}, \ldots, u_j^{\langle -q_k-2\rangle}, j = 1, \ldots, m, j \neq k).$$

**Assumption 2.** Assume that the functions $\psi_i$, $i \geq 1$, are analytic and globally defined on $X \times U$.

Under Assumption 2, the indices $q_1, \dots, q_m$ are globally defined for a given state extension. Frequently, instead of Assumption 1, a milder assumption of generic submersivity is assumed.

**Assumption 3.** Assume that the system (1) is generically submersive, i.e

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial f(x,u)}{\partial(x,u)} = n$$

on $X \times U$.

Assumption 1 implies the generic submersivity property of the system (1) [9] on $X \times U$.

**Remark 1.** Sometimes, for a system of the form (1), instead of submersivity, a more restrictive property of generic state-invertibility of the map $f$ is assumed, i.e.

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial f(x,u)}{\partial x} = n$$

on $X \times U$. In such a case, one can always take $z = u$, meaning that all indices $q_k = 0$, $k = 1, \dots, m$. Definition 1 generalizes the analogous fixed choice for submersive systems.

**Proposition 1.** *Under Assumption 3, Algorithm 1 always ends in at most $n+1$ steps.*

*Proof.* First, we show that Algorithm 1 always ends. Suppose, by contradiction, that it does not stop, i.e. $\xi_i < m$ for all $i > 0$. Since, by construction, $\xi_i \leq \xi_{i+1}$, then at some point $i_*$, one has $\xi_{i_*} = \xi_{i_*+j}$ for $j \geq 0$. The latter may happen if either $\xi_{i_*} = m$ (the maximal value the rank may have) or if

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial \left(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle}, \hat{u}_{i_*}^{\langle -i_*\rangle}\right)^{\mathrm{T}}}{\partial x} = \operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial \left(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle}\right)^{\mathrm{T}}}{\partial x}. \tag{8}$$

Because we assumed that $\xi_i < m$ for all $i > 0$, then (8) must be true. We also note that, since $\xi_{i_*} < m$, then $\dim_{\bar{\mathcal{K}}}(\hat{u}_{i_*}^{\langle -i_*\rangle}) > 0$. By construction,

$$\begin{pmatrix} \mathrm{d}\hat{u}_1^{\langle -1\rangle} \\ \vdots \\ \mathrm{d}\hat{u}_{i_*-1}^{\langle -i_*+1\rangle} \end{pmatrix} = \frac{\partial \left(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle}\right)^{\mathrm{T}}}{\partial x}\mathrm{d}x + \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{i_*-1} \end{pmatrix} \tag{9}$$

for some one-forms, $\omega_i \in \operatorname{span}_{\bar{\mathcal{K}}}\{\mathrm{d}\tilde{u}_j^{\langle -j\rangle}; j = 1, \dots, i\}$. From (9), one gets

$$\frac{\partial \left(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle}\right)^{\mathrm{T}}}{\partial x}\mathrm{d}x = \begin{pmatrix} \mathrm{d}\hat{u}_1^{\langle -1\rangle} - \omega_1 \\ \vdots \\ \mathrm{d}\hat{u}_{i_*-1}^{\langle -i_*+1\rangle} - \omega_{i_*-1} \end{pmatrix}. \tag{10}$$

Since (8) is true, then the rows of the matrix $\partial \hat{u}_{i_*}^{\langle -i_*\rangle}/\partial x$ can be written as linear combinations of the rows of the matrix $\partial(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle})^{\mathrm{T}}/\partial x$, i.e. there exists a matrix $A$ such that $\partial \hat{u}_{i_*}^{\langle -i_*\rangle}/\partial x = A \cdot \left(\partial(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle})^{\mathrm{T}}/\partial x\right)$. Thus, because of (10),

