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Generalized observer forms for discrete-time nonlinear systems

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ABSTRACT

The paper addresses the problem of transforming single-output discrete-time state equations into the generalized observer form, which comprises a linear observable component and a nonlinear injection term, depending on the inputs, output, and a finite number of their known past values. The intrinsic necessary and sufficient transformability conditions are provided, under two mild assumptions, in terms of a certain vector field, defined by the system output and its past values. The first assumption requires the state transition map to be invertible with respect to the state variable, and the second requires the constructibility rank condition to be satisfied. The algorithm is presented to find the required parametrized state transformation. The generalized observer form can be applied, under mild conditions, to also jointly estimate the states and disturbances. Two examples illustrate the theory.

1. Introduction

This paper is devoted to the transformation of discrete-time nonlinear n-dimensional state equations into the generalized observer form, which comprises a linear time-invariant observable component and a nonlinear injection term that depends, besides the input and output, also on their known past values. The so-called parametrized state transformation $Z = \Psi(x, u^{\langle -1\rangle}, \ldots, u^{\langle -r\rangle})$ is used, depending on a few known past input values, called parameters. Since there is no difficulty in storing past input and output values, it is always possible to implement the state observer with linear error dynamics based on the generalized observer form, exactly as in the case of the classical observer form. A similar problem was studied in [16,19], where the nonlinear injection term depends on unknown future values of the inputs and output. An algorithmic approach was proposed in [9], which depends on a correct selection of basis elements of a certain vector space. Various other generalizations of observer forms [1–3,12,13,20–22] complicate the observer design since they allow the linear part to be input-dependent, construct the observer form for a higher dimensional system than that of the original system, propose the necessary transformations that are rarely possible to find in the closed form, or are suitable for high-gain observer construction.

In this paper, necessary and sufficient conditions are found under which the state equations with a single output can be transformed by parametrized state transformation into the generalized observer form. The solvability conditions are developed under two mild assumptions. The first assumption requires the state transition map to be invertible with respect to the state x that is satisfied for all discretized models of the continuous-time systems. The second assumption requires the state variable x to be recovered from the output function and its $n-1$ backward shifts, being an analogue of the classical observability assumption. The solvability conditions are presented in terms of a vector field Ξ, defined uniquely by the system output and its $n-1$ backward shifts. The set of equations from which Ξ is computed is basically the same as in the case of the classical observer form, except that instead of forward shifts the backward shifts of the output are used. The latter is critical for guaranteeing that the nonlinear injection term depends on the past input and output values. The contributions of the paper are: 1) a direct transformation of the state equations into the generalized observer form, depending on the past measurements instead of the 2-step approach from [16]; 2) a constructive algorithm for computation of the parametrized transformation, unlike the one in [9]; 3) a geometric interpretation for coordinate transformation as common invariants and canonical parameters of the vector field Ξ and its forward shifts. 4) If the system involves disturbances with the known dynamics, then it is possible to consider the disturbances as state variables of an extended system, combining the original and disturbance dynamics, and transform the extended system into the generalized observer form. Such a form allows one to construct an observer that estimates jointly the states and disturbances.

In this paper, the generic algebraic setting from [5,15] is applied that differs from the classical differential geometric approach used, for example, in [2]. That is, our results do not hold only locally around an equilibrium point but are valid globally in the open set from which the singular points are removed. The results of the paper are obtained under the generic constructibility assumption.

2. Preliminaries and problem statement

Consider the discrete-time multi-input nonlinear control system

$$x^{\langle 1\rangle}(t) = \Phi(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

where $x^{\langle 1\rangle}(t) := x(t+1)$, $t \in \mathbb{Z}$, the state $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}} \in \bar{X} \subset \mathbb{R}^n$, the input $u(t) = (u_1(t), \ldots, u_m(t))^{\mathrm{T}} \in U \subset \mathbb{R}^m$, the output $y(t) \in Y \subseteq \mathbb{R}$, and $\bar{X}$, U, Y are open sets. The maps Φ and h are supposed to be analytic.

Assumption 1. The extended map $(\Phi^{\mathrm{T}}, \mathrm{Id}^{\mathrm{T}})^{\mathrm{T}} : \bar{X} \times U \to \bar{X} \times U$ has a global analytic inverse on its image.

Let $\mathcal{K}$ be a field of meromorphic functions in a finite number of independent variables from the set $\{x_i, u_j^{\langle k\rangle}; i = 1, \ldots, n; j = 1, \ldots, m; k \in \mathbb{Z}\}$, defined on an open and connected subset $\bar{X} \times U$. Here, $u_j^{\langle k\rangle}$ denotes the kth order shift of u_j. The first-order forward shift of the state variable x is defined by the equations (1). Under Assumption 1, the map Φ is invertible with respect to x, i.e. $x = \Phi^{-1}(x^{\langle 1\rangle}, u)$. Thus, the first-order backward shift of x is $x^{\langle -1\rangle} = \Phi^{-1}(x, u^{\langle -1\rangle})$. The forward and backward shifts of the functions $\varphi \in \mathcal{K}$ are defined respectively by shifting forward or backward the arguments of the function. The higher-order forward and backward shifts are defined recursively, see more from [15]. Let $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi | \varphi \in \mathcal{K}\}$ be the vector space of one-forms and $\mathcal{E}^* = \mathrm{span}_{\mathcal{K}}\{\partial/\partial x_i, \partial/\partial u_j^{\langle k\rangle}; i = 1, \ldots, n; j = 1, \ldots, m; k \in \mathbb{Z}\}$ be the vector space of vector fields. An element in $\mathcal{E}^*$ is in the form

$$\bar{\Xi} = \sum_{i=1}^{n} \zeta_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}} \eta_{jk} \frac{\partial}{\partial u_j^{\langle k\rangle}}. \tag{2}$$

The spaces $\mathcal{E}$ and $\mathcal{E}^*$ are dual. The scalar products of one-forms and vector fields satisfy the relations $\langle \mathrm{d}x_i, \bar{\Xi}\rangle = \zeta_i$, $\langle \mathrm{d}u_j^{\langle k\rangle}, \bar{\Xi}\rangle = \eta_{jk}$. A shift of a one-form $\omega = \sum_i A_i \mathrm{d}\varphi_i$, where $A_i, \varphi_i \in \mathcal{K}$, is defined simply by shifting all the functions A_i and φ_i that define the one-form. Given the vector field (2), its forward shift is another vector field

$$\bar{\Xi}^{\langle 1\rangle} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}} b_{jk} \frac{\partial}{\partial u_j^{\langle k\rangle}}, \tag{3}$$

where

$$a_i = \left\langle \mathrm{d}x_i^{\langle -1\rangle}, \bar{\Xi}\right\rangle^{\langle 1\rangle} = \left\langle \mathrm{d}\Phi_i^{-1}, \bar{\Xi}\right\rangle^{\langle 1\rangle}, \quad b_{jk} = \langle \mathrm{d}u_j^{\langle k-1\rangle}, \bar{\Xi}\rangle^{\langle 1\rangle}, \quad k \in \mathbb{Z}. \tag{4}$$

