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Corresponding author:
Nasir Sohail
nasir.sohail@ut.ee

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Amalgamating inverse semigroups over ample semigroups

Nasir Sohail

Institute of Mathematics and Statistics, University of Tartu, 51009 Tartu, Estonia

ABSTRACT

We consider semigroup amalgams $(S; T_1, T_2)$ in which T_1 and T_2 are inverse semigroups and S is a non-inverse semigroup. They are known to be non-embeddable if T_1 and T_2 are both groups (Clifford semigroups), but S is not such. We prove that $(S; T_1, T_2)$ is non-embeddable if S is a non-inverse ample semigroup. By introducing the notion of rich ampleness, we determine some necessary and sufficient conditions for the weak embedding of $(S; T_1, T_2)$ in an inverse semigroup. In particular, $(S; T_1, T_2)$ is shown to be weakly embeddable in a group if T_1 and T_2 are groups. A rudimentary analysis of the novel classes of rich ample semigroups is also provided.

1. Motivation

The amalgamation problem of semigroups has its origins in the early work of J. M. Howie from the 1960s. The inspiration thereof came from group amalgams, which were considered earlier by O. Schreier. The topic was then extensively studied by various mathematicians during the second half of the previous century. References to this work may be found in Howie's celebrated monograph [4], of which the last chapter is also dedicated to semigroup amalgams. The main emphasis, during all these years, had been on determining the embeddability conditions for semigroup amalgams. Non-embeddable amalgams were discovered sporadically, usually as by-products. One of Howie's pioneering articles [5], however, provided an important class of non-embeddable amalgams that may essentially be viewed as groups intersecting in semigroups. Generalizing Howie's result, Rahkema and Sohail [7] came up in 2014 with two more classes of non-embeddable semigroup amalgams. The current article furthers the same line of research of investigating the (non-embeddability of) amalgams that may essentially be viewed as inverse semigroups intersecting in a non-inverse semigroup. We also consider the question of weak amalgamation for these amalgams.

The study of ample semigroups and their variants has been an active area of research for many decades, see for instance [2] and its references. As every ample semigroup S gives rise to an amalgam $(S; T_1, T_2)$, where T_1 and T_2 are inverse semigroups, it was natural for us to consider the amalgams $(S; T_1, T_2)$ such that T_1 and T_2 are inverse semigroups and S belongs to some class of ample semigroups. In fact, we introduce in this connection the notions of rich and ultra-rich ample semigroups; the intersection of the latter class with that of inverse semigroups is precisely the class of all groups.

2. Introduction and preliminaries

Given a semigroup S, an element $x \in S$ is called *invertible* if there exists a unique element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. We call S an *inverse semigroup* if every $x \in S$ is invertible. *Inverse monoids* are defined similarly. Let X be a non-empty set. Then the set $\mathcal{I}_X$ of all partial bijections of X is an inverse semigroup under the usual composition of partial maps. We call $\mathcal{I}_X$ the *symmetric inverse semigroup* over X. By the Wagner–Preston representation theorem (see for instance [4], Theorem 5.1.7), any inverse semigroup S can be embedded in the symmetric inverse semigroup $\mathcal{I}_S$. If S is a subsemigroup of an inverse semigroup T, then the inverse subsemigroup of T generated by S is called the *inverse hull* of S in T. *Homomorphisms* of inverse semigroups (monoids) are precisely the semigroup homomorphisms. We shall adopt the convention of writing the maps to the right of their arguments throughout this article. Also, we shall omit parentheses around the arguments if there is no risk of confusion. For further details about inverse semigroups and other standard definitions in semigroup theory, the reader may refer to the texts [4,6].

