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A note on automatic continuity of multiplicative linear functionals on Q -algebras

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ABSTRACT

We show that every multiplicative linear functional on a Q -algebra is continuous.

1. Introduction

Throughout the paper, let \mathbb{K} denote either the field \mathbb{R} or \mathbb{C} . For us, the topological algebra (A, τ) is a topological linear space over the field \mathbb{K} , where there has been defined an associative and separately continuous multiplication. The unit element of A is denoted by e_A , and the zero element of A is denoted by θ_A .

A unital topological algebra (A, τ) is a Q -algebra if the set $\text{Inv}(A)$ of all of its invertible elements is open in τ . In the case where (A, τ) is non-unital, one considers the quasi-multiplication $\circ : A \times A \rightarrow A$, which is defined by $a \circ b = a + b - ab$ for all $a, b \in A$. In that case, an element $a \in A$ is called quasi-invertible if there exists $b \in A$ such that $a \circ b = \theta_A = b \circ a$. A non-unital topological algebra (A, τ) is a Q -algebra if the set $\text{Qinv}(A)$ of all of its quasi-invertible elements is open in τ .

During several decades, a number of mathematicians have tried to establish some sufficient conditions on a topological algebra (A, τ) over the field \mathbb{K} that would guarantee that every multiplicative linear functional from A to \mathbb{K} is automatically continuous. It started with showing that one of the sufficient conditions is that A is a Banach algebra (see [1] or [5]). Recently (see [4]), the problem was solved for multiplicative linear polynomial functionals $p : A \rightarrow \mathbb{C}$, where A is either a unital submultiplicative F -algebra, a unital Hausdorff Q -algebra or a unital strongly sequential Hausdorff topological algebra for which, for each $a \in A$, the spectral radius of a does not exceed the radius of boundedness of a .

In the present paper, we generalize the results of [4] to the case of (unital or non-unital) Q -algebras A and all multiplicative linear functionals $f : A \rightarrow \mathbb{K}$.

Let us start with a lemma that can be found in a slightly different setting (usually stated as a list of equivalent conditions for continuity of a linear functional) in some books or papers (see Theorem 1.18 in [5], p. 14, for example). As we will need only one condition that is a consequence of the discontinuity of a linear functional, we will state the result in the setting that best suits our case.

Lemma 1. *Let (A, τ) be a topological linear space over \mathbb{K} and $f : A \rightarrow \mathbb{K}$ a linear functional for which there exists $a \in A$ such that $f(a) \neq 0$. Then f is discontinuous if and only if $\ker(f)$ is dense in A .*

Proof. Let f be a discontinuous map and $a_0 \in A$ such that $f(a_0) \neq 0$. Suppose that $\ker(f)$ is not dense in A . It means that there exists an open neighbourhood $U \subseteq A \setminus (\ker(f))$ of a_0 . As U is open, there exists a balanced neighbourhood O of zero in A such that $a_0 + O \subseteq U$.

Notice that $f(O)$ is a balanced subset of \mathbb{K} because f is linear. It is known (see [3], p. 34, Lemma 3, for example) that the only balanced subsets of \mathbb{K} are \emptyset , $\{\theta_{\mathbb{K}}\}$, \mathbb{K} itself and its bounded subsets. As $O \neq \emptyset$, $f(O) \neq \emptyset$. If $f(O) = \{\theta_{\mathbb{K}}\}$, then $f(O)$ is bounded in $\theta_{\mathbb{K}}$. Suppose that $f(O) = \mathbb{K}$. Then there exists $b \in O$ such that $f(b) = -f(a_0)$, which means that $a_0 + b \in \ker(f)$. But as

$$a_0 + b \in a_0 + O \subseteq U \subseteq A \setminus (\ker(f)),$$

we get a contradiction. Thus, $f(O)$ is a bounded proper subset of \mathbb{K} , which means that there exists $\lambda \in \mathbb{K}$ with $|f(z)| < \lambda$ for all $z \in O$.

Fix any $\epsilon > 0$ and take $W = \frac{\epsilon}{\lambda}O$. Then $|f(w)| < \epsilon$ for each $w \in W$, which means that f is continuous at θ_A . Hence, f must be continuous. But this contradicts the assumptions of the lemma.

With that we have proved that if f is discontinuous, then $\ker(f)$ is dense in A .

Suppose that $\ker(f)$ is dense in A and f is continuous. Then $\ker(f)$, as a pre-image of a closed subset 0 of \mathbb{K} , is also closed. But this means that $\ker(f) = A$. Hence, $f(a) = 0$ for each $a \in A$. But this is a contradiction with our assumption of f .

With that we have proved that if $\ker(f)$ is dense in A , then f is discontinuous.

Thus, the result has been proved. □

Now, we will show that every multiplicative linear functional on a unital Q -algebra is automatically continuous.

Proposition 1. *Let (A, τ) be a unital Q -algebra. If $f : A \rightarrow \mathbb{K}$ is a multiplicative linear functional, then f is continuous.*

Proof. a) If $f(a) = 0$ for each $a \in A$, then f is a constant map, which is continuous.

b) Suppose that there exists $a_0 \in A$ such that $f(a_0) \neq 0$. If f is discontinuous, then, by Lemma 1, $\ker(f)$ is dense in A . Hence, there exists a partially ordered set (I, \preceq) and a family $(x_i)_{i \in I}$ in $\ker(f)$ that converges to the unit element e_A .