Assumption 1 yields that a forward shift of a non-zero vector field $\bar{\Xi} \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$ is a non-zero vector field $\bar{\Xi}^{\langle 1\rangle} \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$. Note also that the forward and backward shift operators commute with the scalar product, i.e. for an arbitrary one-form $\omega \in \mathcal{E}$ and a vector field $\bar{\Xi} \in \mathcal{E}^*$ $\langle \omega, \bar{\Xi}\rangle^{\langle 1\rangle} = \langle \omega^{\langle 1\rangle}, \bar{\Xi}^{\langle 1\rangle}\rangle$, $\langle \omega, \bar{\Xi}\rangle^{\langle -1\rangle} = \langle \omega^{\langle -1\rangle}, \bar{\Xi}^{\langle -1\rangle}\rangle$.

From here on, δ_{ij} denotes the Kronecker delta, which is equal to 1 if $i = j$ and 0 otherwise.

Lemma 1. [16] *Let $\Xi_l = \sum_{q=1}^{\bar{n}} \zeta_{ql}(\bar{x})\partial/\partial\bar{x}_q$, $l = 1, \dots, \bar{m}$, be the vector fields in an $\bar{n}$-dimensional vector space $\overline{C}$ with the coordinates $\bar{x}$. If the vector fields Ξ_l*

(a) are linearly independent,

(b) and commute: $\left[\Xi_l, \Xi_j\right] \equiv 0$, $l, j = 1, \dots, \bar{m}$,

then there exists in $\overline{C}$ the coordinate transformation $X = \Psi(\bar{x})$, adapted to these vector fields, i.e. $\Psi_* \Xi_l = \partial/\partial X_l$, $l = 1, \dots, \bar{m}$, such that $\bar{m}$ independent functions $\Psi_i(\bar{x})$ (as the canonical parameters of Ξ_l) satisfy $\langle \mathrm{d}\Psi_i(\bar{x}), \Xi_l(\bar{x})\rangle \equiv \delta_{il}$, $i, l = 1, \dots, \bar{m}$, and $\bar{n} - \bar{m}$ independent functions $\Psi_i(\bar{x})$ (as the common invariants of Ξ_l) satisfy $\langle \mathrm{d}\Psi_i(\bar{x}), \Xi_l(\bar{x})\rangle \equiv 0$, $i = \bar{m} + 1, \dots, \bar{n}$, $l = 1, \dots, \bar{m}$.

2.1. Problem statement

Assumption 2. The output of the system (1) satisfies the rank condition

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial \left(y, \dots, y^{\langle -n+1\rangle}\right)^{\mathrm{T}}}{\partial x} = n. \tag{5}$$

The condition (5) is a generic analogue of the constructibility property [18] and a special case of the backward observability property [8]. The goal is to find, under Assumptions 1 and 2, the parametrized state transformation $Z = \Psi(x, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j\rangle}, j = 1, \dots, m)$, such that the equations (1) in the new coordinates are in the generalized observer form with the degrees $(s, r_1, \dots, r_m)$, where s and r_j, $j = 1, \dots, m$, take the minimal possible values from the set $\{0, \dots, n-1\}$:

$$\begin{aligned}
Z_i^{\langle 1\rangle} &= Z_{i+1}, \; i = 1, \dots, s, \\
Z_i^{\langle 1\rangle} &= Z_{i+1} + \varphi_i(y, \dots, y^{\langle -s\rangle}, u_j, \dots, u_j^{\langle -r_j\rangle}, j = 1, \dots, m), \; i = s+1, \dots, n-1, \\
Z_n^{\langle 1\rangle} &= \varphi_n(y, \dots, y^{\langle -s\rangle}, u_j, \dots, u_j^{\langle -r_j\rangle}, \; j = 1, \dots, m), \\
y &= Z_{s+1}.
\end{aligned} \tag{6}$$

If s and r_j, $j = 1, ..., m$ are zero, then (6) recovers the classical observer form, in which case the first s equations are missing. Note that under Assumption 2, the problem is always solvable for $s = n - 1$ because y and its backward shifts up to the order $n - 1$ are independent functions, and one can always define the coordinate transformation $Z_i = y^{\langle -n+i\rangle}$, $i = 1, \dots, n$, that results in the equations of the form (6)

$$\begin{aligned}
Z_i^{\langle 1\rangle} &= Z_{i+1}, \quad i = 1, \dots, n-1, \\
Z_n^{\langle 1\rangle} &= \varphi_n(y, \dots, y^{\langle -n+1\rangle}, u_j, \dots, u_j^{\langle -n+1\rangle}, \; j = 1, \dots, m), \\
y &= Z_n.
\end{aligned}$$

3. Main results

Define the one-forms

$$\omega_k = \sum_{i=1}^{n} \frac{\partial y^{\langle -k\rangle}}{\partial x_i} \mathrm{d}x_i, \quad k = 0, \dots, n-1. \tag{7}$$

By (5), $\dim_{\mathcal{K}}(\operatorname{span}_{\mathcal{K}}\{\omega_k, \; k = 0, \dots, n-1\}) = n$, and one can uniquely define, analogously to [2,3], a vector field $\Xi \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x\}$ from

$$\langle \omega_k, \Xi\rangle \equiv \delta_{k0}, \quad k = 0, \dots, n-1. \tag{8}$$

Our main result in Theorem 1 below is given in terms of Ξ and its forward shifts. Since the proof of it is critically based on Lemma 1, we prove first that these vector fields are linearly independent.

Lemma 2. *The vector fields $\Xi^{\langle l\rangle}$, $l = 0, \dots, n-1$, are linearly independent.*

Proof. We prove that $\Pi := \sum_{i=0}^{n-1} \alpha_i \Xi^{\langle i \rangle} = 0$ yields that $\alpha_i = 0$ for $i = 0, \ldots, n-1$. Recall that all $\Xi^{\langle l \rangle}$, $l = 0, \ldots, n-1$, are non-zero vector fields.