$$\begin{aligned} \mathrm{d}\hat{u}_{i_*}^{\langle -i_*\rangle} &= \frac{\partial \hat{u}_{i_*}^{\langle -i_*\rangle}}{\partial x}\mathrm{d}x + \omega_{i_*} = A \cdot \frac{\partial \left(\hat{u}_1^{\langle -1\rangle}, \dots, \hat{u}_{i_*-1}^{\langle -i_*+1\rangle}\right)^{\mathrm{T}}}{\partial x}\mathrm{d}x + \omega_{i_*} \\ &= A \cdot \begin{pmatrix} \mathrm{d}\hat{u}_1^{\langle -1\rangle} - \omega_1 \\ \vdots \\ \mathrm{d}\hat{u}_{i_*-1}^{\langle -i_*+1\rangle} - \omega_{i_*-1} \end{pmatrix} + \omega_{i_*}, \end{aligned}$$

for some $\omega_{i_*} \in \operatorname{span}_{\bar{\mathcal{K}}}\{\mathrm{d}\tilde{u}_j^{\langle -j\rangle}; j = 1,\ldots,i_*\}$. In the latter equality, since the left-hand side is exact, then so must be the right-hand side. Also, the right-hand side belongs to $\operatorname{span}_{\bar{\mathcal{K}}}\{\mathrm{d}u_j^{\langle -k\rangle}; j = 1,\ldots,m; k \geq 1\}$. Therefore, there must exist a function $\varphi(u_j^{\langle -k\rangle}; j = 1,\ldots,m; 1 \leq k \leq i_*)$ such that $\mathrm{d}\hat{u}_{i_*}^{\langle -i_*\rangle} = \mathrm{d}\varphi$, meaning that $\hat{u}_{i_*}^{\langle -i_*\rangle} - \varphi = 0$. We will next show that this contradicts the submersivity assumption.

If the system (1) is submersive, the kernel of the forward shift operator, defined by the equations (1), is trivial, and therefore this property guarantees that independent variables remain independent under the action of forward or backward shift operators. Since the input variables and their forward shifts are all independent, then, by the submersivity assumption, so must be the backward shifts of the input variables. This contradicts the existence of $\varphi(u_j^{\langle -k\rangle}; j = 1,\ldots,m; 1 \leq k \leq i_*)$ such that $\hat{u}_{i_*}^{\langle -i_*\rangle} - \varphi = 0$. Therefore, the assumption that $\xi_i < m$ for all $i \geq 1$ was incorrect, and thus $\xi_i = m$ for some $i \in \mathbb{N}$ must be true.

Second, we are going to show that Algorithm 1 ends in at most $n+1$ steps. We have already shown that Algorithm 1 stops when (8) becomes true, since otherwise one gets a contradiction with the submersivity assumption. The equality (8), however, is achieved in at most $n+1$ steps. Indeed, at every step, one either achieves (8), or the rank of the matrix $\partial\left(\hat{u}_1^{\langle -1\rangle},\ldots,\hat{u}_i^{\langle -i\rangle}\right)^{\mathrm{T}}/\partial x$ increases at least by one. If it increases in every step exactly by one, then (8) is achieved in $n+1$ steps. Thus, Algorithm 1 ends in at most $n+1$ steps. □

Obviously, it follows from Algorithm 1 that the indices $q_k$, $k = 1,\ldots,m$, are not uniquely determined, since the separation of the vector $\hat{u}_{i-1}^{\langle -i\rangle}$ into two parts is, in general, not unique. However, as Corollaries 1 and 2 below show, their sum and the set of their values are invariant with respect to the selection of the variable $z$.

**Proposition 2.** *The ranks $\xi_i$, computed at steps $i = 1,\ldots,\rho$ of Algorithm 1, are invariant with respect to the selection of the function $z = \chi(x,u)$ in* (2).

