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Green's relations for 2×2 matrices over linearly ordered abelian groups

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ABSTRACT

We consider semigroups of 2×2 matrices over linearly ordered abelian groups with respect to multiplication, which is defined similarly to tropical algebra. We study Green's relations on such semigroups. In particular, we describe the $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{H}$-classes of such semigroups and give a simple criterion for determining whether two matrices are $\mathcal{D}$-related. We prove that the $\mathcal{D}$-relation coincides with the $\mathcal{J}$-relation. We also study maximal subgroups of such semigroups. It turns out that if the abelian group is divisible, then these maximal subgroups can have two different forms.

1. Introduction

In this paper, we study the properties of multiplicative semigroups of 2×2 matrices over linearly ordered abelian groups, where the multiplication is defined using the so-called tropical operations. Throughout the text, $\mathbf{A} = (A, +, \leq)$ will denote a linearly ordered abelian group (see [2]). Then $a \vee b = \max\{a, b\}$ for every $a, b \in A$. We note that $(A, \vee, +)$ is a commutative semiring, where $\vee$ is the addition operation and $+$ plays the role of semiring multiplication. It has the multiplicative identity element 0, but it lacks the zero element (i.e. the additive neutral element). However, this semiring has local zeros (see [6]) in the sense that, for every finite subset $B \subseteq A$, there exists an element $z \in A$ such that $b \vee z = b$ for all $b \in B$. Indeed, since the order is linear, we may take z as the smallest element of B.

We will consider matrices over $\mathbf{A}$. Denoting the (i, j)th entry of a matrix $X \in M_{m,n}(A)$ by X_{ij}, the **tropical product** $X \otimes Y \in M_{m,p}(\mathbf{A})$ of $X \in M_{m,n}(A)$ and $Y \in M_{n,p}(A)$ is defined by

$$(X \otimes Y)_{ij} := (X_{i1} + Y_{1j}) \vee \ldots \vee (X_{in} + Y_{nj}).$$

We will often write just XY instead of $X \otimes Y$. The **tropical sum** $X \oplus Y$ of $X, Y \in M_{m,n}(\mathbf{A})$ is defined by $(X \oplus Y)_{ij} := X_{ij} \vee Y_{ij}$. The product of a matrix $X \in M_{m,n}(\mathbf{A})$ and a sclalar $a \in A$ is defined by $(a \cdot X)_{ij} := a + X_{ij}$. With these operations, $(M_n(\mathbf{A}), \oplus, \otimes)$ is a semiring, and $M_{m,n}(\mathbf{A})$ is a left semimodule over the semiring $\mathbf{A}$. In particular, $(M_n(\mathbf{A}), \otimes)$ is a semigroup, and our purpose in this paper is to study the properties of this semigroup for $n = 2$.

An important special case is the linearly ordered abelian group $(\mathbb{R}, +, \leq)$. Matrices over it are called **finitary tropical matrices** (cf. [10]). If a matrix is allowed to have $-\infty$ as an entry, then one speaks about **tropical matrices**, and the study of such matrices belongs to the field called tropical algebra. Investigations in this field are motivated by numerous applications.

In [9], Marianne Johnson and Mark Kambites initiated a systematic study of the multiplicative semigroup of tropical matrices of order 2. Among other things, they described Green's relations in that semigroup. These relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are fundamental tools in semigroup theory. By now, several articles have been published dealing with Green's relations for tropical matrix semigroups: [4,6,10,15].

The aim of this paper is to contribute to that study but in a more general setting. We consider the multiplicative semigroup $M_2(\mathbf{A})$ of 2×2 matrices over a linearly ordered abelian group $\mathbf{A}$. We show what the $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{H}$-classes are (Proposition 2.3, Proposition 2.6, Proposition 2.9), we describe $\mathcal{H}$-classes that contain an idempotent (Proposition 5.3), and we give a necessary and sufficient condition for two matrices to be in the same $\mathcal{D}$-class (Theorem 3.2). We prove that the relations $\mathcal{D}$ and $\mathcal{J}$ coincide in the semigroup $M_2(\mathbf{A})$ (Theorem 4.5). It turns out that the subset of balanced matrices is an ideal in the semigroup $M_2(\mathbf{A})$, which, as a semigroup, is completely simple (Theorem 3.5).

Let us briefly recall the definitions of Green's relations (see e.g. [7]). If S is a semigroup, then S^1 denotes the monoid obtained from S by adjoining an external identity 1. For every $a, b \in S$, $a\mathcal{R}b$ iff $aS^1 = bS^1$, $a\mathcal{L}b$ iff $S^1a = S^1b$ and $a\mathcal{J}b$ iff $S^1aS^1 = S^1bS^1$. In addition, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$.

2. $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{H}$-classes

First, we consider the $\mathcal{R}$-classes and $\mathcal{L}$-classes of the semigroup $M_2(\mathbf{A})$. In [9, Section 3], Johnson and Kambites showed (using methods of tropical geometry) that $\mathcal{R}$-classes of the multiplicative semigroup of 2×2 matrices over the tropical semiring $\mathbb{R} \cup \{-\infty\}$ have eight different forms. We do not consider the externally added element $-\infty$, which makes the situation somewhat simpler.

