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Corresponding author:
Tanel Mullari
tanel.mullari@taltech.ee

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Classical observer form for discrete-time nonlinear system: MIMO case

Arvo Kaldmäe[a], Vadim Kaparin[a], Ülle Kotta[a], Tanel Mullari[b] and Maris Tõnso[a]

[a] Department of Software Science, Tallinn University of Technology, Akadeemia tee 21/1, 12618 Tallinn, Estonia

[b] Department of Cybernetics, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia

ABSTRACT

The paper addresses the problem of transforming multi-input multi-output discrete-time nonlinear state equations into the classical observer form using state transformation. Necessary and sufficient geometric solvability conditions are given in terms of vector fields. The results obtained generalize the previous ones in several aspects. First, the results are also applicable to non-reversible systems. Second, they hold almost everywhere, not only around the equilibrium point of the system. The generalizations are possible due to the use of different mathematical tools. The proof of sufficiency also provides a method for finding the state transformation. The results are illustrated by two examples.

1. Introduction

A nonlinear state observer with linear error dynamics can be easily constructed for nonlinear state equations whenever they are in the observer form [5,14]. Therefore, the ability to transform the equations into such a form is very important. State equations in the observer form have a linear structure up to nonlinear injection terms that depend on measurable inputs and outputs. Since the class of systems that can be transformed into the classical observer form is quite restrictive, various generalizations are proposed in the literature. Some of them suggest extending the state equations by auxiliary dynamics, so that the extended system becomes transformable into the classical observer form [13,21]; others allow the injection terms to also depend on the past or future values of inputs and outputs [6–9,11,20], or imply input dependence of the linear part [3,12]. Different approaches also apply different mathematical frameworks with their own pros and cons. The majority of the results are derived for systems with a single output. However, their extension to a multi-input multi-output (MIMO) case is not straightforward and can be a challenge [2,4,10].

Starting from the classical observer form, this paper extends the single-output results from [15,16] to the MIMO case, thereby laying a foundation for further extension to generalized MIMO observer forms, similar to those from [18,19] for single-output systems. The main contribution of this paper is to establish necessary and sufficient conditions for state equivalence to the classical observer form. The conditions are formulated in terms of certain output-related vector fields and their backward shifts. Unlike the single-output case, in the multi-output counterpart these vector fields are, in general, not uniquely defined, which allows for freedom of choice but can complicate the verification of conditions. However, a similar issue is inherent also in earlier works [2,4]. The suggested approach is compared with the earlier results, and its applicability is demonstrated by several examples.

2. Preliminaries and problem statement

In this paper, we use the generic mathematical setting from [17]. Consider the discrete-time MIMO nonlinear control system

$$x^{\langle 1\rangle}(t) = \overline{\Phi}(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

where $x^{\langle 1\rangle}(t) := x(t+1)$, $t \in \mathbb{Z}$, the state variable $x(t) \in \bar{X} \subseteq \mathbb{R}^n$, the control variable $u(t) \in U \subseteq \mathbb{R}^m$, the output variable $y \in Y \subseteq \mathbb{R}^p$, and the state transition map $\overline{\Phi} : \bar{X} \times U \longrightarrow \bar{X}$ is supposed to be analytic. Both $\bar{X}$ and U are assumed to be open sets. We assume that the map $\overline{\Phi}$ can be extended to the map $\Phi = [\overline{\Phi}^{\mathrm{T}}, \chi^{\mathrm{T}}]^{\mathrm{T}} : (\bar{X} \times U) \longrightarrow (\bar{X} \times \mathbb{R}^m)$ such that Φ has the global analytic inverse $\Phi^{-1} = [\Lambda^{\mathrm{T}}, \lambda^{\mathrm{T}}]^{\mathrm{T}} : \Phi(\bar{X} \times U) \longrightarrow (\bar{X} \times U)$. Introduce the additional variable at time instant t, $z(t) \in \mathbb{R}^m$,

$$z(t) = \chi(x(t), u(t)). \tag{2}$$

The equations (1), (2) define the inversive difference field $\mathcal{K}$ of meromorphic functions in a finite number of variables from the set $C = \{x, u^{\langle k\rangle}, z^{\langle -l\rangle}, k \geq 0, l > 0\}$. Here, $u^{\langle k\rangle}$ denotes the k-th order forward shift of u and $z^{\langle -l\rangle}$ the l-th order backward shift of z. The first order forward shift of x is defined by (1) and the first order backward shifts of x and u by

$$x^{\langle -1\rangle} = \Lambda(x, z^{\langle -1\rangle}), \quad u^{\langle -1\rangle} = \lambda(x, z^{\langle -1\rangle}). \tag{3}$$

The higher order backward shifts of x and u can be found recursively. The forward and backward shifts of functions are defined by shifting the arguments of the functions and replacing the dependent variables $x^{\langle 1\rangle}$, $x^{\langle -1\rangle}$ and $u^{\langle -1\rangle}$ from (1) and (3).

Consider the infinite set of symbols $\mathrm{d}C = \{\mathrm{d}x, \mathrm{d}u^{\langle k\rangle}, \mathrm{d}z^{\langle -l\rangle}, k \geq 0, l \geq 1\}$ and let $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}C\}$ be the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}C$, called the 1-forms

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{j=1}^{m}\sum_{k\geq 0} B_{jk}\mathrm{d}u_j^{\langle k\rangle} + \sum_{q=1}^{m}\sum_{l\geq 1} C_{ql}\mathrm{d}z_q^{\langle -l\rangle},$$

where only a finite number of coefficients differ from zero [1]. Define the space $\mathcal{E}^* = \mathrm{span}_{\mathcal{K}}\{\partial/\partial x$, $\partial/\partial u^{\langle k\rangle}$, $k \geq 0$, $\partial/\partial z^{\langle -l\rangle}$, $l \geq 1\}$, dual to $\mathcal{E}$, whose elements are the vector fields

$$\Gamma = \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m}\sum_{k\geq 0} \overline{\gamma}_{jk}\frac{\partial}{\partial u_j^{\langle k\rangle}} + \sum_{q=1}^{m}\sum_{l\geq 1} \widetilde{\gamma}_{ql}\frac{\partial}{\partial z_q^{\langle -l\rangle}}.$$

Note that in computations the 1-form is typically interpreted as the row vector of its coefficients. Similarly, the vector field is interpreted as the column vector of its coefficients. Therefore, the scalar product of the 1-form and the vector field can be understood as a product of the row and column vectors. By duality between $\mathcal{E}$ and $\mathcal{E}^*$, the scalar products of 1-forms and vector fields satisfy the relations

$$\langle \mathrm{d}x_i, \Gamma\rangle = \gamma_i, \quad \langle \mathrm{d}u_j^{\langle k\rangle}, \Gamma\rangle = \overline{\gamma}_{jk}, \quad \langle \mathrm{d}z_q^{\langle -l\rangle}, \Gamma\rangle = \widetilde{\gamma}_{ql}.$$

Definition 1. [17] *For a vector field $\Gamma \in span_{\mathcal{K}}\{\partial/\partial x\}$, its backward shift is the vector field*

$$\Gamma^{\langle -1\rangle} = \sum_{l=1}^{n} \left\langle \mathrm{d}\overline{\Phi}_l, \Gamma\right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_l} + \sum_{v=1}^{m} \langle \mathrm{d}\chi_v, \Gamma\rangle^{\langle -1\rangle} \frac{\partial}{\partial z_v^{\langle -1\rangle}}, \tag{4}$$

and the projection of $\Gamma^{\langle -1\rangle}$ is the vector field

$$\Gamma^{\langle -1\rangle\pi} = \sum_{l=1}^{n} \left\langle \mathrm{d}\overline{\Phi}_l, \Gamma\right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_l}. \tag{5}$$

Lemma 2. [15] *Let the vector fields $\Gamma_k = \sum_{i=1}^{n} \gamma_{ki}(x)\,\partial/\partial x_i$, $k = 1, \ldots, n$, be linearly independent. If the vector fields Γ_k commute, then generically one can define the state transformation $X_j = \Psi_j(x)$, $\Psi_j \in \mathcal{K}$, $j = 1, \ldots, n$, such that $\left\langle \mathrm{d}\Psi_j, \Gamma_k\right\rangle \equiv \delta_{jk}$, $j, k = 1, \ldots, n$, where δ_{jk} is the Kronecker delta.*

Throughout the paper, the following assumption is made.