*Proof.* Let $z_1 = \chi_1(x,u)$ and $z_2 = \chi_2(x,u)$ be two possible selections for $z = \chi(x,u)$ in (2). We show that they are related through a function $z_1 = \varphi(x^{\langle 1\rangle}, z_2)$, where the matrix $\partial\varphi/\partial z_2$ has full rank over the field $\bar{\mathcal{K}}$. By construction, one has $(x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}} = \Phi_1(x^{\langle 1\rangle}, z_1) = \Phi_2(x^{\langle 1\rangle}, z_2)$ for some analytic functions $\Phi_1$ and $\Phi_2$. Also, since, by Assumption 1, the equations (2) are globally solvable for $x$ and $u$, then the equations (3) are globally solvable for $x^{\langle 1\rangle}$ and $z$. Thus,

$$\operatorname{rank}_{\bar{\mathcal{K}}}\frac{\partial\Phi_i(x^{\langle 1\rangle}, z_i)}{\partial(x^{\langle 1\rangle}, z_i)} = \operatorname{rank}_{\bar{\mathcal{K}}}\begin{bmatrix}\dfrac{\partial\Phi_i(x^{\langle 1\rangle}, z_i)}{\partial x^{\langle 1\rangle}} & \dfrac{\partial\Phi_i(x^{\langle 1\rangle}, z_i)}{\partial z_i}\end{bmatrix} = n+m$$

for $i = 1,2$, meaning that the matrices $\partial\Phi_1/\partial z_1$ and $\partial\Phi_2/\partial z_2$ must have full ranks. Thus, one can solve $\Phi_1(x^{\langle 1\rangle}, z_1) = \Phi_2(x^{\langle 1\rangle}, z_2)$ for $z_1$ to obtain $z_1 = \varphi(x^{\langle 1\rangle}, z_2)$. Since one can also solve $\Phi_1(x^{\langle 1\rangle}, z_1) = \Phi_2(x^{\langle 1\rangle}, z_2)$ for $z_2$, i.e. there must exist $\varphi^{-1}$ such that $z_2 = \varphi^{-1}(x^{\langle 1\rangle}, z_1)$, then obviously the matrix $\partial\varphi/\partial z_2$ must have full rank over the field $\bar{\mathcal{K}}$.

Let $\xi_i^j$, $j = 1,2$, correspond to the rank $\xi_i$ in Algorithm 1 when one has chosen $z_j$, $j = 1,2$, in (2), respectively. Since $z_1 = \varphi(x^{\langle 1\rangle}, z_2)$ or, after applying the backward shift operator, $z_1^{\langle -1\rangle} = \varphi(x, z_2^{\langle -1\rangle})$, then one can compute

$$\begin{aligned}\xi_i^2 &= \operatorname{rank}_{\bar{\mathcal{K}}}\frac{\partial\left(\tilde{u}_1^{\langle -1\rangle},\ldots,\tilde{u}_{i-1}^{\langle -i+1\rangle},\hat{u}_{i-1}^{\langle -i\rangle}\right)^{\mathrm{T}}}{\partial z_2^{\langle -1\rangle}} = \operatorname{rank}_{\bar{\mathcal{K}}}\left[\frac{\partial\left(\tilde{u}_1^{\langle -1\rangle},\ldots,\tilde{u}_{i-1}^{\langle -i+1\rangle},\hat{u}_{i-1}^{\langle -i\rangle}\right)^{\mathrm{T}}}{\partial z_1^{\langle -1\rangle}}\cdot\frac{\partial\varphi}{\partial z_2^{\langle -1\rangle}}\right]\\ &= \operatorname{rank}_{\bar{\mathcal{K}}}\frac{\partial\left(\tilde{u}_1^{\langle -1\rangle},\ldots,\tilde{u}_{i-1}^{\langle -i+1\rangle},\hat{u}_{i-1}^{\langle -i\rangle}\right)^{\mathrm{T}}}{\partial z_1^{\langle -1\rangle}} = \xi_i^1,\end{aligned}$$

where the second to last equality comes from the fact that the matrix $\partial\varphi/\partial z_2^{\langle -1\rangle}$ has full rank over $\bar{\mathcal{K}}$. □

**Corollary 1.** *The sum of the indices $q_k$, $k = 1, \ldots, m$, is invariant with respect to the selection of the function $z = \chi(x, u)$ in* (2).