For a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{A})$, the **column space** of X is defined as

$$C(X) = \left\{ \begin{pmatrix} (\lambda + a) \vee (\mu + b) \\ (\lambda + c) \vee (\mu + d) \end{pmatrix} \mid \lambda, \mu \in A \right\} =: \operatorname{span}\{(a, c)^T, (b, d)^T\}.$$

It is the subsemimodule of $M_{2,1}(\mathbf{A})$ generated by the column vectors of X. Dually, the **row space** $R(X) = \operatorname{span}\{(a, b), (c, d)\}$ is defined. The following important result is a corollary of [6, Proposition 4.1] (an even more general version of this result appears in [13, Proposition 6.6]).

Proposition 2.1. *Let $\mathbf{A}$ be a linearly ordered abelian group and $X, Y \in M_2(\mathbf{A})$. Then $X \mathcal{R} Y$ if and only if $C(X) = C(Y)$.*

To prove the next proposition, we will use the following result.

Lemma 2.2. *Let $\mathbf{A}$ be a linearly ordered abelian group, $x, z, w \in A$ and $z \leq w$. Then*

$$(0, x)^T \in \operatorname{span}\{(0, z)^T, (0, w)^T\} \iff z \leq x \leq w.$$

Proof. ($\Longrightarrow$) If $(0, x)^T \in \operatorname{span}\{(0, z)^T, (0, w)^T\}$, then there exist $\lambda, \mu \in A$ such that

$$0 = \lambda \vee \mu \quad \text{and} \quad x = (\lambda + z) \vee (\mu + w).$$

In particular, $\lambda, \mu \leq 0$, so $x = (\lambda + z) \vee (\mu + w) \leq z \vee w = w$. If $\lambda = 0$, then $x = z \vee (\mu + w)$. Thus, $x \geq z$. If $\mu = 0$, then $x = (\lambda + z) \vee w$. Hence, $x \geq w$, and we have $x = w \geq z$. We have shown that $z \leq x \leq w$.

($\Longleftarrow$) If $z \leq x \leq w$, then

$$\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} (0 + 0) \vee (x - w + 0) \\ (0 + z) \vee (x - w + w) \end{pmatrix} \in \operatorname{span} \left\{ \begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ w \end{pmatrix} \right\}.$$

□

Proposition 2.3. *Let $\mathbf{A}$ be a linearly ordered abelian group. For every $\mathcal{R}$-class R of the semigroup $M_2(\mathbf{A})$ there exist uniquely determined $z, w \in A$ such that $z \leq w$ and*

$$R = \{X \in M_2(\mathbf{A}) \mid C(X) = \operatorname{span}\{(0, z)^T, (0, w)^T\}\} =: R_{zw}.$$

Proof. Let R be the $\mathcal{R}$-class of a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{A})$. Since

$$C(X) = \operatorname{span}\{(a, c)^T, (b, d)^T\} = \operatorname{span}\{(0, c - a)^T, (0, d - b)^T\},$$

we have either $R = R_{c-a,d-b}$ or $R = R_{d-b,c-a}$, depending on whether $c-a \leq d-b$ or $d-b \leq c-a$.

Suppose there is another pair of elements $x, y \in A$ such that $x \leq y$ and $R = R_{xy}$. Then Lemma 2.2 yields

$$z \leq x \leq y \leq w \quad \text{and } x \leq z \leq w \leq y.$$

Hence, $x = z$ and $y = w$. □

Corollary 2.4. *Let* **A** *be a linearly ordered abelian group, $z, w \in A$ and $z \leq w$. Then $R_{zw} = R^1_{zw} \cup R^2_{zw}$, where*

$$R^1_{zw} = \left\{\begin{pmatrix} a & b \\ a+z & b+w \end{pmatrix} \;\middle|\; a, b \in A\right\} \quad \textit{and} \quad R^2_{zw} = \left\{\begin{pmatrix} b & a \\ b+w & a+z \end{pmatrix} \;\middle|\; a, b \in A\right\}.$$

Proof. Suppose that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{zw}$ and denote $x := c-a$, $y := d-b$. We have two possibilities.

1) $x \leq y$. Then

$$C(X) = \operatorname{span}\{(a,c)^T, (b,d)^T\} = \operatorname{span}\{(0,x)^T, (0,y)^T\}.$$

By Proposition 2.3, $X \in R_{xy}$. Due to uniqueness, $x = z$ and $y = w$. Therefore, $c = a + z$, $d = b + w$ and $X = \begin{pmatrix} a & b \\ a+z & b+w \end{pmatrix} \in R^1_{zw}$.

2) $y \leq x$. A similar argument shows that $X = \begin{pmatrix} a & b \\ a+w & b+z \end{pmatrix} \in R^2_{zw}$.