Assumption 3. The system (1) is generically observable with the observability indices ρ_i, $i = 1, \ldots, p$, if $\sum_{i=1}^{p} \rho_i = n$, and

$$\operatorname{rank}_{\mathcal{K}} \left(\left(\frac{\partial y_1}{\partial x}\right)^{\mathrm{T}} \cdots \left(\frac{\partial y_1^{\langle \rho_1 - 1\rangle}}{\partial x}\right)^{\mathrm{T}} \cdots \left(\frac{\partial y_p}{\partial x}\right)^{\mathrm{T}} \cdots \left(\frac{\partial y_p^{\langle \rho_p - 1\rangle}}{\partial x}\right)^{\mathrm{T}} \right)^{\mathrm{T}} = n. \tag{6}$$

Without loss of generality, one can redefine the outputs to guarantee that $\rho_i \geq \rho_k$, if $i < k$.

Problem statement. The aim of this paper is to find, under Assumption 3, the coordinate transformation $X = \Psi(x)$, if it exists, such that in the new coordinates the system takes the observer form

$$\begin{array}{rcl} X_{i,j}^{\langle 1\rangle} & = & X_{i,j+1} + \varphi_{i,j}(y_1, \ldots, y_p, u_1, \ldots, u_m), \quad i = 1, \ldots, p, \quad j = 1, \ldots, \rho_i - 1, \\ X_{i,\rho_i}^{\langle 1\rangle} & = & \varphi_{i,\rho_i}(y_1, \ldots, y_p, u_1, \ldots, u_m), \quad y_i = X_{i,1}. \end{array} \tag{7}$$

Proposition 4. *(Necessary condition) If the state equations* (1) *are transformable by the state transformation* $X = \Psi(x)$ *into the observer form* (7) *with the observability indices* $(\rho_1, \ldots, \rho_p)$*, then*

$$\frac{\partial y_i^{\langle \rho_i\rangle}}{\partial y_k^{\langle j\rangle}} \equiv 0, \quad i = 1, \ldots, p, \quad k = 1, \ldots, i, \quad j = \rho_i, \ldots, \rho_k - 1. \tag{8}$$

Proof. Compute from (7):

$$y_i^{\langle \rho_i\rangle} = \sum_{j=1}^{\rho_i} \varphi_{i,j}(y^{\langle \rho_i - j\rangle}, u^{\langle \rho_i - j\rangle}). \tag{9}$$

Because of (9), the expression of $y_i^{\langle \rho_i\rangle}$ does not depend on shifts of y higher than $\rho_i - 1$, meaning that (8) must be satisfied. □

3. Main result

Define the vector fields $\Xi_k \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x\}$, $k = 1, \ldots, p$, in terms of which the main theorem will be presented, as the solutions of the set of equations

$$\left\langle \mathrm{d}y_i^{\langle j-1\rangle}, \Xi_k \right\rangle \equiv \delta_{ik}\delta_{j\rho_k}, \quad i, k = 1, \ldots, p, \quad j = 1, \ldots, \min(\rho_i, \rho_k). \tag{10}$$

Note that for a fixed $k > 1$, when $\rho_k < \rho_1$, the number of equations in the system (10) that defines Ξ_k is less than the number of the coefficients of Ξ_k. This means that some coefficients of Ξ_k can be chosen freely. The vector fields are uniquely defined only if all observability indices are equal. The unknown functions (coefficients) in the vector fields Ξ_k can be found from the necessary and sufficient solvability conditions of Theorem 8 below. First note that because of the condition (ii), one has to search for coefficients as functions of the argument x only. Next, the condition (i) gives a number of equations to define the unknown functions. Of course, in general, there is certain freedom to choose these unknown functions, and then one may opt for the simplest choice. This approach is described in Section 5 with examples. The equations (10) extend those for the single-output case [16]. The extension is not obvious. The remarks below discuss the problems one faces in the other routes of extension.

Remark 5. Observe that, unlike [4], in the equations (10) that define the vector fields Ξ_k, $k = 1, \ldots, p$, we require $j = 1, \ldots, min(\rho_i, \rho_k)$. If one takes, as in [4], $j = 1, \ldots, \rho_k$, then in the case where $i > k$ and $\rho_k > \rho_i$, the system of equations may become inconsistent. We will demonstrate this on a simple case of two outputs and the observability indices $(3, 2)$. In such a case there is no reason for the scalar product $\langle \mathrm{d}y_2^{\langle 2\rangle}, \Xi_1\rangle$ to be identically equal to zero. This is because $\mathrm{d}y_2^{\langle 2\rangle}$ depends, in principle, linearly on $\mathrm{d}y_1$, $\mathrm{d}y_1^{\langle 1\rangle}$, $\mathrm{d}y_1^{\langle 2\rangle}$, $\mathrm{d}y_2$, and $\mathrm{d}y_2^{\langle 1\rangle}$. Consequently, taking also into account the equations in (10), one has

$$\left\langle \mathrm{d}y_2^{\langle 2\rangle}, \Xi_1 \right\rangle = a_1 \langle \mathrm{d}y_1, \Xi_1\rangle + a_2 \left\langle \mathrm{d}y_1^{\langle 1\rangle}, \Xi_1\right\rangle + a_3 \left\langle \mathrm{d}y_1^{\langle 2\rangle}, \Xi_1\right\rangle + b_1 \langle \mathrm{d}y_2, \Xi_1\rangle + b_2 \left\langle \mathrm{d}y_2^{\langle 1\rangle}, \Xi_1\right\rangle = a_3.$$

In the case where j is given as in (10), no contradictions arise, but the vector fields Ξ_1, $\Xi_1^{\langle -1\rangle\pi}$, $\Xi_1^{\langle -2\rangle\pi}$, Ξ_2, $\Xi_2^{\langle -1\rangle\pi}$ are not necessarily independent. The problem is removable if one additionally requires that $\langle \mathrm{d}y_2^{\langle 2\rangle}, \Xi_1\rangle \equiv 0$. Essentially this means that $\mathrm{d}y_2^{\langle 2\rangle}$ is not allowed to depend on $\mathrm{d}y_1^{\langle 2\rangle}$. In the general case, if $\rho_k > \rho_i$, one has to require that the equations in (10) additionally hold for $j = \rho_i + 1, \ldots, \rho_k$, which imposes restrictions on how $\mathrm{d}y_i^{\langle j-1\rangle}$ can depend on the other output shifts.

Remark 6. If in the equations (10) that define the vector fields Ξ_k, $k = 1, \ldots, p$, one takes $j = 1, \ldots, \rho_i$, then the vector fields Ξ_k are uniquely defined. However, if all observability indices are not equal, then the coefficients of all Ξ_k, $k = 1, \ldots, p$, are not the functions of x only. Consequently, the new states as the functions of old ones will not depend only on x either.

Compute the projections of the backward shifts of Ξ_k, $k = 1, \ldots, p$, up to the order $\rho_k - 1$ according to Definition 1.