*Proof.* Since, by Algorithm 1,

$$\sum_{k=1}^{m} q_k = \sum_{i=2}^{\rho} (\xi_i - \xi_{i-1})\,(i-1)$$

and, by Proposition 2, $\xi_i$, $i = 1, \ldots, \rho$, are invariant with respect to the selection of the function $z = \chi(x, u)$, then so must be the sum of the indices $q_k$. □

**Corollary 2.** *The set of indices $\{q_k,\ k = 1, \ldots, m\}$ is invariant with respect to the selection of the function $z = \chi(x, u)$ in* (2).

*Proof.* By Algorithm 1 and Corollary 1, the set

$$\{q_k,\ k = 1, \ldots, m\} = \{\underbrace{0, \ldots, 0}_{\xi_1}, \underbrace{1, \ldots, 1}_{\xi_2 - \xi_1}, \ldots, \underbrace{\rho - 1, \ldots, \rho - 1}_{\xi_\rho - \xi_{\rho-1}}\}.$$

By Proposition 2, $\xi_i$, $i = 1, \ldots, \rho$, are invariant with respect to the selection of the function $z = \chi(x, u)$. □

To conclude, the indices $q_k$, $k = 1, \ldots, m$, are globally well-defined and, up to reordering the input components, independent of the chosen system extension of the form (3). However, note that one cannot associate with the universal difference field $\mathcal{K}$ the state extension map in the form $z = \chi(x, u)$.

Therefore, a question arises of how to compute the dependent system variables in $\mathcal{K}$. While completing the algorithm, we have found the functions

$$u_k^{\langle -q_k-1\rangle} = \phi_k(x, z^{\langle -1\rangle}, u_j^{\langle -q_j-2\rangle}, \ldots, u_j^{\langle -q_k-2\rangle}, j = 1, \ldots, m, j \neq k), \quad k = 1, \ldots, m, \tag{11}$$

satisfying

$$\text{rank}_{\bar{\mathcal{K}}} = \frac{\partial\,(\phi_1, \ldots, \phi_m)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}} = m.$$

Thus, it is always possible to solve (11) for the variables

$$z^{\langle -1\rangle} = \eta(x, u_k^{\langle -j_k\rangle}, q_k < j_k \leq \rho, k = 1, \ldots, m).$$

Substituting $z^{\langle -1\rangle}$ into the left column of (5) yields:

$$x^{\langle -1\rangle} = \Lambda\left(x, \eta(\cdot)\right), \quad u_k^{\langle -1\rangle} = \lambda_k\left(x, \eta(\cdot)\right) \text{ for such } k \text{ values when } q_k > 0. \tag{12}$$

The new difference field $\mathcal{K}$ in Definition 1 is isomorphic to the difference field $\bar{\mathcal{K}}$. Indeed, the variables from the set $C$ can be mapped one-to-one to the variables from the set $\bar{C}$, and the backward and forward shifts commute with any change of coordinates [7]. Thus, whether one works with $\mathcal{K}$ or $\bar{\mathcal{K}}$ does not affect the vector space $\mathcal{E}$ of one-forms or the vector space $\mathcal{E}^*$ of vector fields.

## 4. Informal discussion

Frequently in the algebraic approach, instead of Assumption 1, a weaker Assumption 3 of generic submersivity is made [1,6] that allows for addressing a larger number of discrete-time nonlinear systems using the algebraic approach. Of course, replacing Assumption 1 with Assumption 3 introduces certain complications, which will be discussed below.

Assumption 1 obviously implies Assumption 3. Whereas Assumption 1 allows one to define the backward shifts of system variables globally, this is not so if only Assumption 3 is made. In the latter case, the equations (1) can be extended exactly as in (2), but it may happen that the equations (2) cannot be solved globally but only generically on $X \times U$. The generic approach ignores singularities, and therefore, for some points in the set $X \times U$, the solution (2) may not exist. Of course, in principle, such points may be removed from the set $X \times U$, and one may work on a restricted subset of $X \times U$ instead. Furthermore, it may also happen that the solution of the equations (2) is not unique globally but defined in different subsets of $X \times U$ by different $\Lambda$ and $\lambda$ in (3). In such a case, the inversive closure of $\bar{\mathcal{K}}$ has to be defined separately for different subsets.