Conversely, if $X \in R^1_{zw} \cup R^2_{zw}$, then

$$C(X) = \operatorname{span}\{(a, a+z)^T, (b, b+w)^T\} = \operatorname{span}\{(0,z)^T, (0,w)^T\}.$$

So, $X \in R_{zw}$. □

Corollary 2.5. *Let* **A** *be a linearly ordered abelian group. In the semigroup $M_2(\mathbf{A})$,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{R} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \iff \{a-c, b-d\} = \{x-z, y-w\}.$$

Dual arguments will give the following results.

Proposition 2.6. *Let* **A** *be a linearly ordered abelian group. For every $\mathcal{L}$-class L of the semigroup $M_2(\mathbf{A})$ there exist uniquely determined $u, v \in A$ such that $u \leq v$ and*

$$L = \{X \in M_2(\mathbf{A}) \mid R(X) = \operatorname{span}\{(0,u), (0,v)\}\} =: L_{uv}.$$

Corollary 2.7. *Let* **A** *be a linearly ordered abelian group, $u, v \in A$ and $u \leq v$. Then $L_{uv} = L^1_{uv} \cup L^2_{uv}$, where*

$$L^1_{uv} = \left\{\begin{pmatrix} a & a+u \\ b & b+v \end{pmatrix} \;\middle|\; a, b \in A\right\} \quad \textit{and} \quad L^2_{uv} = \left\{\begin{pmatrix} b & b+v \\ a & a+u \end{pmatrix} \;\middle|\; a, b \in A\right\}.$$

Corollary 2.8. *Let* **A** *be a linearly ordered abelian group. In the semigroup $M_2(\mathbf{A})$,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{L} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \iff \{a-b, c-d\} = \{x-y, z-w\}.$$

The $\mathcal{H}$-classes of a semigroup are precisely those intersections of $\mathcal{L}$-classes and $\mathcal{R}$-classes that are nonempty. Knowing the $\mathcal{L}$-classes and $\mathcal{R}$-classes allows us to describe the $\mathcal{H}$-classes in $M_2(\mathbf{A})$.

Proposition 2.9. *Let* **A** *be a linearly ordered abelian group. The $\mathcal{H}$-classes of the semigroup $M_2(\mathbf{A})$ have the form $L_{uv} \cap R_{zw}$, satisfying $u \leq v$, $z \leq w$ and $u + w = v + z$. These intersections can be described explicitly as*

$$L_{uv} \cap R_{zw} = (L^1_{uv} \cap R^1_{zw}) \cup (L^2_{uv} \cap R^2_{zw}), \tag{2.1}$$

$$L^1_{uv} \cap R^1_{zw} = \left\{\begin{pmatrix} a & a+u \\ a+z & a+v+z \end{pmatrix} \;\middle|\; a \in A\right\}, \tag{2.2}$$

$$L^2_{uv} \cap R^2_{zw} = \left\{\begin{pmatrix} a & a+v \\ a+w & a+u+w \end{pmatrix} \;\middle|\; a \in A\right\}. \tag{2.3}$$

Proof. Consider an $\mathcal{L}$-class L_{uv}, $u \leq v$, and an $\mathcal{R}$-class R_{zw}, $z \leq w$. First, we prove that the intersection $L_{uv} \cap R_{zw}$ is nonempty if and only if $u+w = v+z$. If $u+w = z+v$, then $u+w-z = v$ and $\begin{pmatrix} 0 & u \\ z & u+w \end{pmatrix} \in L_{uv} \cap R_{zw}$ by Corollary 2.5 and Corollary 2.8. Thus, $L_{uv} \cap R_{zw} \neq \emptyset$.

Conversely, assume that there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L_{uv} \cap R_{zw} = (L^1_{uv} \cup L^2_{uv}) \cap (R^1_{zw} \cup R^2_{zw})$. We have four possibilities.

1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^1_{uv} \cap R^1_{zw}$. Then $b-a = u$, $d-c = v$, $c-a = z$ and $d-b = w$. Hence, $u+w = d-a = z+v$.

2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^2_{uv} \cap R^2_{zw}$. Similar to case 1).

3) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^1_{uv} \cap R^2_{zw}$. Then $b-a = u$, $d-c = v$, $c-a = w$ and $d-b = z$. Now, $u+z = d-a = v+w$. Since $u \leq v$ and $z \leq w$, this implies $u = v$ and $z = w$. Hence, $u+w = v+z$.

4) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^2_{uv} \cap R^1_{zw}$. Similar to case 3).

Finally, we prove the equalities between sets. Assume that $u \leq v$, $z \leq w$ and $u+w = v+z$. Then

$$\text{either} \quad u = v, z = w \quad \text{or} \quad u < v, z < w.$$

In the first case, $L^1_{uv} = L^2_{uv} = L_{uv}$, $R^1_{zw} = R^2_{zw} = R_{zw}$, and the equality (2.1) holds.

Consider the case $u < v$ and $z < w$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^1_{uv} \cap R^2_{zw}$, then, as in 3), $c+v = d$, $a+u = b$ and $a+d = b+c$. Hence, $a+c+v = a+d = b+c = a+u+c$, which implies $v = u$, a contradiction. Thus, $L^1_{uv} \cap R^2_{zw} = \emptyset$. A similar argument shows that $L^2_{uv} \cap R^1_{zw} = \emptyset$. Hence, again, the equality (2.1) holds.