Lemma 7. *Suppose that*

$$\left[\Xi_i, \Xi_k^{\langle -l\rangle\pi}\right] \equiv 0, \quad i, k = 1, \ldots, p, \quad l = 0, \ldots, \rho_k - 1 \tag{11}$$

and the coefficients of the vector fields $\Xi_k^{\langle -l\rangle\pi}$, $k = 1, \ldots, p$, $l = 0, \ldots, \rho_k - 1$, *depend only on the variables* x. *Then the following holds:*

$$\left[\Xi_i^{\langle -j\rangle\pi}, \Xi_k^{\langle -l\rangle\pi}\right] \equiv 0, \quad i, k = 1, \ldots, p, \quad j = 0, \ldots, \rho_i - 1, \quad l = 0, \ldots, \rho_k - 1. \tag{12}$$

Proof. Due to (11), the equality (12) is valid for $j = 0$. Suppose now that (12) holds for a certain index $q \in \{0, \ldots, \rho_i - 2\}$, i.e.

$$\left[\Xi_i^{\langle -q\rangle\pi}, \Xi_k^{\langle -\bar{l}\rangle\pi}\right] \equiv 0, \quad i, k = 1, \ldots, p, \quad \bar{l} = 0, \ldots, \rho_k - 2. \tag{13}$$

Show that the validity of (12) for $j = q + 1$ follows from (13). Shift both sides of (13) backward by one step, obtaining

$$\left[\left(\Xi_i^{\langle -q\rangle\pi}\right)^{\langle -1\rangle}, \left(\Xi_k^{\langle -\bar{l}\rangle\pi}\right)^{\langle -1\rangle}\right] \equiv 0.$$

Due to Definition 1,

$$\left(\Xi_i^{\langle -q\rangle\pi}\right)^{\langle -1\rangle} = \Xi_i^{\langle -q-1\rangle\pi} + \sum_{r=1}^{m} \mu_r \frac{\partial}{\partial z_r^{\langle -1\rangle}}, \quad \left(\Xi_k^{\langle -\bar{l}\rangle\pi}\right)^{\langle -1\rangle} = \Xi_k^{\langle -\bar{l}-1\rangle\pi} + \sum_{r=1}^{m} \bar{\mu}_r \frac{\partial}{\partial z_r^{\langle -1\rangle}}, \tag{14}$$

where

$$\mu_r = \left\langle \mathrm{d}\chi_r, \Xi_i^{\langle -q\rangle\pi}\right\rangle^{\langle -1\rangle}, \quad \bar{\mu}_r = \left\langle \mathrm{d}\chi_r, \Xi_i^{\langle -\bar{l}\rangle\pi}\right\rangle^{\langle -1\rangle}.$$

Taking into account (14), one can rewrite (13) as

$$\begin{aligned}&\left[\Xi_i^{\langle -q-1\rangle\pi}, \Xi_k^{\langle -\bar{l}-1\rangle\pi}\right] + \sum_{r=1}^{m} \left\langle \mathrm{d}\bar{\mu}_r, \Xi_i^{\langle -q-1\rangle\pi}\right\rangle \frac{\partial}{\partial z_r^{\langle -1\rangle}} - \sum_{r=1}^{m} \left\langle \mathrm{d}\mu_r, \Xi_k^{\langle -\bar{l}-1\rangle\pi}\right\rangle \frac{\partial}{\partial z_r^{\langle -1\rangle}} \\ &+\sum_{r=1}^{m} \bar{\mu}_r \left[\frac{\partial}{\partial z_r^{\langle -1\rangle}}, \Xi_k^{\langle -\bar{l}-1\rangle\pi}\right] - \sum_{r=1}^{m} \mu_r \left[\frac{\partial}{\partial z_r^{\langle -1\rangle}}, \Xi_i^{\langle -q-1\rangle\pi}\right] + \sum_{r,s=1}^{m} \left[\mu_r \frac{\partial}{\partial z_r^{\langle -1\rangle}}, \bar{\mu}_s \frac{\partial}{\partial z_s^{\langle -1\rangle}}\right] \equiv 0.\end{aligned} \tag{15}$$

Because the first term on the left-hand side of (15) belongs to $\mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$, then it identically equals zero if the remaining terms either also identically equal zero, which is the case of the 4th and the 5th terms by the assumption of the lemma, or belong to $\mathrm{span}_{\mathcal{K}}\{\partial/\partial z^{\langle -1\rangle}\}$, which is the case of the 2nd, the 3rd and the 6th terms. Then

$$\left[\Xi_i^{\langle -q-1\rangle\pi}, \Xi_k^{\langle -\bar{l}-1\rangle\pi}\right] \equiv 0, \quad i, k = 1, \ldots, p, \quad \bar{l} = 0, \ldots, \rho_k - 2.$$

Denoting $l := \bar{l} + 1$, we see that (12) really holds in the case $j = q + 1$. □

In Theorem 8 below, we assume the system observability because the observability property as well as the observability indices are invariant under state transformation. Since the observer form is observable by definition, it makes no sense to address non-observable systems.

Theorem 8. *Under Assumption 3, the state equations* (1) *are transformable by a state transformation* $X = \Psi(x)$ *into the observer form* (7) *with the observability indices* $(\rho_1, \ldots, \rho_p)$ *if and only if among the solutions of* (10) *there exist the vector fields* $\Xi_1, \ldots, \Xi_p$ *that satisfy the following conditions:*

(i)
$$\left[\Xi_i, \Xi_k^{\langle -l\rangle \pi}\right] \equiv 0, \quad i, k = 1, \ldots, p, \quad l = 0, \ldots, \rho_k - 1,$$

(ii) *the coefficients of the vector fields* $\Xi_k^{\langle -l\rangle \pi}$, $k = 1, \ldots, p$, $l = 0, \ldots, \rho_k - 1$, *depend only on the variables* x,

(iii)
$$\left\langle \mathrm{d}y_i^{\langle j-1\rangle}, \Xi_k \right\rangle \equiv \delta_{ik}\delta_{j\rho_k}, \quad k = 1, \ldots, p-1, \quad i = k+1, \ldots, p, \quad j = \rho_i + 1, \ldots, \rho_k.$$

Proof. Sufficiency. The sufficiency proof consists of two parts. (a) If (iii) holds, then the vector fields $\Xi_k^{\langle -l\rangle \pi}$, $k = 1, \ldots, p$, $l = 0, \ldots, \rho_k - 1$, are linearly independent. (b) If also (i) and (ii) are true, then one can define the new coordinates as the canonical parameters of n vector fields $\Xi_k^{\langle -l\rangle \pi}$, $k = 1, \ldots, p$, $l = 0, \ldots, \rho_k - 1$, in terms of which the state equations take the observer form (7).

(a) To prove the linear independence of $\Xi_k^{\langle -l\rangle \pi}$, $k = 1, \ldots, p$, $l = 0, \ldots, \rho_k - 1$, we prove first that the following holds:

$$\left\langle \mathrm{d}y_i^{\langle j-1\rangle}, \Xi_k^{\langle -l\rangle \pi} \right\rangle \equiv \delta_{ik}\delta_{j,\rho_k - l}, \quad i, k = 1, \ldots, p, \quad l = 0, \ldots, \rho_k - 1, \quad j = 1, \ldots, \rho_k - l. \quad (16)$$

Since $\mathrm{d}y_i^{\langle j-1\rangle} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{\langle j-2\rangle}\}$, but $\Xi_k^{\langle -l\rangle \pi} \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x, \partial/\partial z^{\langle -1\rangle}, \ldots, \partial/\partial z^{\langle -l\rangle}\}$, one can rewrite (16) as $\langle \mathrm{d}y_i^{\langle j-1\rangle}, \Xi_k^{\langle -l\rangle}\rangle \equiv \delta_{ik}\delta_{j,\rho_k-l}$. Shifting the latter forward l times and taking into account that $\delta_{j,\rho_k-l} = \delta_{j+l,\rho_k}$, one gets

$$\left\langle \mathrm{d}y_i^{\langle j+l-1\rangle}, \Xi_k \right\rangle \equiv \delta_{ik}\delta_{j+l,\rho_k}, \quad l = 0, \ldots, \rho_k - 1, \quad j = 1, \ldots, \rho_k - l.$$

Denoting $q := j + l$, then $q = 1, \ldots, \rho_k$, and one finally obtains

$$\left\langle \mathrm{d}y_i^{\langle q-1\rangle}, \Xi_k \right\rangle \equiv \delta_{ik}\delta_{q\rho_k}, \quad i, k = 1, \ldots, p, \quad q = 1, \ldots, \rho_k, \quad (17)$$

the validity of which follows directly from (10), (iii) and system observability. Note that for (17) to hold, $\mathrm{d}y_i^{\langle j\rangle}$, $i = 1, \ldots, p$, $j = 0, \ldots, \rho_k - 1$ for each $k = 1, \ldots, p$, should be linearly independent. If $i > k$, then $\rho_k \le \rho_i$, but this is not a problem since the condition (iii) has to be satisfied too. Therefore, for (17) to hold, one needs $\mathrm{d}y_i^{\langle j\rangle}$, $i = 1, \ldots, p$, $j = 0, \ldots, \rho_i - 1$, to be linearly independent, which is true under the observability assumption.