Consider, for instance, the simple example

$$\begin{aligned} x^{\langle 1\rangle} &= x^2 + u, \\ z &= u. \end{aligned} \tag{13}$$

Obviously, the solution (3) now takes the form

$$x = \pm\sqrt{x^{\langle 1\rangle} - u}. \tag{14}$$

Here we face the situation where the equations in the form (2), now given by (13), have two solutions. That is, one cannot define $\bar{\mathcal{K}}$ globally but separately for positive and negative $x$ values. To conclude, we have to consider separately two subsets of $X \times U$, restricted by $x > 0$ and $x < 0$, respectively.[1]

The restriction of the set $X \times U$ to, say, $\hat{X} \times \hat{U}$ is not without its drawbacks. If there are no different inverses in different regions, then the region is invariant with respect to system dynamics. This allows one to compute infinite trajectories $x(t)$, $t \in \mathbb{Z}$. If, however, there are different functions $\Lambda$ and $\lambda$ in (3) for different regions, and one restricts the system to one such region, then the invariance property is lost, and the trajectory may leave from the restricted set $\hat{X} \times \hat{U}$ at a certain time instant $t^*$. The same may happen if the solution (3) is not defined everywhere in $X \times U$. That is, special care is needed to study such systems using the algebraic approach. These aspects are typically handled on the level of examples.

To conclude, for simplicity of presentation, we made Assumptions 1 and 2. Of course, if only Assumption 3 is made, one has to take the above-mentioned aspects into account in the computation of the indices $q_k$, $k = 1, \dots, m$.

## 5. Example

Consider the extended state equations

$$\begin{aligned} &x_1^{\langle 1\rangle} = u_2 + x_3, \quad x_2^{\langle 1\rangle} = u_1 + x_1 x_4, \quad x_3^{\langle 1\rangle} = u_2, \quad x_4^{\langle 1\rangle} = u_3 - x_1 x_4, \\ &z_1 = x_1, \qquad z_2 = x_2, \qquad z_3 = x_4. \end{aligned} \tag{15}$$

To obtain the inverse system in the form (3), we solve the equations (15) for $x$, $u$:

$$\begin{aligned} &x_1 = z_1, \qquad x_2 = z_2, \qquad x_3 = x_1^{\langle 1\rangle} - x_3^{\langle 1\rangle}, \quad x_4 = z_3, \\ &u_1 = x_2^{\langle 1\rangle} - z_1 z_3, \quad u_2 = x_3^{\langle 1\rangle}, \quad u_3 = x_4^{\langle 1\rangle} + z_1 z_3. \end{aligned} \tag{16}$$

Shifting the result backwards yields

$$x_1^{\langle -1\rangle} = z_1^{\langle -1\rangle}, \qquad x_2^{\langle -1\rangle} = z_2^{\langle -1\rangle}, \qquad x_3^{\langle -1\rangle} = x_1 - x_3, \qquad x_4^{\langle -1\rangle} = z_3^{\langle -1\rangle}, \tag{17a}$$

$$u_1^{\langle -1\rangle} = x_2 - z_1^{\langle -1\rangle} z_3^{\langle -1\rangle}, \quad u_2^{\langle -1\rangle} = x_3, \qquad u_3^{\langle -1\rangle} = x_4 + z_1^{\langle -1\rangle} z_3^{\langle -1\rangle}. \tag{17b}$$

We now apply Algorithm 1.