Let us prove the equality (2.2) (the proof of (2.3) is similar). The inclusion $\supseteq$ follows from the fact that $a+v+z-a-u = z+v-u = w$. For the inclusion $\subseteq$ we take a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^1_{uv} \cap R^1_{zw}$. Then $b-a = u$, $d-c = v$, $c-a = z$ and $d-b = w$. Hence, $b = a+u$, $c = a+z$, $d = b+w = (a+u)+(v+z-u) = a+v+z$ and $X = \begin{pmatrix} a & a+u \\ a+z & a+v+z \end{pmatrix}$. □

3. $\mathcal{D}$-classes

Recall that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. To every 2×2 matrix we will associate a certain element of A.

Definition 3.1. *We call the element $a+d-b-c \in A$ the **deviation** of a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{A})$ and denote it by* $\operatorname{dev}(X)$. *We say that X is **balanced*** (see [11]) *if* $\operatorname{dev}(X) = 0$, *i.e.* $a+d = b+c$.

It is easy to check that $\operatorname{dev}(X) = 0$ if and only if the column space (equivalently, the row space) of X is 1-generated. Such matrices are often referred to as matrices of rank 1.

In [6, Theorem 5.5], it is proved that two $n \times n$ tropical matrices are $\mathcal{D}$-related if and only if their row spaces (or column spaces) are isomorphic as semimodules. This result is further generalized in [14, Theorem 7.1] for matrices over exact semirings. We will show that in our setting it suffices to compare the deviations of the matrices.

Theorem 3.2. *Let* $\mathbf{A}$ *be a linearly ordered abelian group and* $X, Y \in M_2(\mathbf{A})$. *Then*

$$X \,\mathcal{D}\, Y \iff \operatorname{dev}(X) = \operatorname{dev}(Y) \ \ \text{or} \ \ \operatorname{dev}(X) = -\operatorname{dev}(Y).$$

Proof. ($\Longrightarrow$) From Corollary 2.4 we see that if $X \,\mathcal{R}\, Y$, then $\operatorname{dev}(X) = \pm \operatorname{dev}(Y)$. By Corollary 2.7, $X \,\mathcal{L}\, Y$ implies $\operatorname{dev}(X) = \pm \operatorname{dev}(Y)$. Hence, $X \,\mathcal{D}\, Y$ implies $\operatorname{dev}(X) = \pm \operatorname{dev}(Y)$.

($\Longleftarrow$) Suppose that we have $\operatorname{dev}(X) = \pm \operatorname{dev}(Y)$. Let $X \in R_{zw}$ and $Y \in L_{uv}$, where $z \leq w$ and $u \leq v$. Then

$$w - z = \operatorname{dev}\begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix} = \pm \operatorname{dev}(X) = \pm \operatorname{dev}(Y) = \operatorname{dev}\begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix} = v - u.$$

Since $w - z, v - u \geq 0$, we have $w - z = v - u$ or, equivalently, $w + u = z + v$. Now,

$$X \,\mathcal{R} \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix} \mathcal{R} \begin{pmatrix} 0 & u \\ z & z+v \end{pmatrix} \mathcal{L} \begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix} \mathcal{L}\, Y,$$

yielding $X(\mathcal{R} \circ \mathcal{L})Y$. □

Corollary 3.3. *Let* $\mathbf{A}$ *be a linearly ordered abelian group. For every* $X \in M_2(\mathbf{A})$, $X\mathcal{D}X^T$.

Recall that if $\mathbf{A}$ is an ordered abelian group, then the set $A^+ = \{a \in A \mid a \geq 0\}$ is called the **positive cone** of $\mathbf{A}$ [2]. For $a \in A$, $|a|$ will denote the **absolute value** of a, i.e. $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

Proposition 3.4. *Let* $\mathbf{A}$ *be a linearly ordered abelian group. The set of all* $\mathcal{D}$*-classes of the semigroup* $M_2(\mathbf{A})$ *is in one-to-one correspondence with the positive cone of* $\mathbf{A}$.

Proof. Consider the mappings

$$f : M_2(\mathbf{A})/\mathcal{D} \to A^+, \quad [X]_{\mathcal{D}} \mapsto |\operatorname{dev}(X)|,$$
$$g : A^+ \to M_2(\mathbf{A})/\mathcal{D}, \quad a \mapsto \left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{D}}.$$

Note that f is well defined due to Theorem 3.2. Clearly, $fg = id_{A^+}$. To prove that gf is the identity mapping, observe that, for every $X \in M_2(\mathbf{A})$,

$$(gf)\left([X]_{\mathcal{D}}\right) = g(|\operatorname{dev}(X)|) = \left[\begin{pmatrix} |\operatorname{dev}(X)| & 0 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{D}} = \left[\begin{pmatrix} \operatorname{dev}(X) & 0 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{D}} = [X]_{\mathcal{D}}.$$