Now, using (16), prove that

$$\dim_{\mathcal{K}} \operatorname{span}_{\mathcal{K}}\{\Xi_k^{\langle -l\rangle \pi}, \ k = 1, \ldots, p, \ l = 0, \ldots, \rho_k - 1\} = n. \quad (18)$$

For this purpose, we will prove that in the equality below all coefficients a_{kl} are identically equal to zero:

$$\sum_{k=1}^{p} \sum_{l=0}^{\rho_k - 1} a_{kl} \Xi_k^{\langle -l\rangle \pi} \equiv 0. \quad (19)$$

The proof is done step by step with ρ_1 steps, the step r showing that $a_{k,\rho_k - r} \equiv 0$ for the values of k for which $\rho_k \ge r$.

Step 1. Multiply both sides of (19) by $\mathrm{d}y_i$, $i = 1, \ldots, p$:

$$\sum_{k=1}^{p} \sum_{l=0}^{\rho_k - 1} a_{kl} \left\langle \mathrm{d}y_i, \Xi_k^{\langle -l\rangle \pi} \right\rangle \equiv 0.$$

Due to (16), the latter takes the form

$$\sum_{k=1}^{p} \sum_{l=0}^{\rho_k - 1} a_{kl}\delta_{ik}\delta_{1,\rho_k - l} = a_{i,\rho_i - 1} \equiv 0$$

since $\delta_{1,\rho_k-l} = \delta_{l,\rho_k-1}$.

Step r. Assume that the steps up to $r-1$ are completed and $a_{i,\rho_i-r+1} = 0$ holds. One can rewrite (19) as

$$\sum_{k=1}^{p_{r-1}} \sum_{l=0}^{\rho_k-r} a_{kl} \Xi_k^{\langle -l \rangle \pi} \equiv 0, \tag{20}$$

where p_{r-1} is the number of outputs whose observability index is greater than $r-1$. Multiply both sides of (20) by $\mathrm{d}y_i^{\langle r-1 \rangle}$, $i = 1, \dots, p_{r-1}$:

$$\sum_{k=1}^{p_{r-1}} \sum_{l=0}^{\rho_k-r} a_{kl} \left\langle \mathrm{d}y_i^{\langle r-1 \rangle}, \Xi_k^{\langle -l \rangle \pi} \right\rangle \equiv 0.$$

Due to (16), the latter takes the form

$$\sum_{k=1}^{p_{r-1}} \sum_{l=0}^{\rho_k-r} a_{kl} \delta_{ki} \delta_{r,\rho_k-l} = a_{i,\rho_i-r} \equiv 0.$$

(b) According to Lemma 7, the validity of (12) follows from (i) and (ii). Since the vector fields $\Xi_k^{\langle -l \rangle \pi}$ are linearly independent, one can apply Lemma 2 to define the new states $X_{i,j} = \Psi_{i,j}(x)$ as the canonical parameters of the vector fields in (18):

$$\left\langle \mathrm{d}\Psi_{i,j}, \Xi_k^{\langle -l \rangle \pi} \right\rangle \equiv \delta_{ik} \delta_{j,\rho_k-l}, \quad i,k = 1,\dots,p, \quad j = 1,\dots,\rho_i, \quad l = 0,\dots,\rho_k - 1. \tag{21}$$

Since the 1-forms $\mathrm{d}\Psi_{i,j}$ as the total differentials of the state transformation functions form a new basis for $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}$, it is clear that the 1-forms $\mathrm{d}\Psi_{i,j}^{\langle 1 \rangle}$ can be written as the linear combinations

$$\mathrm{d}\Psi_{i,j}^{\langle 1 \rangle} = \sum_{s=1}^{p} \sum_{q=1}^{\rho_s} \alpha_{isjq} \mathrm{d}\Psi_{s,q} + \sum_{v=1}^{m} \beta_{ijv} \mathrm{d}u_v, \quad i = 1,\dots,p, \quad j = 1,\dots,\rho_i. \tag{22}$$

Next, we will show that

$$\alpha_{isjq} = \delta_{is}\delta_{q,j+1}, \quad i,s = 1,\dots,p, \quad j = 1,\dots,\rho_i, \quad q = 2,\dots,\rho_s. \tag{23}$$

Multiply both sides of (22) by the vector field $\Xi_k^{\langle -l \rangle \pi}$, $k = 1,\dots,p$, $l = 0,\dots,\rho_k - 1$. Recall that $\Xi_k^{\langle -l \rangle \pi} \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x\}$; therefore, $\langle \mathrm{d}u_v, \Xi_k^{\langle -l \rangle \pi} \rangle \equiv 0$, $v = 1,\dots,m$, and so

$$\left\langle \mathrm{d}\Psi_{i,j}^{\langle 1 \rangle}, \Xi_k^{\langle -l \rangle \pi} \right\rangle = \sum_{s=1}^{p} \sum_{q=1}^{\rho_s} \alpha_{isjq} \left\langle \mathrm{d}\Psi_{s,q}, \Xi_k^{\langle -l \rangle \pi} \right\rangle. \tag{24}$$

Applying (21) to the scalar product on the right-hand side of (24), we get $\langle \mathrm{d}\Psi_{s,q}, \Xi_k^{\langle -l \rangle \pi} \rangle \equiv \delta_{sk}\delta_{q,\rho_k-l}$. Substituting this into (24), we get, by the definition of the Kronecker delta, $\langle \mathrm{d}\Psi_{i,j}^{\langle 1 \rangle}, \Xi_k^{\langle -l \rangle \pi} \rangle = \langle \mathrm{d}\Psi_{i,j}^{\langle 1 \rangle}, \Xi_k^{\langle -l \rangle} \rangle = \alpha_{ikj,\rho_k-l}$. The backward shift of this equality yields $\alpha_{ikj,\rho_k-l}^{\langle -1 \rangle} = \langle \mathrm{d}\Psi_{i,j}, \Xi_k^{\langle -l-1 \rangle} \rangle = \langle \mathrm{d}\Psi_{i,j}, \Xi_k^{\langle -l-1 \rangle \pi} \rangle$, $l = 0,\dots,\rho_k - 2$. Due to (21), we obtain $\alpha_{ikj,\rho_k-l}^{\langle -1 \rangle} \equiv \delta_{ik}\delta_{j,\rho_k-l-1}$. Denoting now $q := \rho_k - l$ for a fixed value of l, we get from the last equality $\alpha_{ikjq}^{\langle -1 \rangle} \equiv \delta_{ik}\delta_{j,q-1}$, $j = 1,\dots,\rho_i$ for $q = 2,\dots,\rho_k$. Next, shift the obtained result forward and take into account that 1) the Kronecker delta is shift invariant and 2) $\delta_{j,q-1} = \delta_{q,j+1}$, to get $\alpha_{ikjq} = \delta_{ik}\delta_{q,j+1}$. That is, (23) holds.