As A is a Q -algebra, there exists a neighbourhood U of e_A in A such that $U \subseteq \text{Inv}(A)$. As $(x_i)_{i \in I}$ converges to e_A , there exists $i_U \in I$ such that $x_i \in U$ for all $i \in I$ with $i_U \preceq i$. Fix any $i_0 \in I$ such that $i_U \preceq i_0$. Then $x_{i_0} \in U \subseteq \text{Inv}(A)$, which means that there exists $x_{i_0}^{-1} \in A$ with $e_A = x_{i_0} x_{i_0}^{-1}$.

Therefore, $f(e_A) = f(x_{i_0} x_{i_0}^{-1}) = f(x_{i_0}) f(x_{i_0}^{-1}) = 0$ and $f(a_0) = f(e_A a_0) = f(e_A) f(a_0) = 0$, which contradicts our assumption that $f(a_0) \neq 0$. Hence, f must be continuous. □

Next, we will prove a similar result without assuming the existence of a unital element in a Q -algebra. In order to do that, we have to recall some notions connected with the unitisation of a topological algebra.

Let (A, τ) be any non-unital topological algebra over the field \mathbb{K} . Consider the unitisation $(A \times \mathbb{K}, \tau_{A \times \mathbb{K}})$, where $A \times \mathbb{K}$ is the Dorroh extension (for algebras) of A , and $\tau_{A \times \mathbb{K}}$ is the product topology constructed from the topology τ on A and the natural topology on \mathbb{K} . Then the algebraic operations of $A \times \mathbb{K}$ are defined as follows:

$$(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu),$$

$$k(a, \lambda) = (ka, k\lambda),$$

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$

for all $(a, \lambda), (b, \mu) \in A \times \mathbb{K}$ and all $k \in \mathbb{K}$.

It is known that this unitisation is again a topological algebra with the unit element $(\theta_A, 1)$.

In what follows, by the ‘unitisation of an algebra (A, τ) ’, we will always mean the Dorroh extension described above.

Theorem 1. *Let (A, τ) be a Q -algebra. If $f : A \rightarrow \mathbb{K}$ is a multiplicative linear functional, then f is continuous.*

Proof. If (A, τ) is unital, then the result follows by Proposition 1.

Suppose that (A, τ) is non-unital. Consider the unitisation $(A \times \mathbb{K}, \tau_{A \times \mathbb{K}})$ of (A, τ) and the map $F : A \times \mathbb{K} \rightarrow \mathbb{K}$, defined by $F((a, \lambda)) = f(a) + \lambda$ for all $(a, \lambda) \in A \times \mathbb{K}$.

Let us show that F is a multiplicative linear functional on $A \times \mathbb{K}$.

As f is linear, we have

$$\begin{aligned} F((a, \lambda) + (b, \mu)) &= F((a + b, \lambda + \mu)) = f(a + b) + (\lambda + \mu) \\ &= f(a) + \lambda + f(b) + \mu = F((a, \lambda)) + F((b, \mu)), \end{aligned}$$

$$F(k(a, \lambda)) = F((ka, k\lambda)) = f(ka) + k\lambda = k(f(a)) + k\lambda = k(f(a) + \lambda) = kF((a, \lambda))$$

for all $(a, \lambda), (b, \mu) \in A \times \mathbb{K}$ and all $k \in \mathbb{K}$. Hence, F is a linear functional.

Take any $(a, \lambda), (b, \mu) \in A \times \mathbb{K}$. As f is a multiplicative linear functional, then

$$\begin{aligned} F((a, \lambda)((b, \mu))) &= F((ab + \mu a + \lambda b, \lambda\mu)) = f(ab + \mu a + \lambda b) + \lambda\mu \\ &= f(a)f(b) + \mu f(a) + \lambda f(b) + \lambda\mu = (f(a) + \lambda)(f(b) + \mu) = F((a, \lambda))F((b, \mu)). \end{aligned}$$

Hence, F is a multiplicative linear functional.

It is known (see [2], p. 156, Lemma 3.6.26) that (A, τ) is a Q -algebra if and only if $(A \times \mathbb{K}, \tau_{A \times \mathbb{K}})$ is a Q -algebra. As (A, τ) is a Q -algebra by assumption, $(A \times \mathbb{K}, \tau_{A \times \mathbb{K}})$ is a unital Q -algebra and $F : A \times \mathbb{K} \rightarrow \mathbb{K}$ is a multiplicative linear functional on $A \times \mathbb{K}$. Hence, F is continuous by Proposition 1.

Take any $a \in A$ and a neighbourhood O of $f(a)$ in \mathbb{K} . Then O is also a neighbourhood of $F((a, 0)) = f(a)$ in \mathbb{K} . By the continuity of F , there exists a neighbourhood W of $(a, 0)$ in $A \times \mathbb{K}$ such that $F(W) \subseteq O$. By the properties of the product topology on $A \times \mathbb{K}$, there exist a neighbourhood U of a in A and a neighbourhood Z of 0 in \mathbb{K} such that $U \times Z \subseteq W$. Hence, $F((b, \lambda)) \in F(W) \subseteq O$ for each $b \in U$ and each $\lambda \in Z$. This means that $f(b) = F((b, 0)) \in O$ for each $b \in U$, which implies that there exists a neighbourhood U of a in A such that $f(U) \subseteq O$. Hence, f is continuous. \square

Data availability statement

All data are available in the article.

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Q -algebratel defineeritud multiplikatiivsete lineaarsete funktsionaalide automaatsest pidevusest

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Töös näidatakse, et iga multiplikatiivne lineaarne funktsionaal, mis on defineeritud Q -algebral, on pidev.