Note that by (7) and the fact that $\Xi \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$, the definition of Ξ yields that $\left\langle \mathrm{d}y^{\langle -l \rangle}, \Xi \right\rangle = \delta_{l0}$ for $l = 0, \ldots, n-1$. Shifting forward $i = 0, \ldots, n-1$ times, the latter relation gives $\left\langle \mathrm{d}y^{\langle -l+i \rangle}, \Xi^{\langle i \rangle} \right\rangle = \delta_{l0}$ for $l = 0, \ldots, n-1$. Now, if one defines $j = -l + i$, and disregarding the negative values of j, one obtains

$$\left\langle \mathrm{d}y^{\langle j \rangle}, \Xi^{\langle i \rangle} \right\rangle = \delta_{ji}, \quad i = 0, \ldots, n-1; j = 0, \ldots, i. \tag{9}$$

Compute the scalar product $\langle \mathrm{d}y, \Pi \rangle$. By taking $j = 0$ in (9), one gets that $\langle \mathrm{d}y, \Pi \rangle = \alpha_0$. At the same time, since $\Pi = 0$, then $\langle \mathrm{d}y, \Pi \rangle \equiv 0$, which results in $\alpha_0 = 0$. Next, compute the scalar product $\left\langle \mathrm{d}y^{\langle 1 \rangle}, \Pi \right\rangle$. By taking $j = 1$ in (9) and remembering that $\alpha_0 = 0$, one gets that $\left\langle \mathrm{d}y^{\langle 1 \rangle}, \Pi \right\rangle = \alpha_1$. Again, since $\Pi = 0$, then $\alpha_1 = 0$. Continuing in the similar way, one gets $\alpha_i = 0$ for $i = 0, \ldots, n-1$, meaning that $\Xi^{\langle l \rangle}$, $l = 0, \ldots, n-1$, are linearly independent. □

Remark 1. The parametrized state transformation can be interpreted as a coordinate transformation in the $(n + \sum_{j=1}^{m} r_j)$-dimensional subspace $\bar{C}$ of C with the coordinates $\bar{x} = (x, u_j^{\langle -1 \rangle}, \ldots, u_j^{\langle -r_j \rangle}; j = 1, \ldots, m)$ such that the coordinates $u_j^{\langle -1 \rangle}, \ldots, u_j^{\langle -r_j \rangle}$ remain unchanged. The conditions (i) and (ii) in Theorem 1 define the smallest possible value of s, related to the number of commuting vector fields, and the conditions (iii) and (iv) define the smallest possible values of r_j. Observe that all the conditions hold automatically for the limiting case $s = n - 1$. In the special case where $s = r_j = 0$, $j = 1, \ldots, m$, the theorem yields the necessary and sufficient conditions for the classical observer form.

Theorem 1. *Under Assumptions 1 and 2, the equations* (1) *can be transformed, via a parametrized state transformation* $Z = \Psi(x, u_j, \ldots, u_j^{\langle -r_j \rangle}, j = 1, \ldots, m)$*, into the generalized observer form* (6) *with degrees* $(s, r_1, \ldots, r_m)$ *if and only if the following conditions are satisfied for* $j = 1, \ldots, m$*:*

(i) $\left[\partial/\partial u_j, \Xi^{\langle l \rangle}\right] \equiv 0$, $l = 1, \ldots, n-s-1$,

(ii) $\left[\Xi, \Xi^{\langle q \rangle}\right] \equiv 0$, $q = 0, \ldots, n-s-1$,

(iii) $\partial y^{\langle -s \rangle}/\partial u_j^{\langle -k_j \rangle} \equiv 0$, $k_j = r_j + 1, \ldots, s$,

(iv) $\left[\partial/\partial u_j^{\langle -k_j \rangle}, \Xi\right] \equiv 0$, $k_j = r_j + 1, \ldots, n-1$.

Proof. Sufficiency. Throughout the proof, let $j = 1, \ldots, m$. The proof consists of two steps. First, we define the parametrized state transformation, which is then shown to transform the state equations into the generalized observer form.

We will show that from (ii) follows $\left[\Xi^{\langle l \rangle}, \Xi^{\langle q \rangle}\right] \equiv 0$, $l, q = 0, \ldots, n-s-1$. By the properties of the Lie bracket, the condition $\left[\Xi^{\langle j \rangle}, \Xi^{\langle l \rangle}\right] = 0$, $j, l = 0, \ldots, n-s-1$, is equivalent to the condition $\left[\Xi^{\langle j \rangle}, \Xi^{\langle l \rangle}\right] = 0$, $j = 0, \ldots, n-s-1$, and $l = j+1, \ldots, n-s-1$. Then the claim follows from $\left[\Xi^{\langle j \rangle}, \Xi^{\langle l \rangle}\right] = \left[\Xi, \Xi^{\langle l-j \rangle}\right]^{\langle j \rangle} = 0$ by the condition (ii) and assuming that $l > j$.

Consider the vector fields $\Xi^{\langle l \rangle}$, $l = 0, \ldots, n-s-1$. If the condition (i) holds, then the coefficients of $\Xi^{\langle l \rangle}$ do not depend on u_j and its forward shifts. The condition (iv) guarantees that the coefficients of $\Xi^{\langle l \rangle}$, $l = 0, \ldots, n-s-1$, do not depend on the backward shifts of u_j higher than the order r_j. Therefore, the vector fields $\Xi^{\langle l \rangle}$, $l = 0, \ldots, n-s-1$, are well-defined in the space $\bar{C}$. Since, by Lemma 2, the vector fields $\Xi^{\langle l \rangle}$, $l = 0, \ldots, n-s-1$, are linearly independent, and by the condition (ii) they commute, then Lemma 1 guarantees that there exists a coordinate transformation $X = \Psi(\bar{x})$ in the space $\bar{C}$ adapted to these vector fields. Because $\Xi^{\langle l \rangle} \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$, $l = 0, \ldots, n-s-1$, then $u_j^{\langle -k_j \rangle}$, $k_j = 1, \ldots, r_j$, are some of the common invariants of the corresponding vector fields, and they remain unchanged. The rest of the coordinate transformation defines a parametrized state transformation, which, as shown below, transforms the equations (1) into the generalized observer form (6) with the degrees $(s, r_1, \ldots, r_m)$.

Next, we show that the additional common invariants of the vector fields $\Xi^{\langle l\rangle}$, $l = 0, \dots, n-s-1$, are $y^{\langle -i\rangle}$, $i = 1, \dots, s$. From (8), it follows that $\langle \omega_k, \Xi\rangle \equiv 0$, $k = 1, \dots, n-1$. According to (7) and the fact that $\Xi \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x\}$, the last equality is equivalent to the identity $\left\langle \mathrm{d}y^{\langle -k\rangle}, \Xi\right\rangle \equiv 0$. Writing k as the sum of two indices, $k = i + l$, where $i = 1, \dots, s$ and $l = 0, \dots, n-s-1$, the last identity takes the form $\left\langle \mathrm{d}y^{\langle -i-l\rangle}, \Xi\right\rangle \equiv 0$. Shifting the result forward l times, one gets

$$\left\langle \mathrm{d}y^{\langle -i\rangle}, \Xi^{\langle l\rangle}\right\rangle \equiv 0, \quad i = 1, \dots, s; l = 0, \dots, n-s-1. \tag{10}$$

The relation (10) proves that $y^{\langle -i\rangle}$, $i = 1, \dots, s$, are the common invariants of the vector fields $\Xi^{\langle l\rangle}$, $l = 0, \dots, n-s-1$. By Lemma 1, these invariants are defined as new state coordinates. Thus, let

$$Z_i = \Psi_i = y^{\langle -s+i-1\rangle}, \quad i = 1, \dots, s. \tag{11}$$

The condition (iii) guarantees that Ψ_i, $i = 1, \dots, s$, depend only on the variables $x, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j\rangle}$.