**Step 1.** The backward shifts $u^{\langle -1\rangle}$ are given by (17b), thus the rank

$$\xi_1 := \operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial u^{\langle -1\rangle}}{\partial z^{\langle -1\rangle}} = \operatorname{rank}_{\bar{\mathcal{K}}} \begin{pmatrix} -z_3^{\langle -1\rangle} & 0 & -z_1^{\langle -1\rangle} \\ 0 & 0 & 0 \\ z_3^{\langle -1\rangle} & 0 & z_1^{\langle -1\rangle} \end{pmatrix} = 1.$$

Decompose $u^{\langle -1\rangle} = \left(\tilde{u}_1^{\langle -1\rangle}, \hat{u}_1^{\langle -1\rangle}\right)$ so that $\tilde{u}_1^{\langle -1\rangle} = \left(u_1^{\langle -1\rangle}\right)$ and $\hat{u}_1^{\langle -1\rangle} = \left(u_2^{\langle -1\rangle}, u_3^{\langle -1\rangle}\right)$. Obviously, $\dim_{\bar{\mathcal{K}}} \tilde{u}_1^{\langle -1\rangle} = \xi_1 = 1$ and

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial \tilde{u}_1^{\langle -1\rangle}}{\partial z^{\langle -1\rangle}} = \xi_1 = 1.$$

[1] The value $x = 0$ has been excluded because including it into a subset would make the result a non-open set.

Since $\partial \hat{u}_1^{\langle -1\rangle}/\partial z^{\langle -1\rangle} \not\equiv 0$, we express $\hat{u}_1^{\langle -1\rangle}$ as a function of $\tilde{u}_1^{\langle -1\rangle}$ and $x$, i.e. $u_3^{\langle -1\rangle} = x_2 + x_4 - u_1^{\langle -1\rangle}$. Since $\tilde{u}_1^{\langle -1\rangle} = \left(u_1^{\langle -1\rangle}\right)$, we obtain $\sigma(1) = 1$; thus, for the input variable $u_1$, define $q_1 = 0$.

**Step 2.** Take $i = 2$ and compute

$$\hat{u}_1^{\langle -2\rangle} = \left(u_2^{\langle -2\rangle}, u_3^{\langle -2\rangle}\right) = \left(x_1 - x_3,\ z_2^{\langle -1\rangle} + z_3^{\langle -1\rangle} - u_1^{\langle -2\rangle}\right)$$

and

$$\xi_2 := \operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial\left(\tilde{u}_1^{\langle -1\rangle}, \hat{u}_1^{\langle -2\rangle}\right)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}} = \operatorname{rank}_{\bar{\mathcal{K}}} \begin{pmatrix} -z_3^{\langle -1\rangle} & 0 & -z_1^{\langle -1\rangle} \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2.$$

Decompose $\hat{u}_1^{\langle -2\rangle} = \left(\tilde{u}_2^{\langle -2\rangle}, \hat{u}_2^{\langle -2\rangle}\right)$ so that $\tilde{u}_2^{\langle -2\rangle} = \left(u_3^{\langle -2\rangle}\right)$ and $\hat{u}_2^{\langle -2\rangle} = \left(u_2^{\langle -2\rangle}\right)$. Clearly, $\dim_{\bar{\mathcal{K}}} \tilde{u}_2^{\langle -2\rangle} = \xi_2 - \xi_1 = 1$ and

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial\left(\tilde{u}_1^{\langle -1\rangle}, \tilde{u}_2^{\langle -2\rangle}\right)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}} = \xi_2 = 2.$$

Since $\tilde{u}_2^{\langle -2\rangle} = \left(u_3^{\langle -2\rangle}\right)$, the permutation $\sigma(2) = 3$ allows one to define for $u_3$ the index $q_3 := 1$.

**Step 3.** Take $i = 3$ and compute $\hat{u}_2^{\langle -3\rangle} = \left(u_2^{\langle -3\rangle}\right) = \left(z_1^{\langle -1\rangle} - x_1 + x_3\right)$. Now,

$$\xi_3 := \operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial\left(\tilde{u}_1^{\langle -1\rangle}, \tilde{u}_2^{\langle -2\rangle}, \hat{u}_2^{\langle -3\rangle}\right)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}} = \operatorname{rank}_{\bar{\mathcal{K}}} \begin{pmatrix} -z_3^{\langle -1\rangle} & 0 & -z_1^{\langle -1\rangle} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3.$$