□

Thus, $M_2(\mathbf{A})/\mathcal{D} = \{D_a \mid a \in A^+\}$, where

$$D_a = \{X \in M_2(\mathbf{A}) \mid |\operatorname{dev}(X)| = a\}.$$

In particular, D_0 is the set of all balanced matrices. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a balanced matrix, then $A = \begin{pmatrix} a & c \end{pmatrix}^T \otimes \begin{pmatrix} 0 & b-a \end{pmatrix}$. Conversely, if $u = \begin{pmatrix} u_1 & u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \in M_{1,2}(\mathbf{A})$, then $u^T \otimes v = \begin{pmatrix} u_1 + v_1 & u_1 + v_2 \\ u_2 + v_1 & u_2 + v_2 \end{pmatrix}$ is balanced. Hence, $D_0 = \{u^T \otimes v \mid u, v \in M_{1,2}(\mathbf{A})\}$. More generally, for any natural number $n \geq 2$, we can consider the set

$$B_n(\mathbf{A}) = \{u^T \otimes v \mid u, v \in M_{1,n}(\mathbf{A})\} \subseteq M_n(\mathbf{A}).$$

Recall that completely simple semigroups are precisely those that are isomorphic to Rees matrix semigroups over groups [7, Theorem 3.3.1].

Theorem 3.5. *If* $\mathbf{A}$ *is a linearly ordered abelian group, then* $B_n(\mathbf{A})$ *is an ideal of the semigroup* $M_n(\mathbf{A})$. *The semigroup* $B_n(\mathbf{A})$ *is completely simple.*

Proof. If $u^T \otimes v \in B_n(\mathbf{A})$ and $X \in M_n(\mathbf{A})$, then

$$(u^T \otimes v) \otimes X = u^T \otimes (v \otimes X) \in B_n(\mathbf{A}) \;\text{ and }\; X \otimes (u^T \otimes v) = (X \otimes u^T) \otimes v \in B_n(A)$$

because $v \otimes X \in M_{1,n}(\mathbf{A})$ and $X \otimes u^T \in M_{n,1}(\mathbf{A})$. Thus, $B_n(\mathbf{A})$ is an ideal.

We will prove that the semigroup $B_n(\mathbf{A})$ is isomorphic to a Rees matrix semigroup over the abelian group $(A, +)$. Consider the Rees matrix semigroup $\mathcal{M} = \mathcal{M}(\mathbf{A}, I, I, p)$, where $I = M_{1,n-1}(\mathbf{A})$, and the sandwich matrix $p : I \times I \to A$ is defined by

$$p(i, j) := 0 \vee a, \quad \text{where } i \otimes j^T = (a) \in M_1(\mathbf{A}).$$

For every $u = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \in M_{1,n}(\mathbf{A})$ we denote $i_u := \begin{pmatrix} u_2 - u_1 & \dots & u_n - u_1 \end{pmatrix} \in I$. We define a mapping $\varphi : B_n(\mathbf{A}) \to \mathcal{M}$ by

$$\varphi\left(u^T \otimes v\right) := (i_u, u_1 + v_1, i_v).$$

Take any $u, v, w, z \in M_{1,n}(\mathbf{A})$ and let $v \otimes w^T = (a) \in M_1(\mathbf{A})$. Then

$$\begin{aligned}
&\varphi\left((u^T \otimes v) \otimes (w^T \otimes z)\right) = \varphi\left(u^T \otimes (v \otimes w^T) \otimes z\right) = \varphi\left(u^T \otimes (a \cdot z)\right) \\
&= \varphi\left(u^T \otimes \begin{pmatrix} a + z_1 & \dots & a + z_n \end{pmatrix}\right) = (i_u, u_1 + a + z_1, i_z) \\
&= (i_u, u_1 + z_1 + ((v_1 + w_1) \vee (v_2 + w_2) \vee \dots \vee (v_n + w_n)), i_z) \\
&= (i_u, u_1 + v_1 + w_1 + z_1 + (0 \vee (v_2 - v_1 + w_2 - w_1) \vee \dots \vee (v_n - v_1 + w_n - w_1)), i_z) \\
&= \left(i_u, u_1 + v_1, \begin{pmatrix} v_2 - v_1 & \dots & v_n - v_1 \end{pmatrix}\right)\left(\begin{pmatrix} w_2 - w_1 & \dots & w_n - w_1 \end{pmatrix}, w_1 + z_1, i_z\right) \\
&= \varphi\left(u^T \otimes v\right) \varphi\left(w^T \otimes z\right).
\end{aligned}$$

So, φ is a semigroup homomorphism. If $\left(\begin{pmatrix} i_2 & \dots & i_n \end{pmatrix}, a, \begin{pmatrix} j_2 & \dots & j_n \end{pmatrix}\right)$ is any element in $\mathcal{M}$, then

$$\varphi\left(\begin{pmatrix} a & i_2 + a & \dots & i_n + a \end{pmatrix}^T \otimes \begin{pmatrix} 0 & j_2 & \dots & j_n \end{pmatrix}\right) = \left(\begin{pmatrix} i_2 & \dots & i_n \end{pmatrix}, a, \begin{pmatrix} j_2 & \dots & j_n \end{pmatrix}\right),$$

proving that φ is surjective.