The rest of the new state coordinates are defined as the canonical parameters of $\Xi^{\langle l\rangle}$, $l = 0, \dots n-s-1$, i.e. by

$$\left\langle \mathrm{d}\Psi_i, \Xi^{\langle l\rangle}\right\rangle \equiv \delta_{i,l+s+1}, \quad i = s+1, \dots, n \tag{12}$$

for $l = 0, \dots, n-s-1$. Note that one can always take

$$Z_{s+1} = \Psi_{s+1} = y. \tag{13}$$

Indeed, from (8) and the fact that $\Xi \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x\}$ follows $\left\langle \mathrm{d}y^{\langle -l\rangle}, \Xi\right\rangle \equiv \delta_{l0}$, $l = 0, \dots, n-s-1$. Shifting the last equality forward l times yields $\left\langle \mathrm{d}y, \Xi^{\langle l\rangle}\right\rangle \equiv \delta_{l0}$, $l = 0, \dots, n-s-1$, which is equivalent to (12) for $i = s+1$. Now, one has $Z_i^{\langle 1\rangle} = Z_{i+1}$, $i = 1, \dots, s$, that is, we obtained the first set of s equations in (6).

Next, we show that by defining the remaining coordinates $Z_{s+2}, \dots, Z_n$ as in (12), we obtain the second and third set of rows in (6). Since $\mathrm{d}\Psi_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_j^{\langle -1\rangle}, \dots, \mathrm{d}u_j^{\langle -r_j\rangle}\}$, $i = 1, \dots, n$, then $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_j^{\langle -1\rangle}, \dots, \mathrm{d}u_j^{\langle -r_j\rangle}\} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\Psi, \mathrm{d}u_j^{\langle -1\rangle}, \dots, \mathrm{d}u_j^{\langle -r_j\rangle}\}$, and because all $\mathrm{d}\Psi_i^{\langle 1\rangle}$ belong to $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_j, \dots, \mathrm{d}u_j^{\langle -r_j\rangle}\}$, then one may write $\mathrm{d}\Psi_i^{\langle 1\rangle}$, $i = s+1, \dots, n$, as the linear combinations

$$\mathrm{d}\Psi_i^{\langle 1\rangle} = \sum_{q=1}^{n} \alpha_{iq}(x, u_j, \dots, u_j^{\langle -r_j\rangle})\mathrm{d}\Psi_q + \sum_{l=1}^{m}\sum_{k=0}^{r_l} \tilde{\alpha}_{ilk}(x, u_j, \dots, u_j^{\langle -r_j\rangle})\mathrm{d}u_l^{\langle -k\rangle}. \tag{14}$$

Compute the scalar product of both sides of (14) with the vector fields $\Xi^{\langle l\rangle}$, $l = 1, \dots, n-s-1$, taking into account that, because $\Xi^{\langle l\rangle} \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x\}$, $\langle \mathrm{d}u_j^{\langle -k\rangle}, \Xi^{\langle l\rangle}\rangle \equiv 0$ for $k = 0, \dots, r_j$ and $l = 0, \dots, n-s-1$. One obtains $\langle \mathrm{d}\Psi_i^{\langle 1\rangle}, \Xi^{\langle l\rangle}\rangle = \sum_{q=1}^{n} \alpha_{iq}\langle \mathrm{d}\Psi_q, \Xi^{\langle l\rangle}\rangle$. From (10)–(12) and the definition of the Kronecker delta, the last formula takes the form $\langle \mathrm{d}\Psi_i^{\langle 1\rangle}, \Xi^{\langle l\rangle}\rangle = \alpha_{i,l+s+1}$, which can be rewritten as

$$\left\langle \mathrm{d}\Psi_i, \Xi^{\langle l-1\rangle}\right\rangle^{\langle 1\rangle} = \alpha_{i,l+s+1} \tag{15}$$

for $i = s+1, \dots, n$ and $l = 1, \dots, n-s-1$. Due to (12), $\langle \mathrm{d}\Psi_i, \Xi^{\langle l-1\rangle}\rangle = \delta_{i,l+s}$, and so, from (15), one gets $\alpha_{i,l+s+1} = \delta_{i,l+s}$. Taking $q = l+s+1$, the last expression yields $\alpha_{iq} = \delta_{i+1,q}$, $i = s+1, \dots, n$, $q = s+2, \dots, n$. Now, (14) becomes $\mathrm{d}\Psi_i^{\langle 1\rangle} = \mathrm{d}\Psi_{i+1} + \sum_{q=1}^{s+1} \alpha_{iq}(x, u_j, \dots, u_j^{\langle -r_j\rangle})\mathrm{d}\Psi_q + \sum_{l=1}^{m}\sum_{k=0}^{r_l} \tilde{\alpha}_{ilk}(x, u_j, \dots, u_j^{\langle -r_j\rangle})\mathrm{d}u_l^{\langle -k\rangle}$, which, by (11) and (13), yields (6).

Necessity. First, recall that since the property of the Lie bracket being equal to zero is invariant under a parametrized state transformation, see the property (iv) of Proposition 6 in [7], then the validity of the conditions (i), (ii) and (iv) can be checked in terms of the new Z coordinates. We show that in the Z coordinates the vector fields $\Xi^{\langle l\rangle}$, $l = 0, \dots, n-s-1$, are the corresponding partial derivative operators $\Xi^{\langle l\rangle} = \partial/\partial Z_{s+l+1}$.