A semigroup S is called *right ample* if it can be embedded in an inverse semigroup T (typically, in the symmetric inverse semigroup $\mathcal{I}_X$ of a non-empty set X) such that the image of S is closed under the unary operation $s \longmapsto s^{-1}s$, where S is identified with its isomorphic copy in T, and $s^{-1} \in T$ denotes the inverse of $s \in S$. We shall call T an inverse semigroup *associated* with S. *Left ample* semigroups are defined analogously. We say that S is *ample* if it is both right and left ample. If S is a subsemigroup of an associated inverse semigroup T, then we shall say that S is (*right*, *left*) *ample in* T. Given a semigroup S, we denote by $E(S)$ the set of idempotents of S. A subsemigroup S of a semigroup T is called *full* if $E(T) \subseteq S$. Every full subsemigroup of an inverse semigroup T is ample in T. The converse is not true; for example, $\mathbb{N}$ is ample but not full in the multiplicative monoid $\mathbb{Q}$. It is possible that S is made into a left and a right ample semigroup by different associated inverse semigroups. In such a case, the problem of finding a single (associated) inverse semigroup making S into a left as well as a right ample semigroup is, in general, undecidable ([3], Theorem 3.4 and Corollary 4.3). If T_1 and T_2 are inverse semigroups admitting a homomorphism $\phi : T_1 \longrightarrow T_2$, and S is right (respectively, left) ample in T_1, then one can easily verify that $S\phi$ is right (respectively, left) ample in T_2. More information about ample semigroups may be found in [2] and the references contained therein.

A semigroup *amalgam* is a 5-tuple $\mathcal{A} \equiv (S; T_1, T_2; \phi_1, \phi_2)$ comprising pair-wise disjoint semigroups S, T_1, T_2 and monomorphisms:

$$\phi_i : S \longrightarrow T_i,\ 1 \leq i \leq 2.$$

We say that $\mathcal{A}$ is *embeddable* (or *strongly embeddable*, for emphasis) if there exists a semigroup T admitting monomorphisms $\psi_i : T_i \longrightarrow T$, $1 \leq i \leq 2$, such that

(*i*) $\phi_1\psi_1 = \phi_2\psi_2$,

(*ii*) $\forall\, t_1 \in T_1, \forall\, t_2 \in T_2, t_1\psi_1 = t_2\psi_2 \Longrightarrow \exists\, s \in S$ such that $t_1 = s\phi_1, t_2 = s\phi_2$.

If condition (*ii*) is not necessarily satisfied, then $\mathcal{A}$ is said to be *weakly embeddable*. We call $(S; T_1, T_2; \phi_1, \phi_2)$ a *special amalgam* if T_1 and T_2 are isomorphic, say, via $\nu : T_1 \longrightarrow T_2$, such that $s\phi_1\nu = s\phi_2$ for all $s \in S$. Any special amalgam is weakly embeddable, for instance in T_1. It is customary to denote a semigroup amalgam by $(S; T_1, T_2)$ if no explicit mention of ϕ_1 and ϕ_2 is needed. We shall also call $(S; T_1, T_2)$ an *amalgam over* S. Every ample semigroup S gives rise to an amalgam $(S; T_1, T_2)$ in which S is right (respectively, left) ample in the inverse semigroup T_i (respectively, T_j), where $\{i, j\} = \{1, 2\}$. We shall consider these amalgams in Theorem 3.4.

Let $T_1 * T_2$ denote the free product of semigroups T_1 and T_2. Then, by the *amalgamated coproduct* of $(S; T_1, T_2; \phi_1, \phi_2)$ we mean the quotient semigroup $(T_1 * T_2)/\theta_R$, where θ_R denotes the congruence on $T_1 * T_2$ generated by the relation

$$R = \{(s\phi_1, s\phi_2) : s \in S\}.$$

We denote $(T_1 * T_2)/\theta_R$ by $T_1 *_S T_2$. In fact, the following diagram is a pushout in the category of all semigroups, where the homomorphisms

$$\eta_i : T_i \longrightarrow T_1 *_S T_2,\ 1 \leq i \leq 2$$

send $x \in T_i$ to the congruence class $(x)_{\theta_R} \in T_1 *_S T_2$.