To show that φ is injective, suppose that $\varphi\left(u^T \otimes v\right) = \varphi\left(w^T \otimes z\right)$, where $u, v, w, z \in M_{1,n}(\mathbf{A})$. Then $(-u_1) \cdot u = (-w_1) \cdot w$, $(-v_1) \cdot v = (-z_1) \cdot z$ and $u_1 + v_1 = w_1 + z_1$. Hence,

$$\begin{aligned}
u^T \otimes v &= (u_1 \cdot (-u_1) \cdot u)^T \otimes (v_1 \cdot (-v_1) \cdot v) \\
&= u_1 \cdot ((-u_1) \cdot u)^T \otimes v_1 \cdot ((-v_1) \cdot v) \\
&= (u_1 + v_1) \cdot \left(((-u_1) \cdot u)^T \otimes (-v_1) \cdot v\right) \\
&= (w_1 + z_1) \cdot \left(((-w_1) \cdot w)^T \otimes (-z_1) \cdot z\right) \\
&= (w_1 \cdot (-w_1) \cdot w)^T \otimes (z_1 \cdot (-z_1) \cdot z) = w^T \otimes z.
\end{aligned}$$

Thus, φ is an isomorphism. □

Using e.g. [12, Proposition 2], we obtain the following.

Corollary 3.6. *A linearly ordered abelian group* $\mathbf{A}$ *is Morita equivalent to semigroups* $B_n(\mathbf{A})$, $n \geq 2$.

4. On $\mathcal{J}$-relation

In this section, our aim is to prove that $\mathcal{J} = \mathcal{D}$ in the matrix semigroup $M_2(\mathbf{A})$. For the semigroup $M_2(\overline{\mathbb{R}})$ of tropical matrices this is proved in [9, Theorem 3.7]. In [10, Theorem 6.1], it is shown that $\mathcal{J} = \mathcal{D}$ in the semigroup $M_n(\mathbb{R})$ of finitary tropical matrices.

Recall that there is a natural partial order on the set $E(S)$ of all idempotents of a semigroup S: $f \leq e$ iff $ef = f = fe$. One writes $f < e$ when $f \leq e$ and $f \neq e$.

In [9, Theorem 4.1], Johnson and Kambites proved that the idempotents in the semigroup of 2×2 matrices over the tropical semiring are of exactly four types. This result was generalized in [11, Theorem 2.1] to the case of linearly ordered abelian groups.

Proposition 4.1. *Let* $\mathbf{A}$ *be a linearly ordered abelian group. The set of idempotents of the semigroup* $M_2(\mathbf{A})$ *is* $\mathcal{A} \cup \mathcal{B} \cup C$, *where*

$$\mathcal{A} = \left\{\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \middle| \, x, y \in A, x + y \leq 0\right\},$$

$$\mathcal{B} = \left\{\begin{pmatrix} 0 & x \\ y & x + y \end{pmatrix} \middle| \, x, y \in A, x + y < 0\right\},$$

$$C = \left\{\begin{pmatrix} x + y & x \\ y & 0 \end{pmatrix} \middle| \, x, y \in A, x + y < 0\right\}.$$

We say that e is an idempotent of **type** $\mathcal{A}$ (**type** $\mathcal{B}$, **type** $\mathcal{C}$) if $e \in \mathcal{A}$ (resp. $e \in \mathcal{B}$, $e \in \mathcal{C}$).

Definition 4.2 ([5, Definition 1.9])**.** *An idempotent element in a semigroup is called $\mathcal{D}$-**minimal** if it is minimal in the set of all idempotents in its $\mathcal{D}$-class. A $\mathcal{D}$-class is called **locally minimal** if it contains a $\mathcal{D}$-minimal idempotent.*

We need the following two results.

Theorem 4.3 ([5, Theorem 1.16])**.** *Every locally minimal $\mathcal{D}$-class in a semigroup is a $\mathcal{J}$-class.*

Lemma 4.4 ([11, Lemma 3.1])**.** *Let* **A** *be a linearly ordered abelian group. If $f < e$ for two idempotents $e, f \in E(M_2(\mathbf{A}))$, then $e \in \mathcal{A}$.*

Theorem 4.5. *If* **A** *is a linearly ordered abelian group, then $\mathcal{J} = \mathcal{D}$ in the semigroup $M_2(\mathbf{A})$.*

Proof. If $A = \{0\}$, then the claim is clear, so we assume that $A \neq \{0\}$. In view of Theorem 4.3, it suffices to prove that every $\mathcal{D}$-class is locally minimal. Then every $\mathcal{D}$-class is a $\mathcal{J}$-class. Hence, $\mathcal{J} = \mathcal{D}$.