In order to compute the vector fields $\Xi^{\langle l\rangle}$, $l = 0, \dots, n-s-1$, we first find the expressions for the backward shifts of the state variables Z. By shifting back the first s equations in (6), one gets

$$Z_i^{\langle -1\rangle} = Z_{i-1}, \quad i = 2, \dots, s+1. \tag{16}$$

Then, $y^{\langle -i\rangle} = Z_{s-i+1}$, $i = 0, \dots, s$, and the last $n - s$ state equations can be rewritten as

$$
\begin{aligned}
Z_i^{\langle 1\rangle} &= Z_{i+1} + \varphi_i(Z_{s+1}, \dots, Z_1, u_j, \dots, u_j^{\langle -r_j\rangle}), \quad i = s+1, \dots, n-1,\\
Z_n^{\langle 1\rangle} &= \varphi_n(Z_{s+1}, \dots, Z_1, u_j, \dots, u_j^{\langle -r_j\rangle}).
\end{aligned}
$$

When shifting the latter equations back once and considering (16), one gets

$$
\begin{aligned}
Z_i &= Z_{i+1}^{\langle -1\rangle} + \varphi_i(Z_s, \dots, Z_1, Z_1^{\langle -1\rangle}, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j-1\rangle}), \quad i = s+1, \dots, n-1,\\
Z_n &= \varphi_n(Z_s, \dots, Z_1, Z_1^{\langle -1\rangle}, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j-1\rangle}).
\end{aligned}
\tag{17}
$$

By Assumption 1, $\mathrm{rank}_{\mathcal{K}}(\partial\Phi/\partial x) = n$. Since the latter condition is invariant with respect to the parametrized state transformation, the equations (6) must also satisfy the rank condition. One may assume that $\partial\varphi_n/\partial y^{\langle -s\rangle} = \partial\varphi_n/\partial Z_1 \not\equiv 0$; otherwise the first s equations and the last equation in (6) (i.e. $s+1$ different state equations of (6)) would depend only on s different state variables, and thus the rank condition cannot be satisfied. Now, one can solve the equations (17) in variables $Z_1^{\langle -1\rangle}, Z_{s+2}^{\langle -1\rangle}, \dots, Z_n^{\langle -1\rangle}$ to get

$$
\begin{aligned}
Z_i^{\langle -1\rangle} &= Z_{i-1} - \varphi_{i-1}(Z_s, \dots, Z_1, \eta(\cdot), u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j-1\rangle}), \quad i = s+2, \dots, n,\\
Z_1^{\langle -1\rangle} &= \eta(Z_n, Z_s, \dots, Z_1, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j-1\rangle}),
\end{aligned}
\tag{18}
$$

where η is a function obtained by solving the last equation in (17) for $Z_1^{\langle -1\rangle}$. Now, the backward shifts of the state variables of (6) are defined by (16) and (18).

Next, find the vector field Ξ for (6). For that, compute the backward shifts of y up to the order $n-1$: $y^{\langle -i\rangle} = Z_{s-i+1}$, $i = 0, \dots, s$, $y^{\langle -s-1\rangle} = \eta(Z_n, Z_1, \dots, Z_s, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j-1\rangle})$, $y^{\langle -k\rangle} = \zeta_k(Z_{n+s-k+1}, \dots, Z_n, Z_1, \dots, Z_s, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j-k+s\rangle})$, $k = s+2, \dots, n-1$, for some functions ζ_k, $k = s+2, \dots, n-1$. The vector field Ξ in the matrix form must satisfy $P\Xi = (1, 0, \dots, 0)^T$, where the matrix $P := \partial(y, \dots, y^{\langle -n+1\rangle})/\partial Z$. Since $y = Z_{s+1}$, and none of the backward shifts of y depend on y itself, then one gets $\Xi = \partial/\partial Z_{s+1}$. Compute the forward shift of Ξ. Because only the backward shift of Z_{s+2} depends on Z_{s+1} and $\partial Z_{s+2}^{\langle -1\rangle}/\partial Z_{s+1} = 1$, then, by the definition of the forward shift of a vector field, one gets $\Xi^{\langle 1\rangle} = \partial/\partial Z_{s+2}$. Similarly, one can compute the forward shifts of Ξ up to the order $n-s-1$ as $\Xi^{\langle l\rangle} = \partial/\partial Z_{s+l+1}$ for $l = 0, \dots, n-s-1$. Now, it is obvious that the conditions (i), (ii) and (iv) are satisfied. Also, because $y^{\langle -s\rangle} = Z_1 = \Psi_1(x, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j\rangle})$, the condition (iii) is satisfied. □

The sufficiency proof of Theorem 1 is constructive and can be summarized as the following algorithm.

Algorithm:

1. Let $\Xi^{\langle l-1\rangle} = \sum_{i=1}^{n} M_{il}\partial/\partial x_i$ for $l = 1, \dots, n-s$, and construct a matrix M with the element M_{il} at the ith row and the lth column.
2. Compute the left inverse M_L^{-1} of M as $(\theta_{s+1}^T, \dots, \theta_n^T)^T = (M^TM)^{-1}M^T$.
3. Find the functions $\beta_{iq}(x, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j\rangle}, j = 1, \dots, m)$ and $\tilde\beta_{ilk}(x, u_j^{\langle -1\rangle}, \dots, u_j^{\langle -r_j\rangle}, j = 1, \dots, m)$ such that $\mathrm{d}\Psi_i = \theta_i\mathrm{d}x + \sum_{q=1}^{s}\beta_{iq}\mathrm{d}y^{\langle -q\rangle} + \sum_{l=1}^{m}\sum_{k=1}^{r_l}\tilde\beta_{ilk}\mathrm{d}u_l^{\langle -k\rangle}$ for some functions Ψ_i, $i = s+2, \dots, n$.
4. Define $Z_i = y^{\langle i-s-1\rangle}$ for $i = 1, \dots, s+1$ and $Z_j = \Psi_j$ for $j = s+2, \dots, n$.

Remark 2. The equations (1) depend on $n+m$ variables, i.e. more than the number of equations n. Since these equations define the backward shifts of the states x and the inputs u, then there are always m variables z, whose backward shifts remain free independent variables in the field $\mathcal{K}$. In this paper, we have implicitly chosen $z = u$, a natural choice when Assumption 1 is satisfied. We would like to stress that the results of the paper are valid only for such a choice. Other choices of variables z may lead to a non-unique vector field Ξ or even to wrong conclusions about the solvability of the problem.

4. Joint state and disturbance estimation

It is well known that for a system in the generalized observer form (6) it is relatively easy to construct a state observer, which guarantees that the error between the actual and the estimated states asymptotically decays to zero, see for example [6,19]. However, the transformation of a system into the generalized observer form can also frequently be used to construct disturbance observers to estimate disturbances affecting the system dynamics. In this section, we present the basics on how to use the results from Section 3 to construct disturbance observers.

Consider a discrete-time single output nonlinear control system

$$\begin{aligned} x^{\langle 1\rangle}(t) &= \Phi(x(t), w(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{19}$$

where $x(t)$, $u(t)$, and $y(t)$ are as in (1), and $w(t) \in W \subset \mathbb{R}$ is the disturbance input. It is relatively common in studying the disturbance estimation problem to assume that some finite time-derivative of the disturbance is zero [4,10,14].

Assumption 3. The kth derivative of disturbance w satisfies $w^{(k)} = 0$, $k \geq 1$.