Decompose $\hat{u}_2^{\langle -3\rangle} = \left(\tilde{u}_3^{\langle -3\rangle}, \hat{u}_3^{\langle -3\rangle}\right)$ so that $\tilde{u}_3^{\langle -3\rangle} = \left(u_2^{\langle -3\rangle}\right)$, and $\hat{u}_3^{\langle -3\rangle}$ is an empty vector. Such decomposition satisfies $\dim_{\bar{\mathcal{K}}} \tilde{u}_3^{\langle -3\rangle} = \xi_3 - \xi_2 = 1$ and

$$\operatorname{rank}_{\bar{\mathcal{K}}} \frac{\partial\left(\tilde{u}_1^{\langle -1\rangle}, \tilde{u}_2^{\langle -2\rangle}, \tilde{u}_3^{\langle -3\rangle}\right)^{\mathrm{T}}}{\partial z^{\langle -1\rangle}} = \xi_3 = 3.$$

Since $\tilde{u}_3^{\langle -3\rangle} = \left(u_2^{\langle -3\rangle}\right)$, the permutation $\sigma(3) = 2$ allows one to define for $u_2$ the index $q_2 := 2$.

The algorithm results in the indices $q_1 = 0, q_2 = 2, q_3 = 1$. Define $\mathcal{K}$ as a difference field of meromorphic functions in a finite number of independent variables from the set

$$C := \left\{x, u^{\langle i\rangle}, u_1^{\langle -j_1\rangle}, u_2^{\langle -j_2\rangle}, u_3^{\langle -j_3\rangle};\ i \geq 0, j_1 > 0, j_2 > 2, j_3 > 1\right\}.$$

While completing the algorithm, we have computed $u_k^{\langle -q_k-1\rangle}$, $k = 1, 2, 3$:

$$u_1^{\langle -1\rangle} = x_2 - z_1^{\langle -1\rangle} z_3^{\langle -1\rangle}, \quad u_2^{\langle -3\rangle} = z_1^{\langle -1\rangle} - x_1 + x_3, \quad u_3^{\langle -2\rangle} = z_2^{\langle -1\rangle} + z_3^{\langle -1\rangle} - u_1^{\langle -2\rangle}. \tag{18}$$

Solving the system (18) for $z^{\langle -1\rangle}$ yields:

$$z_1^{\langle -1\rangle} = x_1 - x_3 + u_2^{\langle -3\rangle},\ z_2^{\langle -1\rangle} = u_1^{\langle -2\rangle} + u_3^{\langle -2\rangle} - \frac{x_2 - u_1^{\langle -1\rangle}}{x_1 - x_3 + u_2^{\langle -3\rangle}},\ z_3^{\langle -1\rangle} = \frac{x_2 - u_1^{\langle -1\rangle}}{x_1 - x_3 + u_2^{\langle -3\rangle}}. \tag{19}$$

It remains to substitute $z^{\langle -1\rangle}$ from (19) into (17):

$$\begin{aligned} x_1^{\langle -1\rangle} &= x_1 - x_3 + u_2^{\langle -3\rangle}, & x_2^{\langle -1\rangle} &= u_1^{\langle -2\rangle} + u_3^{\langle -2\rangle} - \frac{x_2 - u_1^{\langle -1\rangle}}{x_1 - x_3 + u_2^{\langle -3\rangle}}, & & \\ x_3^{\langle -1\rangle} &= x_1 - x_3, & x_4^{\langle -1\rangle} &= \frac{x_2 - u_1^{\langle -1\rangle}}{x_1 - x_3 + u_2^{\langle -3\rangle}}, & & \\ u_1^{\langle -1\rangle} &= u_1^{\langle -1\rangle}, & u_2^{\langle -1\rangle} &= x_3, & u_3^{\langle -1\rangle} &= x_2 + x_4 - u_1^{\langle -1\rangle}. \end{aligned} \tag{20}$$