Balanced matrices form one $\mathcal{D}$-class. The set of all idempotents in that $\mathcal{D}$-class is $\mathcal{B} \cup \mathcal{C} \cup \left\{\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \mid a \in A\right\}$. Due to Lemma 4.4, all idempotents in $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{D}$-minimal. Since $A \neq \{0\}$, $\mathcal{B}$ and $\mathcal{C}$ are nonempty. Thus, this $\mathcal{D}$-class is locally minimal.

Let us prove that all idempotents in a $\mathcal{D}$-class D of non-balanced matrices are $\mathcal{D}$-minimal. To this end, suppose that $f \leq e$, where $f, e \in D$. Non-balanced idempotent matrices must belong to $\mathcal{A}$, so $f, e \in \mathcal{A}$, say $f = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ and $e = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}$, where $a + b < 0$ and $c + d < 0$. Note that $\operatorname{dev}(f) = -a - b > 0$ and $\operatorname{dev}(e) = -c - d > 0$, so the equality $\operatorname{dev}(f) = -\operatorname{dev}(e)$ is not possible. It follows that $\operatorname{dev}(f) = \operatorname{dev}(e)$ and, hence, $a + b = c + d$. Now, $f = fe$ means that

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 \vee (a+d) & a \vee c \\ b \vee d & 0 \vee (b+c) \end{pmatrix}$$

and, therefore, $c \leq a$ and $d \leq b$. We conclude that $c + d \leq a + d \leq a + b = c + d$. Hence, $c + d = a + d$ and $a = c$. Also, $a + d = a + b$ and, so, $b = d$. Thus, $e = f$, as needed.

Finally, we mention that each $\mathcal{D}$-class D_a, where $a > 0$, contains at least one idempotent $\begin{pmatrix} 0 & -a \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. □

5. Maximal subgroups

Maximal subgroups of a semigroup are those $\mathcal{H}$-classes that contain idempotents. For the tropical matrix semigroups over $\mathbb{R}$ or $\mathbb{R} \cup \{-\infty\}$, maximal subgroups have been studied in several papers, e.g. [9,8,16].

The next proposition describes those $\mathcal{H}$-classes of $M_2(\mathbf{A})$ that contain idempotents.

Proposition 5.1. *Let* **A** *be a linearly ordered abelian group. An $\mathcal{H}$-class $L_{uv} \cap R_{zw}$ of the semigroup $M_2(\mathbf{A})$ contains an idempotent if and only if*

$$u = v, z = w \quad \text{or} \quad z = -v, w = -u.$$

Proof. **Necessity.** Suppose that $L_{uv} \cap R_{zw}$ contains an idempotent e. Then e must have one of the types $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ given in Proposition 4.1.

Suppose e is an idempotent of type $\mathcal{A}$, i.e. $e = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, where $a + b \leq 0$. If $e \in L^1_{uv} \cap R^1_{zw}$, then $b = z$, $-a = w$, $a = u$ and $-b = v$. It follows that $z = -v$ and $w = -u$. If $e \in L^2_{uv} \cap R^2_{zw}$, then we obtain the same equalities.

If $e = \begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix} \in \mathcal{B}$, then $x = u = v$ and $y = z = w$.

If $e = \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} \in \mathcal{C}$, then $-y = u = v$ and $-x = z = w$.

Sufficiency. We show that each $\mathcal{H}$-class $L_{uu} \cap R_{zz}$ contains an idempotent. Since the order is linear, we have two possibilities for $u+z$.

If $u+z \geq 0$, then $-u-z \leq 0$ and $e = \begin{pmatrix} -u-z & -z \\ -u & 0 \end{pmatrix} \in L_{uu} \cap R_{zz}$. It follows from Proposition 4.1 that the matrix e is idempotent.

If $u+z < 0$, then $e = \begin{pmatrix} 0 & u \\ z & u+z \end{pmatrix} \in L_{uu} \cap R_{zz}$. By Proposition 4.1, the matrix e is idempotent.

Suppose now that $z = -v$ and $w = -u$. Then the matrix $e = \begin{pmatrix} 0 & u \\ z & 0 \end{pmatrix} \in L_{uv} \cap R_{zw}$. Since $z+u = u-v \leq 0$, we see that e is an idempotent by Proposition 4.1. □

Thus, the $\mathcal{H}$-classes that contain an idempotent have two possible forms: $L_{uu} \cap R_{zz}$ and $L_{uv} \cap R_{-v,-u}$.

Due to Theorem 3.2, each idempotent E in the semigroup $M_2(\mathbf{A})$ is $\mathcal{D}$-related to the idempotent $\begin{pmatrix} 0 & -|\operatorname{dev}(E)| \\ 0 & 0 \end{pmatrix}$ of type $\mathcal{A}$. Since all maximal subgroups of $M_2(\mathbf{A})$ within the same $\mathcal{D}$-class are isomorphic to each other [7, Proposition 2.3.6], each maximal subgroup of $M_2(\mathbf{A})$ is isomorphic to an $\mathcal{H}$-class H_x of an idempotent $E_x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, where $x \in A$, $x \leq 0$.