Under Assumption 3, the disturbance dynamics can be described by the state equations

$$\begin{aligned} \dot{\xi}_1(\tau) &= \xi_2(\tau) \\ &\vdots \\ \dot{\xi}_{k-1}(\tau) &= \xi_k(\tau) \\ \dot{\xi}_k(\tau) &= 0, \end{aligned} \tag{20}$$

where $\tau \in \mathbb{R}$, the vector $\xi(\tau) = (\xi_1(\tau), \ldots, \xi_k(\tau))^{\mathrm{T}} \subseteq \mathbb{R}^k$, and $\xi_1(\tau) = w(\tau), \xi_2(\tau) = \dot{w}(\tau), \ldots,$ $\xi_k(\tau) = w^{(k-1)}(\tau)$. Since the systems under study are not continuous- but discrete-time, we first have to find the exact discrete-time model of (20). The system (20) admits the Lie-Taylor discretization in the form [11]:

$$\xi^{\langle 1\rangle}(t) = \sum_{r=0}^{k-1} \frac{T^r}{r!} \mathcal{L}_f^r \xi \Big|_{\xi(t)}, \tag{21}$$

where T is the sampling period, $f = (\xi_2, \ldots, \xi_k, 0)^{\mathrm{T}}$, and the Lie derivatives $\mathcal{L}_f^0 g = g$, $\mathcal{L}_f g = \partial g/\partial \xi \cdot f$ for some function $g = g(\xi)$, $\mathcal{L}_f^r = \mathcal{L}_f(\mathcal{L}_f^{r-1})$ for $r \geq 2$. The summation index r in (21) goes only up to $k-1$ because all the Lie derivatives of the order k or higher equal to zero, as shown below. Compute for the system (20):

$$\mathcal{L}_f^1 \xi = f = \begin{pmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_k \\ 0 \end{pmatrix}, \quad \mathcal{L}_f^2 \xi = \frac{\partial f}{\partial \xi} f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_k \\ 0 \end{pmatrix} = \begin{pmatrix} \xi_3 \\ \vdots \\ \xi_k \\ 0 \\ 0 \end{pmatrix}, \quad \cdots,$$

$$\mathcal{L}_f^{k-1} \xi = \begin{pmatrix} \xi_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathcal{L}_f^k \xi = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

To conclude, discretized equations of the disturbance dynamics (20) are

$$\xi_i^{\langle 1\rangle}(t) = \sum_{r=0}^{k-i} \frac{T^r}{r!} \xi_{r+i}(t), \quad i = 1, \ldots, k. \tag{22}$$

If the dynamics of the disturbance is known, the system (19) can be extended to

$$\begin{aligned} x^{\langle 1\rangle}(t) &= \Phi(x(t), \xi_1(t), u(t)) \\ \xi^{\langle 1\rangle}(t) &= \psi(\xi(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{23}$$

where $\xi(t) = (w(t), w^{\langle 1\rangle}(t), \dots, w^{\langle k-1\rangle}(t))^{\mathrm{T}}$, and the components of $\psi(\xi(t))$ are defined by the right-hand side of (22). Note that in (23), the disturbance w and its derivatives up to the order $k-1$ at the time instance t are considered as the state variables of the extended system, and thus the equations (23) are in the form (1). Therefore, in order to estimate the disturbance, it is sufficient to construct a state observer for the system (23).

Assumption 1*. The extended map $(\Phi^{\mathrm{T}}, \psi^{\mathrm{T}}, \mathrm{Id}^{\mathrm{T}})^{\mathrm{T}}\colon \bar{X}\times W\times U \to \bar{X}\times W\times U$ has a global analytic inverse on its image.

Note also that if the system (19) satisfies Assumption 1, then the system (23) satisfies Assumption 1*. In order to use the results from Section 3, one also has to make the following assumption.

Assumption 2*. System (23) satisfies the generic constructibility condition.

Now, under Assumption 1*, a procedure to estimate the disturbance w jointly with its forward shifts and the state x can be given as:

1. Construct the system (23) and check whether it satisfies Assumption 2*. If yes, then go to step 2; if not, then the procedure cannot be applied.
2. Transform the equation (23) into the generalized observer form (6), as explained in the algorithm in Section 3.

The form (6) for the extended system (23) allows one, jointly with the state x of the system (19), to estimate the disturbance w and its forward shifts up to the order k.

5. Examples

The main motivation behind the academic Example 1 below is to illustrate all the steps of the proposed method with explicitly given intermediate computations. Note that in the case of the real-life Example 2, this is not possible since the intermediate expressions are too long to be given in the paper.

Example 1. Consider the reversible state equations

$$\begin{aligned} x_1^{\langle 1\rangle} &= x_3, & x_2^{\langle 1\rangle} &= x_2u_1 + x_3 + x_4, \\ x_3^{\langle 1\rangle} &= x_2 - x_1, & x_4^{\langle 1\rangle} &= x_1x_3u_2, \quad y = x_3. \end{aligned} \tag{24}$$

We assume that $x_1(t) \neq 0$ and $u_2(t) \neq 0$ for all $t \in \mathbb{Z}$. Then the equations (24) guarantee that also $x_3(t) \neq 0$ and $x_4(t) \neq 0$ for all $t \in \mathbb{Z}$, and so all computations below are valid. Our task is to transform the equations (24) into the generalized observer form (6). From (24), one can compute $x_1^{\langle -1\rangle} = x_4/(x_1u_2^{\langle -1\rangle})$, $x_2^{\langle -1\rangle} = (x_4 + x_1x_3u_2^{\langle -1\rangle})/(x_1u_2^{\langle -1\rangle})$, $x_3^{\langle -1\rangle} = x_1$, $x_4^{\langle -1\rangle} = x_2 - x_1 - x_3u_1^{\langle -1\rangle} - (u_1^{\langle -1\rangle}x_4)/(x_1u_2^{\langle -1\rangle})$, and the backward shifts of y up to the order 3:

$$y = x_3, \quad y^{\langle -1\rangle} = x_1, \quad y^{\langle -2\rangle} = \frac{x_4}{x_1u_2^{\langle -1\rangle}}, \quad y^{\langle -3\rangle} = \frac{-u_1^{\langle -1\rangle}x_4 - x_1u_2^{\langle -1\rangle}(u_1^{\langle -1\rangle}x_3 + x_1 - x_2)}{u_2^{\langle -2\rangle}x_4}.$$

One can easily check that Assumption 2 is satisfied, which guarantees the linear independence of the one-forms ω_k, $k = 0, \dots, 3$, constructed according to (7):

$$\begin{aligned} &\omega_0 = \mathrm{d}x_3, \quad \omega_1 = \mathrm{d}x_1, \quad \omega_2 = -\frac{x_4}{x_1^2u_2^{\langle -1\rangle}}\mathrm{d}x_1 + \frac{1}{x_1u_2^{\langle -1\rangle}}\mathrm{d}x_4, \quad \omega_3 = \frac{x_1u_2^{\langle -1\rangle}}{x_4u_2^{\langle -2\rangle}}\mathrm{d}x_2 \\ &\quad + \frac{\left(x_2 - 2x_1 - x_3u_1^{\langle -1\rangle}\right)u_2^{\langle -1\rangle}}{x_4u_2^{\langle -2\rangle}}\mathrm{d}x_1 - \frac{x_1u_1^{\langle -1\rangle}u_2^{\langle -1\rangle}}{x_4u_2^{\langle -2\rangle}}\mathrm{d}x_3 + \frac{x_1\left(x_3u_1^{\langle -1\rangle} - x_2 + x_1\right)u_2^{\langle -1\rangle}}{x_4^2u_2^{\langle -2\rangle}}\mathrm{d}x_4. \end{aligned}$$