According to (23), the equality (22) takes for $i = 1,\dots,p$, $j = 1,\dots,\rho_i - 1$ the form

$$\begin{aligned} \mathrm{d}\Psi_{i,j}^{\langle 1 \rangle} &= \mathrm{d}\Psi_{i,j+1} + \sum_{s=1}^{p} \alpha_{isj1} \mathrm{d}\Psi_{s,1} + \sum_{v=1}^{m} \beta_{ijv} \mathrm{d}u_v, \\ \mathrm{d}\Psi_{i,\rho_i}^{\langle 1 \rangle} &= \sum_{s=1}^{p} \alpha_{is\rho_i 1} \mathrm{d}\Psi_{s,1} + \sum_{v=1}^{m} \beta_{i\rho_i 1} \mathrm{d}u_v. \end{aligned} \tag{25}$$

We prove next that in the new coordinates

$$y_k = X_{k,1}, \quad k = 1,\dots,p. \tag{26}$$

Taking $j = 1$ in (21), we see that for the validity of (26) the outputs must satisfy the conditions

$$\left\langle \mathrm{d}y_i, \Xi_k^{\langle -\rho_k+l\rangle\pi} \right\rangle \equiv \delta_{ik}\delta_{l1}, \quad i,k = 1,\dots,p, \quad l = 1,\dots,\rho_k. \tag{27}$$

We will prove that (27) is equivalent to (17). Since $\mathrm{d}y_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}$, and by (4), $\Xi_k^{\langle -\rho_k+l\rangle} \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x,\ \partial/\partial z^{\langle -1\rangle},\dots,\partial/\partial z^{\langle -\rho_k+l\rangle}\}$, in (27) the operator π may be omitted: $\langle \mathrm{d}y_i, \Xi_k^{\langle -\rho_k+l\rangle}\rangle \equiv \delta_{ik}\delta_{l1}$. Shifting both sides of the obtained equality $\rho_k - l$ times forward, while l is fixed, we get the equivalent equality $\langle \mathrm{d}y_i^{\langle \rho_k - l\rangle}, \Xi_k\rangle \equiv \delta_{ik}\delta_{l1}$. Denote $q := \rho_k - l + 1$, then $l = \rho_k - q + 1$. Because $l = 1,\dots,\rho_k$, then also $q = 1,\dots,\rho_k$, and the last equality takes the form

$$\left\langle \mathrm{d}y_i^{\langle q-1\rangle}, \Xi_k \right\rangle \equiv \delta_{ik}\delta_{\rho_k-q+1,1}, \quad i,k = 1,\dots,p, \quad q = 1,\dots,\rho_k.$$

Since $\delta_{\rho_k-q+1,1} \equiv \delta_{q\rho_k}$, we get (17). Consequently, (27) is valid and therefore (26) holds.

Because of (26), the equations (25) yield, after integration, the state equations in the form (7).

Necessity. Show first that the vector fields Ξ_k, defined by (10), are in the new coordinates the partial derivative operators

$$\Xi_k = \frac{\partial}{\partial X_{k,\rho_k}}, \quad k = 1,\dots,p. \tag{28}$$

To prove the validity of (28), one has to show that

$$\left\langle \mathrm{d}X_{i,j}, \Xi_k \right\rangle \equiv \delta_{ik}\delta_{j\rho_k}, \quad i,k = 1,\dots,p, \quad j = 1,\dots,\rho_i. \tag{29}$$

Express from (7) the new coordinates in terms of inputs, outputs and their forward shifts:

$$\begin{aligned} X_{i,1} &= y_i, \\ X_{i,j} &= y_i^{\langle j-1\rangle} - \varphi_{i,1}(y^{\langle j-2\rangle}, u^{\langle j-2\rangle}) - \dots - \varphi_{i,j-1}(y,u), \quad i = 1,\dots,p, \quad j = 2,\dots,\rho_i. \end{aligned} \tag{30}$$

Recall that by definition $\Xi_k \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial X\}$ and compute for $i = 1,\dots,p$, $j = 2,\dots,\rho_i$,

$$\begin{aligned} \left\langle \mathrm{d}X_{i,1}, \Xi_k \right\rangle &= \left\langle \mathrm{d}y_i, \Xi_k \right\rangle, \\ \left\langle \mathrm{d}X_{i,j}, \Xi_k \right\rangle &= \left\langle \mathrm{d}y_i^{\langle j-1\rangle}, \Xi_k \right\rangle - \sum_{q=1}^{p} \frac{\partial \varphi_{i,1}(y^{\langle j-2\rangle}, u^{\langle j-2\rangle})}{\partial y_q^{\langle j-2\rangle}} \left\langle \mathrm{d}y_q^{\langle j-2\rangle}, \Xi_k \right\rangle - \dots \\ &\quad - \sum_{q=1}^{p} \frac{\partial \varphi_{i,j-1}(y,u)}{\partial y_q} \left\langle \mathrm{d}y_q, \Xi_k \right\rangle. \end{aligned} \tag{31}$$

Using the definition of the vector fields Ξ_k in (10), the validity of (29) follows from (31). Therefore, also (28) holds.

Next, prove that

$$\Xi_k^{\langle -l\rangle\pi} = \frac{\partial}{\partial X_{k,\rho_k-l}}, \quad k = 1,\dots,p, \quad l = 1,\dots,\rho_k - 1. \tag{32}$$

Compute first $\Xi_k^{\langle -1\rangle\pi}$, $k = 1,\dots,p$, using (5) and (29):

$$\Xi_k^{\langle -1\rangle\pi} = \sum_{i=1}^{p}\sum_{j=1}^{\rho_i} \left\langle \mathrm{d}X_{i,j}^{\langle 1\rangle}, \frac{\partial}{\partial X_{k,\rho_k}} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial X_{i,j}}.$$

Computing the total differentials of both sides of (7), one can easily see that $\langle \mathrm{d}X_{i,j}^{\langle 1\rangle}, \partial/\partial X_{k,\rho_k}\rangle \equiv \delta_{ik}\delta_{j+1,\rho_k}$. Since the Kronecker delta is shift invariant, we get (32) for $l = 1$. Suppose now that (32) holds for $r \in \{1,\dots,\rho_k - 2\}$ and prove that it holds for $r + 1$. Due to (5) and (32), one has

$$\Xi_k^{\langle -r-1\rangle\pi} = \sum_{i=1}^{p}\sum_{j=1}^{\rho_i} \left\langle \mathrm{d}X_{i,j}^{\langle 1\rangle}, \frac{\partial}{\partial X_{k,\rho_k-r}} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial X_{i,j}}.$$

As above, according to (7), we get $\langle \mathrm{d}X_{i,j}^{\langle 1\rangle}, \partial/\partial X_{k,\rho_k-r}\rangle \equiv \delta_{ik}\delta_{j+1,\rho_k+r}$ that will yield (32).

Since in the coordinates X the vector fields $\Xi_k^{\langle -l\rangle\pi}$, $k = 1,\dots,p$, $l = 0,\dots,\rho_k - 1$, are linearly independent partial derivative operators, the conditions (i) and (ii) hold for them.