Since $E_x \in L^1_{x0} \cap R^1_{0,-x}$, by Proposition 2.9, we know that $H_x = L_{x0} \cap R_{0,-x}$. The equalities (2.2) and (2.3) imply that

$$H_x = (L^1_{x0} \cap R^1_{0,-x}) \cup (L^2_{x0} \cap R^2_{0,-x}) = \{X_a \mid a \in A\} \cup \{Y_a \mid a \in A\},$$

where

$$X_a = \begin{pmatrix} a & a+x \\ a & a \end{pmatrix} \quad \text{and} \quad Y_a = \begin{pmatrix} a & a \\ a-x & a \end{pmatrix}.$$

Straightforward calculations show that, for all $a, b \in A$,

$$X_a X_b = X_{a+b}, \quad X_a Y_b = Y_{a+b} = Y_b X_a \quad \text{and} \quad Y_a Y_b = X_{a+b-x}. \tag{5.1}$$

Hence, H_x is an abelian group. Now, $\mathbf{A}' = \{X_a \mid a \in A\}$ is a subgroup of H_x, which is isomorphic to $\mathbf{A}$. Thus, we have the following result.

Proposition 5.2. *Let $\mathbf{A}$ be a linearly ordered abelian group. Then the maximal subgroup H_0 of $M_2(\mathbf{A})$ is isomorphic to the group $(\mathbf{A}, +)$.*

With the $\mathcal{H}$-classes of the form H_x, where $x < 0$, the situation is more complicated. The group structure of such $\mathcal{H}$-classes depends upon the properties of x and $\mathbf{A}$. In the next result, $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ is the additive group of integers modulo 2.

Proposition 5.3. *If the element $x < 0$ is divisible by 2 in a linearly ordered abelian group $\mathbf{A}$, then the $\mathcal{H}$-class H_x of the semigroup $M_2(\mathbf{A})$ is isomorphic to the group $\mathbf{A} \times \mathbb{Z}_2$.*

Proof. The subgroup $\mathbf{A}'$ of H_x has index 2. We can find an element $a \in A$ such that $2a = x$. Then the element $Y_a \in H_x \setminus \mathbf{A}'$ has order 2. Therefore, $H_x \cong \mathbf{A}' \times \mathbb{Z}_2 \cong \mathbf{A} \times \mathbb{Z}_2$. □

Corollary 5.4. *If $\mathbf{A}$ is a divisible linearly ordered abelian group, then each maximal subgroup of $M_2(\mathbf{A})$ is isomorphic to $\mathbf{A}$ or $\mathbf{A} \times \mathbb{Z}_2$.*

We will also consider the non-divisible linearly ordered abelian group $(\mathbb{Z}, +)$.

Proposition 5.5. *Up to isomorphism, the maximal subgroups of the semigroup $(M_2(\mathbb{Z}), \cdot)$ are $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}_2$.*

Proof. By Proposition 5.2, the maximal subgroup H_0 is isomorphic to $\mathbb{Z}$.

Consider now maximal subgroups H_x, where $x \in \mathbb{Z}$, $x < 0$. If x is even, then $H_x \cong \mathbb{Z} \times \mathbb{Z}_2$, by Proposition 5.3. Assume that x is odd and consider the mapping

$$\gamma : H_x \to H_x, \quad Z \mapsto Z^2.$$

Since H_x is commutative, γ is a group homomorphism. Now, $X_{2a} = X_a^2 = X_0$ if and only if $a = 0$ and $Y_{2a-x} = Y_a^2 = X_0$ cannot hold (because x is odd). We see that $\operatorname{Ker}(\gamma) = \{X_0\}$. Thus, γ is injective. It is easy to see that $\gamma(H_x) = \{X_a \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$, so $H_x \cong \mathbb{Z}$. □

6. Conclusion

In this paper, we gave descriptions of Green's relations for semigroups of 2×2 matrices over any linearly ordered abelian group with respect to tropical multiplication. We also studied maximal subgroups in such semigroups. In future investigations, it would be interesting to see if such descriptions can be extended to matrices of bigger order or to matrices over linearly ordered abelian groups with an externally added bottom element.

Data availability statement

All data are available in the article.

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Greeni seosed teist järku ruutmaatriksitele üle lineaarselt järjestatud Abeli rühmade

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Me vaatleme teist järku ruutmaatriksite poolrühmi üle lineaarselt järjestatud Abeli rühmade sellise korrutamistehte suhtes, mis on defineeritud sarnaselt troopilise algebraga. Uurime selliste poolrühmade Greeni seoseid. Muuhulgas kirjeldame ära poolrühmade $\mathcal{R}$-, $\mathcal{L}$- ja $\mathcal{H}$-klassid ning esitame lihtsa kriteeriumi, mille abil saab tuvastada, kas kaks maatriksit on $\mathcal{D}$-seoses. Tõestame, et $\mathcal{D}$-seos langeb kokku $\mathcal{J}$-seosega. Lisaks uurime selliste poolrühmade maksimaalseid alamrühmi. Tuleb välja, et kui vaadeldav Abeli rühm on jaguv, siis nendel maksimaalsetel alamrühmadel on kaks võimalikku kuju.