Next, by (8), find the vector field $\Xi = u_1^{\langle -1\rangle}\partial/\partial x_2 + \partial/\partial x_3$ as the solution of the system of the equations $\langle \omega_k, \Xi\rangle \equiv \delta_{k0}$, $k = 0, \dots, 3$. Compute the forward shifts of Ξ up to the order 3:

$$\Xi^{\langle 1\rangle} = \frac{\partial}{\partial x_2}, \quad \Xi^{\langle 2\rangle} = \frac{\partial}{\partial x_4}, \quad \Xi^{\langle 3\rangle} = \frac{1}{x_3u_2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - u_1\frac{\partial}{\partial x_4}\right).$$

The condition (i) of Theorem 1 is satisfied for $l = 0, 1, 2$, suggesting that s may be 1. But to define s, one has also to check the condition (ii). The vector fields $\Xi^{\langle l\rangle}$ commute for $l = 0, 1, 2$, but $\Xi^{\langle 3\rangle}$ does not commute with Ξ; therefore one may conclude that $s = 1$.

According to Step 1 of the algorithm, construct the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \\ u_1^{\langle -1\rangle} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, as in Step 2, compute its left inverse

$$M_L^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -u_1^{\langle -1\rangle} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

defining the one-forms $\theta_3 \mathrm{d}x = \mathrm{d}x_2 - u_1^{\langle -1\rangle}\mathrm{d}x_3$, $\theta_4 \mathrm{d}x = \mathrm{d}x_4$. In Step 3, since $\theta_4 \mathrm{d}x$ is a total differential, one can take $\Psi_4 = x_4$. In order to turn $\theta_3 \mathrm{d}x$ into a total differential, take $\tilde{\beta}_{311} = -x_3$ to get $\mathrm{d}\Psi_3 = \theta_3 \mathrm{d}x - x_3 \mathrm{d}u_1^{\langle -1\rangle} = \mathrm{d}(x_2 - u_1^{\langle -1\rangle} x_3)$.

According to Step 4, taking into account that $s = 1$, define the parametrized state transformation $Z_1 = x_1$, $Z_2 = x_3$, $Z_3 = x_2 - x_3 u_1^{\langle -1\rangle}$, $Z_4 = x_4$ with the inverse $x_1 = Z_1$, $x_2 = Z_2 u_1^{\langle -1\rangle} + Z_3$, $x_3 = Z_2$, $x_4 = Z_4$. Finally, taking into account that $Z_1 = x_1 = y^{\langle -1\rangle}$ and $Z_2 = x_3 = y$, we get the equations in the form (6):

$$\begin{aligned} Z_1^{\langle 1\rangle} &= Z_2, \quad Z_2^{\langle 1\rangle} = Z_3 + y u_1^{\langle -1\rangle} - y^{\langle -1\rangle}, \\ Z_3^{\langle 1\rangle} &= Z_4 + u_1 y^{\langle -1\rangle} + y, \quad Z_4^{\langle 1\rangle} = y y^{\langle -1\rangle} u_2, \quad y = Z_2. \end{aligned}$$

Example 2. Examine a hydraulic press with the vertical cylinder, lifting the load upwards via pumping oil into the chamber under the piston. The state equations of this system can be described by

$$\begin{aligned} x_1^{\langle 1\rangle} &= x_1 + x_2 T, \\ x_2^{\langle 1\rangle} &= x_2 + \frac{[S(x_3 - x_4) - mg - \mu x_2]\, T}{m}, \\ x_3^{\langle 1\rangle} &= x_3 + \frac{\beta(u - x_2)T}{l_0 + x_1}, \\ x_4^{\langle 1\rangle} &= C - x_3, \qquad y = x_1, \end{aligned} \tag{25}$$

see the equations (1), adapted to upward movement, in [17]. In (25), x_1 and x_2 are the position of the piston and its velocity, respectively; x_3 and x_4 are the pressures under and above the piston, respectively; and T is the sampling time. The constants have the following meaning: m is the mass loaded on the piston, S is the effective piston area, μ is the damping coefficient, $l_0 > 0$ is the height of the chamber under the piston, and $\beta > 0$ is the isothermal bulk modulus of the oil. The parameter C shows how the pressure above the piston is automatically adjusted according to the pressure under the piston. We consider here the so-called saving circuit, where C is kept constant. Since $x_1 \geq 0$, then $l_0 + x_1 > 0$ in the computations below. The sum $\beta + C - x_4 - x_3$ also cannot be equal to zero since $\beta > 0$ and $C - x_4 - x_3 = 0$ once the piston stops moving. The input u in (25) is the volume of oil, pumped into the chamber under the piston during the time T, divided by S. Compute

$$\begin{aligned} x_1^{\langle -1\rangle} &= -\frac{(Tu^{\langle -1\rangle} - x_1)\beta + (C - x_3 - x_4) l_0}{\beta + C - x_3 - x_4}, \\ x_2^{\langle -1\rangle} &= \frac{u^{\langle -1\rangle}\beta}{\beta + C - x_3 - x_4} + \frac{(C - x_3 - x_4)(l_0 + x_1)}{T(\beta + C - x_3 - x_4)}, \\ x_3^{\langle -1\rangle} &= C - x_4, \\ x_4^{\langle -1\rangle} &= \frac{\beta(\mu T - m)(Tu^{\langle -1\rangle} - l_0 - x_1)}{ST^2(x_3 + x_4 - C - \beta)} - x_4 - \frac{m x_2}{ST} + \frac{(CS - mg)T^2 + (m - \mu T)(l_0 + x_1)}{ST^2}. \end{aligned} \tag{26}$$