Finally, show that also the condition (iii) is valid for (7). Note that if $i > k$, but $\rho_i = \rho_k$, then (iii) holds by (10). Next, consider the case $\rho_i < \rho_k$. Taking $j = \rho_i$, one gets from (30) $X_{i,\rho_i} = y_i^{\langle \rho_i - 1\rangle} - \varphi_{i,1}(y^{\langle \rho_i-2\rangle}, u^{\langle \rho_i - 2\rangle}) - \cdots - \varphi_{i,\rho_i-1}(y,u)$. Shift the latter forward, substitute $X_{i,\rho_i}^{\langle 1\rangle}$ from (7) and shift the result again s steps forward, where $s = 0,\dots,\rho_k - \rho_i - 1$:

$$y_i^{\langle \rho_i + s\rangle} = \sum_{r=1}^{\rho_i} \varphi_{i,r}(y^{\langle \rho_i - r + s\rangle}, u^{\langle \rho_i - r + s\rangle}).$$

Denoting $j := \rho_i + s + 1$ yields

$$y_i^{\langle j-1\rangle} = \sum_{r=1}^{\rho_i} \varphi_{i,r}(y^{\langle j-r-1\rangle}, u^{\langle j-r-1\rangle}), \quad j = \rho_i + 1,\dots,\rho_k.$$

Recall that $\Xi_k \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial X\}$ and compute

$$\left\langle \mathrm{d}y_i^{\langle j-1\rangle}, \Xi_k \right\rangle = \sum_{r=1}^{\rho_i}\sum_{q=1}^{p} \frac{\partial \varphi_{i,r}(y^{\langle j-r-1\rangle}, u^{\langle j-r-1\rangle})}{\partial y_q^{\langle j-r-1\rangle}} \left\langle \mathrm{d}y_q^{\langle j-r-1\rangle}, \Xi_k \right\rangle,$$
$$k = 1,\dots,p-1, \quad i = k+1,\dots,p, \quad j = \rho_i + 1,\dots,\rho_k.$$

Due to (10), all scalar products on the right-hand side identically equal zero since for all possible j and r values $j - r < \rho_k$. Consequently, (iii) holds. □

Remark 9. Note that the earlier result for the single-input single-output (SISO) case follows from Theorem 8 (see [16]).

4. Comparison with earlier result

The problem of transforming the state equations (1) into the form (7) by a state transformation has been studied before for MIMO discrete-time systems in [4]. First, compare the assumptions made in this paper with those from [4]. Note that in [4], the problem statement was slightly different, allowing the coefficients of $X_{i,j+1}$ in (7) to also depend on the input variable u. However, the paper [4] additionally gave a solution to the problem statement given in this paper as a special case.

Working point. The results of the paper [4] are valid locally around the equilibrium point $(x_0, u_0) = (0,0)$. In this paper, however, a generic solution is given, meaning that the results are valid in an open and dense subset of the set $\bar{X} \times U$. Thus, the results of this paper are applicable to a larger domain than the solution in [4].

Assumptions. Both papers consider discrete-time control systems of the form (1) with $\overline{\Phi}$ analytic. However, the paper [4] assumes that $\overline{\Phi}$ is reversible in the neighbourhood of the point $(x_0, u_0) = (0,0)$, while in this paper a less restrictive assumption is made, and a more general class of systems is studied. In this paper, it is assumed that the equations (1) are observable, whereas in [4], observability was given as part of the necessary and sufficient solvability conditions. The last difference is, of course, irrelevant.

Thus, in order to compare the results of this paper with those in [4], we make the following assumption.

Assumption 10. Assume that

- the equilibrium point $(x_0, u_0) = (0,0)$ is the regular point of the observability matrix, meaning that the dimension of the observability space does not drop at this point (is not less than n);
- the state transition map $\overline{\Phi}$ is reversible in the neighbourhood of the point $(x_0, u_0) = (0,0)$.

Solvability conditions. The solvability conditions in [4] are given in terms of two sets of vector fields: $r_{k,l}$, $k = 1,\dots,p$, $l = 1,\dots,\rho_k$, and $r_{k,l}(u)$, $k = 1,\dots,p$, $l = 2,\dots,\rho_k$. The vector fields

$r_{k,1}$ are defined from the set of equations below in the neighbourhood of the point $(x_0, u_0) = (0, 0)$ under the assumption that the state transition map $\overline{\Phi}_0$ is invertible in this neighbourhood:

$$\left\langle \mathrm{d}(h_i \circ \overline{\Phi}_0^{j-1}(x)), r_{k,1} \right\rangle = \delta_{i,k}\delta_{j,\rho_k},$$

where $\overline{\Phi}_0^0 = \mathrm{Id}$, $\overline{\Phi}_0^1(x) = \overline{\Phi}_0 = \overline{\Phi}(x, 0)$ and $\overline{\Phi}_0^{r+1}(x) = \overline{\Phi}_0(\overline{\Phi}_0^r(x))$ for $r > 1$. Then the vector fields $r_{k,l}$, $l = 2, \ldots, \rho_k$, are computed by transporting $r_{k,1}$ along $\overline{\Phi}_0(x)$ iteratively:

$$r_{k,l} = \mathrm{Ad}_{\overline{\Phi}_0} r_{k,l-1} := \left(\frac{\partial \overline{\Phi}_0}{\partial x} r_{k,l-1} \right)_{\overline{\Phi}_0^{-1}(x)}, \quad l > 1.$$

The vector fields $r_{k,l}(u)$, however, are found by transporting $r_{k,l-1}$ along $\overline{\Phi}(x, u)$:

$$r_{k,l}(u) = \mathrm{Ad}_{\overline{\Phi}} r_{k,l-1} := \left(\frac{\partial \overline{\Phi}}{\partial x} r_{k,l-1} \right)_{\overline{\Phi}^{-1}(x,u)}, \quad l > 1,$$

where $\overline{\Phi}^{-1}(x, u)$ is the inverse of $\overline{\Phi}(x, u)$ under the constant u, computed at the point (x, u).

Now, the solution in [4] is stated as follows.

Theorem 11. *Under Assumption 10, the state equations* (1) *are transformable by a state transformation $X = \Psi(x)$ into the observer form* (7) *in the neighbourhood of the point* $(x_0, u_0) = (0, 0)$ *with the observability indices* $(\rho_1, \ldots, \rho_p)$ *if and only if*

*A*1 $[r_{i,1}, r_{k,l}] = 0$ *for* $i, k = 1, \ldots, p$ *and* $l = 1, \ldots, \rho_k$*;*

*A*2 $r_{k,l}(u) = r_{k,l}$ *for* $k = 1, \ldots, p$ *and* $l = 2, \ldots, \rho_k$*;*

*A*3 $\operatorname{span} O_i = \operatorname{span}\{O_i \cap O\}$, *where* $O = \{\mathrm{d}y_j, \ldots, \mathrm{d}y_j^{\langle \rho_j - 1 \rangle}; j = 1, \ldots, p\}$ *and* $O_i = \{\mathrm{d}y_j, \ldots, \mathrm{d}y_j^{\langle \rho_i - 1 \rangle}; j = 1, \ldots, p\} - \{\mathrm{d}y_i^{\langle \rho_i - 1 \rangle}\}$.

Note that the definition of O_i in [4] was incorrect, yielding that $A3$ is always satisfied, and the correct definition was given in the paper [2] by the same authors.

Recall that under Assumption 10, if z is chosen as $z = u$ and the conditions of Theorem 8 are satisfied, then under the reversibility assumption, one has that $r_{k,l} = \Xi_k^{\langle -l+1 \rangle}$, and so the condition $A1$ of Theorem 11 is satisfied. A detailed proof of the equality $r_{k,l} = \Xi_k^{\langle -l+1 \rangle}$ of the vector fields is given in [15] for the SISO case. The proof for the MIMO case is similar. Now, because of the equality $r_{k,l} = \Xi_k^{\langle -l+1 \rangle}$ and the fact that the $\mathrm{Ad}_{\overline{\Phi}}$ operator corresponds to the backward shift of a vector field, one has $r_{k,l}(u) = \mathrm{Ad}_{\overline{\Phi}} r_{k,l-1} = \mathrm{Ad}_{\overline{\Phi}} \Xi_k^{\langle -l+2 \rangle} = \Xi_k^{\langle -l+1 \rangle} = r_{k,l}$. Clearly, the condition $A2$ of Theorem 11 is satisfied. It remains to show that the condition (iii) of Theorem 8 yields that $A3$ is satisfied. First, note that the condition (iii) of Theorem 8 is equivalent to the condition (8). If the latter is satisfied, then $\mathrm{d}y_j^{\langle l \rangle} \in \operatorname{span} O$ for all $j = 1, \ldots, p$ and $l \geq 0$. Thus, $O_i \subseteq O$ and the condition $A3$ is satisfied.