## 6. Conclusion

Under Assumptions 1 and 2 for a discrete-time nonlinear control system, a universal inversive difference field $\mathcal{K}$ is constructed that always exists. Assumption 1 is slightly more restrictive than the standard submersivity Assumption 3. Assumptions 1 and 2 allow one to define the field $\mathcal{K}$ globally on the domain of definition of a system. The discussion section explains what may happen if, instead of Assumption 1, only the system's submersivity is assumed. The field $\mathcal{K}$ is unique up to reordering the input components, and the uniqueness will allow one to simplify the proofs in the study of a nonlinear system using the algebraic approach. The construction extends the existing one for state-invertible systems. An algorithm is given to find the independent variables of the universal inversive difference field. Moreover, it is proven that Algorithm 1 stops in at most $n + 1$ steps, where $n$ is the state dimension of the control system.

### Data availability statement

All data are available in the article.

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## References

1. Aranda-Bricaire, E., Kotta, Ü. and Moog, C. H. Linearization of discrete-time systems. *SIAM Journal on Control and Optimization*, 1996, **34**(6), 1999–2023. https://doi.org/10.1137/s0363012994267315
2. Conte, G., Moog, C. H. and Perdon, A. M. *Algebraic Methods for Nonlinear Control Systems. Theory and Applications*. 2nd ed. In *Communications and Control Engineering*. Springer, London, 2007. https://doi.org/10.1007/978-1-84628-595-0
3. Fliess, M. Automatique en temps discret et algèbre aux différences (Discrete-time automatic and difference algebra). *Forum Mathematicum*, 1990, **2**, 213–232. https://doi.org/10.1515/form.1990.2.213
4. Fliess, M. Automatique et corps différentiels (Automatic and differential bodies). *Forum Mathematicum*, 1989, **1**, 227–238. https://doi.org/10.1515/form.1989.1.227
5. Fliess, M. Generalized controller canonical form for linear and nonlinear dynamics. *IEEE Transactions on Automatic Control*, 1990, **35**(9), 994–1001. https://doi.org/10.1109/9.58527
6. Grizzle, J. W. A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM Journal on Control and Optimization*, 1993, **31**(4), 1026–1044. https://doi.org/10.1137/0331046
7. Kaldmäe, A., Kaparin, V., Kotta, Ü., Mullari, T. and Pawluszewicz, E.. On the properties of forward and backward shifts of vector fields. *Proceedings of the Estonian Academy of Sciences*, 2022, **71**(4), 314–325. https://doi.org/10.3176/proc.2022.4.02
8. Kaldmäe, A., Kotta, Ü. and Tõnso, M. Transformation of backward observable discrete-time systems into a generalized observer form. In *European Control Conference (ECC)*. Accepted for publication. 2026.
9. Mullari, T., Kotta, Ü., Bartosiewicz, Z., Pawłuszewicz, E. and Moog, C. H. Forward and backward shifts of vector fields: towards the dual algebraic framework. *IEEE Transactions on Automatic Control*, 2017, **62**(6), 3029–3033. https://doi.org/10.1109/tac.2016.2608718

## Mittelinaarse juhtimisüsteemiga seotud diferentskorpuse definitsioonist: erinevad võimalused

**Arvo Kaldmäe, Ülle Kotta ja Maris Tõnso**

Artiklis on konstrueeritud diskreetse ajaga mittelineaarse submersiivse juhtimissüsteemi jaoks universaalne diferentskorpus, mis eksisteerib alati. Korpus on ühene kuni juhttoimete järjestuseni. Ühesusest tingitult lihtsustab sellise korpuse kasutamine teoreemide ja lemmade tõestamist oluliselt. Esitatud konstruktsioon laiendab teadaolevat konstruktsiooni oleku suhtes pööratavate süsteemide jaoks, mis moodustavad submersiivsete süsteemide alamhulga. Esitatud on algoritm korpuse sõltumatute muutujate leidmiseks ja tõestatud, et algoritm peatub hiljemalt $(n+1)$-l sammul, kus $n$ on juhtimissüsteemi dimensioon.