Using (26) and symbolic software, the backward shifts of y up to the order 3 can be computed. One gets

$$y^{\langle -1\rangle} = -\frac{(Tu^{\langle -1\rangle} - x_1)\beta + (C - x_3 - x_4)l_0}{\beta + C - x_3 - x_4},$$

whereas the expressions of $y^{\langle -2\rangle}$ and $y^{\langle -3\rangle}$ are too long to present here. Note that Assumption 2 is satisfied. Next, we find the one-forms ω_k, $k = 0, \dots, 3$, according to (7), getting

$$\begin{aligned}
\omega_0 &= \mathrm{d}x_1,\\
\omega_1 &= \frac{\beta}{\beta + C - x_3 - x_4}\mathrm{d}x_1 - \frac{(Tu^{\langle -1\rangle} - l_0 - x_1)\beta}{(\beta + C - x_3 - x_4)^2}(\mathrm{d}x_3 + \mathrm{d}x_4),
\end{aligned}$$

omitting ω_2 and ω_3 due to their length. Using the symbolic software, compute from the equations (8) the vector field

$$\Xi = \frac{\partial}{\partial x_1} + \frac{2m - \mu T}{mT}\frac{\partial}{\partial x_2} + \left(\frac{\beta + C - x_3 - x_4}{Tu^{\langle -1\rangle} - l_0 - x_1} + \frac{m}{2ST^2}\right)\frac{\partial}{\partial x_3} - \frac{m}{2ST^2}\frac{\partial}{\partial x_4}. \tag{27}$$

Since $n - 1 = 3$, to check the conditions of Theorem 1, we have to compute the forward shifts of Ξ up to the order 3. Shifting the vector field Ξ forward according to (3) and (4), we get

$$\begin{aligned}
\Xi^{\langle 1\rangle} &= \frac{1}{T}\frac{\partial}{\partial x_2} + \frac{m}{2ST^2}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right),\\
\Xi^{\langle 2\rangle} &= \frac{m}{2ST^2}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right),\\
\Xi^{\langle 3\rangle} &= \frac{m}{2ST^2}\left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right).
\end{aligned} \tag{28}$$

The coefficients of the vector fields (27) and (28) do not depend on u. The vector fields Ξ, $\Xi^{\langle 1\rangle}$, and $\Xi^{\langle 2\rangle}$ also commute, which is not the case for $\Xi^{\langle 3\rangle}$ because $[\Xi, \Xi^{\langle 3\rangle}] = m/(ST^2(Tu^{\langle -1\rangle} - l_0 - x_1))\,\partial/\partial x_3$. That is, the conditions (i) and (ii) of Theorem 1 hold for $l, q = 0, 1, 2$, suggesting $s = 1$. The function $y^{\langle -1\rangle}$ and the coefficients of Ξ depend on $u^{\langle -1\rangle}$ but not on $u^{\langle -2\rangle}$. Therefore, the conditions (iii) and (iv) of Theorem 1 are satisfied for $r = 1$. Thus, we have established that the conditions of Theorem 1 are satisfied for $s = r = 1$.

Let us now compute the parametrized state transformation that takes (25) into the form (6). For this, we use the algorithm from Section 3. Construct the matrix M according to Step 1:

$$M = \begin{pmatrix} 1 & 0 & 0\\ \frac{2m-\mu T}{Tm} & \frac{1}{T} & 0\\ M_{3,1} & \frac{m}{2ST^2} & \frac{m}{2ST^2}\\ -\frac{m}{2ST^2} & -\frac{m}{2ST^2} & -\frac{m}{2ST^2} \end{pmatrix},$$

where

$$M_{3,1} := \frac{\beta + C - x_3 - x_4}{Tu^{\langle -1\rangle} - l_0 - x_1} + \frac{m}{2ST^2},$$

and compute, according to Step 2, its left inverse. The latter can be found with the help of symbolic software, but again, its form is too complicated to be presented here. Therefore, also Step 3 cannot be described due to the complicated expressions. However, computations with symbolic software yield in Step 4 the following parametrized state transformation:

$$\begin{aligned}
Z_1 &= y^{\langle -1\rangle} = -\frac{(Tu^{\langle -1\rangle} - x_1)\beta + (C - x_4 - x_3)l_0}{\beta + C - x_4 - x_3},\\
Z_2 &= y = x_1, \quad Z_3 = Tx_2 - \frac{x_1(2m - \mu T)}{m},\\
Z_4 &= -x_1 - x_2T + \frac{x_1(2m - \mu T)}{m} - \frac{2x_4ST^2}{m}.
\end{aligned}$$

Taking into account that $Z_1 = y^{\langle -1\rangle}$ and $Z_2 = y$, the state equations in the new coordinates are

$$
\begin{aligned}
Z_1^{\langle 1\rangle} &= Z_2,\\
Z_2^{\langle 1\rangle} &= Z_3 + \frac{y(3m-\mu T)}{m},\\
Z_3^{\langle 1\rangle} &= Z_4 + \frac{ST^2(Tu^{\langle -1\rangle} - y + y^{\langle -1\rangle})\beta}{(l_0 + y^{\langle -1\rangle})m} - \frac{T^2(gm - CS) + y(3m - 2\mu T)}{m},\\
Z_4^{\langle 1\rangle} &= \frac{ST^2(Tu^{\langle -1\rangle} - y + y^{\langle -1\rangle})\beta}{(l_0 + y^{\langle -1\rangle})m} - \frac{(CST + y\mu)T}{m} + T^2 g + y,\\
y &= Z_2.
\end{aligned}
$$

6. Conclusion

The paper provides constructive, necessary and sufficient conditions under which the state equations of a single-output discrete-time nonlinear system can be transformed by parametrized state transformation into the generalized observer form, whose nonlinear terms are allowed to depend on the current and past values of the inputs and output. The conditions are given in terms of certain vector fields, specified by the system output and its backward shifts. The main result can also be applied to systems with disturbances if the dynamics of the disturbance is known. The future research will be devoted to relaxing Assumptions 1 and 2 made in the current paper. These assumptions are not necessary for the solvability of the problem, as shown by Example 4 in [8].

Data availability statement

All data are available in the article.

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Diskreetse mittelineaarse juhtimissüsteemi üldistatud vaatlejakujud

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Artiklis käsitletakse ühe väljundiga diskreetse ajaga olekuvõrrandite teisendamist üldistatud vaatlejakujule. Nimetatud vaatlejakuju sisaldab lineaarset vaadeldavat alamsüsteemi ja mittelineaarset osa, mis sõltub sisenditest, väljundist ja lõplikust arvust nende varasematest teadaolevatest väärtustest. Teisenduseks kasutatakse klassikalise olekuteisenduse asemel nn parameetrilist olekuteisendust, kus parameetrite rollis on sisendite varasemad teadaolevad väärtused. Tarvilikud ja piisavad teisendatavuse tingimused on esitatud kahel nõrgal eeldusel teatud vektorvälja abil, mis on defineeritud süsteemi väljundi ja selle varasemate väärtuste kaudu. Esimene eeldus nõuab olekuvõrrandite pööratavust olekumuutuja suhtes ja teine seda, et olekud oleksid leitavad väljundfunktsioonist ja selle $n-1$ tagasinihkest. Antakse algoritm parameetrilise olekuteisenduse leidmiseks, mis viib olekuvõrrandid otsitavale kujule. Üldistatud vaatlejakuju saab ühtlasi kasutada olekute ja häiringute üheaegseks hindamiseks teatud nõrkadel eeldustel. Teooriat illustreerivad kaks näidet.