5. Examples

Example 1. Consider the state equations

$$\begin{aligned} &x_1^{\langle 1 \rangle} = x_3, \quad x_2^{\langle 1 \rangle} = x_1(x_4 + 1), \quad x_3^{\langle 1 \rangle} = u + x_2 + x_5, \quad x_4^{\langle 1 \rangle} = \frac{u}{x_3}, \\ &x_5^{\langle 1 \rangle} = ux_2 - x_1x_4 - x_1, \quad y_1 = x_1, \quad y_2 = x_2. \end{aligned} \tag{33}$$

The system (33) is not reversible but submersive and taking $z = x_1$ allows one to find the backward shifts

$$\begin{aligned} &x_1^{\langle -1 \rangle} = z^{\langle -1 \rangle}, \quad x_2^{\langle -1 \rangle} = \frac{x_2 + x_5}{x_1x_4}, \quad x_3^{\langle -1 \rangle} = x_1, \\ &x_4^{\langle -1 \rangle} = \frac{x_2}{z^{\langle -1 \rangle}} - 1, \quad x_5^{\langle -1 \rangle} = x_3 - x_1x_4 - \frac{x_2 + x_5}{x_1x_4}, \quad u^{\langle -1 \rangle} = x_1x_4. \end{aligned}$$

Computing $\mathrm{d}y_1 = \mathrm{d}x_1, \mathrm{d}y_1^{\langle 1\rangle} = \mathrm{d}x_3, \mathrm{d}y_1^{\langle 2\rangle} = \mathrm{d}x_2 + \mathrm{d}x_5 + \mathrm{d}u, \mathrm{d}y_2 = \mathrm{d}x_2, \mathrm{d}y_2^{\langle 1\rangle} = (x_4+1)\,\mathrm{d}x_1 + x_1\mathrm{d}x_4$ reveals that the rank of the observability matrix (6) is generically equal to $n = 5$; thus, the system is generically observable with the observability indices $\rho_1 = 3, \rho_2 = 2$.

Next, find the vector fields $\Xi_1 = \sum_{q=1}^{5} \xi_{1q}\partial/\partial x_q$ and $\Xi_2 = \sum_{q=1}^{5} \xi_{2q}\partial/\partial x_q$, where the functions ξ_{1q}, ξ_{2q} should be determined from (10). For $k = 1$, the set of equations (10) results in

$$\langle \mathrm{d}y_1, \Xi_1\rangle = 0, \quad \left\langle \mathrm{d}y_1^{\langle 1\rangle}, \Xi_1\right\rangle = 0, \quad \left\langle \mathrm{d}y_1^{\langle 2\rangle}, \Xi_1\right\rangle = 1, \quad \langle \mathrm{d}y_2, \Xi_1\rangle = 0, \quad \left\langle \mathrm{d}y_2^{\langle 1\rangle}, \Xi_1\right\rangle = 0,$$

yielding the unique solution

$$\Xi_1 = \frac{\partial}{\partial x_5}.$$

However, for $k = 2$, the set of equations (10) results in four equations

$$\langle \mathrm{d}y_1, \Xi_2\rangle = 0, \quad \left\langle \mathrm{d}y_1^{\langle 1\rangle}, \Xi_2\right\rangle = 0, \quad \langle \mathrm{d}y_2, \Xi_2\rangle = 0, \quad \left\langle \mathrm{d}y_2^{\langle 1\rangle}, \Xi_2\right\rangle = 1,$$

while involving five unknown functions. Therefore, its solution Ξ_2 is not unique and depends on the unknown function ξ_{25}:

$$\Xi_2 = \frac{1}{x_1}\frac{\partial}{\partial x_4} + \xi_{25}\frac{\partial}{\partial x_5}.$$

The next task is to find $\xi_{25}(x)$ from the conditions (i)–(iii) of Theorem 8 that have to be satisfied. Consider, for instance,

$$[\Xi_1, \Xi_2] = \left(0,\ 0,\ 0,\ 0,\ \frac{\partial \xi_{25}}{\partial x_5}\right)^{\mathrm{T}}.$$

Equalizing the last coordinate of the above Lie bracket to zero gives a partial differential equation, whose solution is $\xi_{25} = C(x_1, x_2, x_3, x_4)$, with C being an arbitrary function of its arguments. For the sake of simplicity, we take $C \equiv 0$; thus, $\xi_{25} = \xi_{25}^{\langle -1\rangle} = 0$, leading to the updated vector field

$$\Xi_2 = \frac{1}{x_1}\frac{\partial}{\partial x_4}.$$

In order to check the condition (i), the following projections of the backward shifts of Ξ_1 and Ξ_2 are required:

$$\Xi_1^{\langle -1\rangle\pi} = \frac{\partial}{\partial x_3}, \quad \Xi_1^{\langle -2\rangle\pi} = \frac{\partial}{\partial x_1} - \frac{x_4}{x_1}\frac{\partial}{\partial x_4}, \quad \Xi_2^{\langle -1\rangle\pi} = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5}.$$

All Lie brackets in (i) equal zero; thus, the condition (i) is satisfied. The condition (ii) is also fulfilled since the coefficients of $\Xi_1, \Xi_1^{\langle -1\rangle\pi}, \Xi_1^{\langle -2\rangle\pi}, \Xi_2, \Xi_2^{\langle -1\rangle\pi}$ depend only on the variable x. The condition (iii) reduces to the requirement that $\left\langle \mathrm{d}y_2^{\langle 2\rangle}, \Xi_1\right\rangle = 0$, which is valid since $\mathrm{d}y_2^{\langle 2\rangle} = \mathrm{d}x_3$.

Next, construct the state coordinates as the canonical parameters of the vector fields $\Xi_1, \Xi_1^{\langle -1\rangle\pi}, \Xi_1^{\langle -2\rangle\pi}, \Xi_2$ and $\Xi_2^{\langle -1\rangle\pi}$ by (21). Defining matrices

$$P := \begin{pmatrix} \mathrm{d}\Psi_{1,1} \\ \mathrm{d}\Psi_{1,2} \\ \mathrm{d}\Psi_{1,3} \\ \mathrm{d}\Psi_{2,1} \\ \mathrm{d}\Psi_{2,2} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \Xi_1^{\langle -2\rangle\pi} & \Xi_1^{\langle -1\rangle\pi} & \Xi_1 & \Xi_2^{\langle -1\rangle\pi} & \Xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{x_4}{x_1} & 0 & 0 & 0 & \frac{1}{x_1} \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

allows one to write (21) as a matrix equation $PM = I_5$, with I_5 being the identity matrix. The matrix P can be found as the inverse of M,

$$P = M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ x_4 & 0 & 0 & x_1 & 0 \end{pmatrix},$$

the rows of which define the exact 1-forms (since the vector fields in M commute), the integration of which yields the new state coordinates $X_{1,1} = x_1$, $X_{1,2} = x_3$, $X_{1,3} = x_2 + x_5$, $X_{2,1} = x_2$, $X_{2,2} = x_1x_4$. In these coordinates, the state equations (33) take the observer form (7):

$$X_{1,1}^{\langle 1\rangle} = X_{1,2}, \quad X_{1,2}^{\langle 1\rangle} = u + X_{1,3}, \quad X_{1,3}^{\langle 1\rangle} = uy_2, \quad X_{2,1}^{\langle 1\rangle} = X_{2,2} + y_1, \quad X_{2,2}^{\langle 1\rangle} = u,$$
$$y_1 = X_{1,1}, \quad y_2 = X_{2,1}.$$

The purpose of the example below is to demonstrate that in the MIMO case, unlike the SISO case, the conditions (i) and (ii) of Theorem 8 are not sufficient to transform the state equations (1) into the observer form (7). The reason is that in the case where $\rho_k > \rho_i$ in (10), it is not enough to take $j = 1, \ldots, \min(\rho_i, \rho_k)$; (10) must also hold for $j = \rho_i + 1, \ldots, \rho_k$ to guarantee $y_i = X_{i,1}$, $i = 1, \ldots, p$.

Example 2. Consider the state equations

$$x_1^{\langle 1\rangle} = x_1 + x_2, \quad x_2^{\langle 1\rangle} = x_1u + x_2, \quad x_3^{\langle 1\rangle} = x_3u^2 + x_2, \quad y_1 = x_1, \qquad y_2 = x_3 \tag{34}$$

that define the backward shifts

$$x_1^{\langle -1\rangle} = \frac{x_2 - x_1}{u^{\langle -1\rangle} - 1}, \quad x_2^{\langle -1\rangle} = \frac{x_1u^{\langle -1\rangle} - x_2}{u^{\langle -1\rangle} - 1}, \quad x_3^{\langle -1\rangle} = \frac{(x_3 - x_1)u^{\langle -1\rangle} - x_3 + x_2}{(u^{\langle -1\rangle} - 1)(u^{\langle -1\rangle})^2}.$$

Compute the total differentials $\mathrm{d}y_i^{\langle j-1\rangle}$, $i, j = 1, 2$, obtaining $\mathrm{d}y_1 = \mathrm{d}x_1$, $\mathrm{d}y_1^{\langle 1\rangle} = \mathrm{d}x_1 + \mathrm{d}x_2$, $\mathrm{d}y_2 = \mathrm{d}x_3$, $\mathrm{d}y_2^{\langle 1\rangle} = \mathrm{d}x_2 + u^2\mathrm{d}x_3 + 2x_3u\mathrm{d}u$. So, the system (34) is observable with the observability indices $\rho_1 = 2$, $\rho_2 = 1$.

Compute the vector fields Ξ_1 and Ξ_2 according to (10). First, find Ξ_1 from $\langle \mathrm{d}y_1, \Xi_1\rangle \equiv 0$, $\langle \mathrm{d}y_1^{\langle 1\rangle}, \Xi_1\rangle \equiv 1$, $\langle \mathrm{d}y_2, \Xi_1\rangle \equiv 0$, which yields

$$\Xi_1 = \frac{\partial}{\partial x_2}.$$

The equations to compute $\Xi_2 = \sum_{q=1}^{3} \xi_q \partial/\partial x_q$ now consist of two equations $\langle \mathrm{d}y_1, \Xi_2\rangle = \xi_1 = 0$, $\langle \mathrm{d}y_2, \Xi_2\rangle = \xi_3 = 1$, whereas the number of unknown functions is three, resulting in

$$\Xi_2 = \xi_2(x)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3},$$

where the unknown function ξ_2 is a function of the argument x only by the condition (ii) of Theorem 8. Compute also

$$\Xi_1^{\langle -1\rangle\pi} = \sum_{q=1}^{3} \left\langle \mathrm{d}x_q^{\langle -1\rangle}, \Xi_1\right\rangle \frac{\partial}{\partial x_q} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Define ξ_2 so that the condition (i) is satisfied, meaning that all three vector fields commute:

$$[\Xi_1, \Xi_2] = \frac{\partial \xi_2}{\partial x_2}\frac{\partial}{\partial x_2} \equiv 0, \left[\Xi_1, \Xi_1^{\langle -1\rangle\pi}\right] \equiv 0, \left[\Xi_2, \Xi_1^{\langle -1\rangle\pi}\right] = -\left(\frac{\partial \xi_2}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_2}{\partial x_3}\right)\frac{\partial}{\partial x_2} \equiv 0. \tag{35}$$

From (35), it immediately follows that all Lie brackets identically equal zero only if ξ_2 is an arbitrary function of the argument $x_1 - x_3$. For simplicity, take $\xi_2 = 0$, which results in $\Xi_2 = \partial/\partial x_3$.

Define the new coordinates $X = \Psi(x)$ by (21) as the canonical parameters of $\Xi_1^{\langle -1\rangle\pi}$, Ξ_1 and Ξ_2: $X_{1,1} = x_1$, $X_{1,2} = x_2 - x_1$, $X_{2,1} = x_3 - x_1$. The inverse transformation is $x_1 = X_{1,1}$, $x_2 = X_{1,1} + X_{1,2}$, $x_3 = X_{1,1} + X_{2,1}$. In the new coordinates, the state equations read

$$X_{1,1}^{\langle 1\rangle} = X_{1,2} + 2y_1, \quad X_{1,2}^{\langle 1\rangle} = y_1(u - 1), \quad X_{2,1}^{\langle 1\rangle} = y_2u^2 - y_1. \tag{36}$$

The equations (36) are similar to the observer form (7), except that now $X_{2,1} \neq y_2$, but $X_{2,1} = y_2 - y_1$. The reason lies in the following. The validity of (iii) for $i = j = 2$ is not guaranteed since

$$\left\langle \mathrm{d}y_2^{\langle 1\rangle}, \Xi_1\right\rangle = \left\langle \mathrm{d}x_2 + u^2\mathrm{d}x_3 + 2x_3u\mathrm{d}u, \frac{\partial}{\partial x_2}\right\rangle = 1 \not\equiv 0.$$

6. Conclusion

The paper studied the problem of transforming, by the state transformation, the multi-input multi-output discrete-time nonlinear state equations into the classical observer form. The necessary and sufficient solvability conditions were given that generalize those from [15,16] for the single-output case. The extension was not straightforward; quite the opposite, it was a challenging task in many extension steps. First, it was not obvious how to generalize (from the single-output case) the equations that define the vector fields Ξ_k, $k = 1, \ldots, p$, in terms of which the main theorem is formulated. Second, and more important, is the fact that, in general, the vector fields Ξ_k are not uniquely defined, unlike in the single-output case. The only exception is when all the observability indices are equal. Thus, even if a particular set of the vector fields Ξ_k (fixed solution) fails to satisfy the conditions of Theorem 8, there may still exist another solution that fulfils the conditions. The method to construct the required state transformation was also given. The obtained conditions address a larger class of systems than the earlier conditions. The comparison with the earlier results was also done. An interesting research perspective is to extend the results of this paper to the case where, instead of the state transformation, a parametrized state transformation that depends on a few known past input values, called the parameters, is used.

Data availability statement

All data are available in the article.

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Mitme sisendi ja väljundiga diskreetse mittelineaarse juhtimissüsteemi klassikaline vaatlejakuju

Arvo Kaldmäe, Vadim Kaparin, Ülle Kotta, Tanel Mullari ja Maris Tõnso

Artikkel käsitleb mitme sisendi ja väljundiga mittelineaarsete diskreetaja olekuvõrrandite olekuteisendusega viimist klassikalisele vaatlejakujule. Tarvilikud ja piisavad geomeetrilised lahenduvustingimused on esitatud vektorväljade kaudu. Leitud tulemused üldistavad varasemaid mitmes aspektis: esiteks on tulemused rakendatavad ka mittepööratavatele süsteemidele; teiseks kehtivad need peaaegu kõikjal, mitte ainult süsteemi tasakaalupunkti ümbruses. Üldistused on võimalikud tänu teistsuguse matemaatilise aparatuuri kasutamisele. Piisavuse tõestus annab ühtlasi meetodi olekuteisenduse leidmiseks. Tulemusi illustreeritakse kahe näite abil.