Theorem 2.1 ([4], Theorem 8.2.4)**.** *A semigroup amalgam* $(S; T_1, T_2)$ *is (weakly) embeddable if and only if it is (weakly) embedded in* $T_1 *_S T_2$ *via the homomorphisms* $\eta_i : T_i \longrightarrow T_1 *_S T_2$, $i \in \{1, 2\}$, *defined above.*

Proof. Follows immediately from the properties of a pushout. □

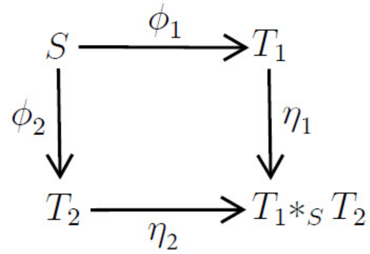


Fig. 1. Amalgamated coproduct.

A semigroup S is called an *amalgamation base* for a class (equivalently, category) C of semigroups if every amalgam $(S; T_1, T_2)$, with $T_1, T_2 \in C$, is embeddable in some $T \in C$. Given a semigroup $T_1 \in C$ containing an isomorphic copy of a semigroup S, we say that $(S; T_1)$ is an *amalgamation pair* for C if for all $T_2 \in C$ the amalgam $(S; T_1, T_2)$ is embeddable in some $T \in C$. *Weak amalgamation bases* (*pairs*) are defined similarly.

Theorem 2.2 ([4], Theorems 8.6.1 and 8.6.4)**.** *Inverse semigroups are amalgamation bases for the classes of all semigroups and inverse semigroups.*

Let S be an inverse semigroup and T be an arbitrary semigroup. Then, by the above theorem, $(S; T)$ is an amalgamation pair for the class of all semigroups. If S and T are both inverse, then $(S; T)$ is also an amalgamation pair for the class of inverse semigroups. All of the assertions made in this section about semigroups are also true for monoids.

3. Amalgamation over ample semigroups

It was shown by Howie [5] that a semigroup amalgam $(S; T_1, T_2)$ does not embed if T_1 and T_2 are both groups but S is not such. Generalizing this result, Rahkema and Sohail [7] showed that $(S; T_1, T_2)$ is non-embeddable if T_1 and T_2 are both completely regular (respectively, Clifford) semigroups, but S is not completely regular (respectively, Clifford). In this section, we shall consider the amalgams $(S; T_1, T_2)$ in which T_1 and T_2 are both inverse semigroups but S is not such – the non-embeddability of such amalgams was left as an open problem in [7]. We begin by introducing the notion of an antiamalgamation pair.

Definition 3.1. Let C be a class of semigroups. Suppose that $T_1 \in C$ contains an isomorphic copy of a semigroup S via $\phi_1 : S \longrightarrow T_1$. Then the pair $(S; T_1)$ will be called an *antiamalgamation pair* for C if for every $T_2 \in C$ and every monomorphism $\phi_2 : S \longrightarrow T_2$ the amalgam $(S; T_1, T_2; \phi_1, \phi_2)$ is non-embeddable (in any semigroup).

Recall that in every inverse semigroup, the idempotents commute (see for instance [4], Theorem 5.1.1).

Theorem 3.2. *Let T_1 be an inverse semigroup and $\phi_1 : S \longrightarrow T_1$ be a monomorphism such that $S\phi_1$ is right as well as left ample in T_1. If S is non-inverse, then $(S; T_1)$ is an antiamalgamation pair for the class of inverse semigroups.*

Proof. Let S, T_1 and ϕ_1 be as described in the statement of the theorem. Let T_2 be an inverse semigroup admitting a monomorphism $\phi_2 : S \longrightarrow T_2$. Given $s \in S$, let us denote $s\phi_1$ and $s\phi_2$ by s_1 and s_2, respectively. Identifying S with its isomorphic copies $S\phi_1$ and $S\phi_2$ and using the properties of inverses, we may calculate in $T_1 *_S T_2$:

$$\begin{aligned} ss_1^{-1}ss_2^{-1} = ss_2^{-1}, \quad ss_2^{-1}ss_1^{-1} = ss_1^{-1}, \\ s_2^{-1}ss_1^{-1}s = s_2^{-1}s, \quad s_1^{-1}ss_2^{-1}s = s_1^{-1}s. \end{aligned} \tag{1}$$

Since $S\phi_1$ is right and left ample in T_1, the identification of S with $S\phi_1$ also gives

$$ss_1^{-1},\ s_1^{-1}s \in S.$$

By the commutativity of idempotents in T_2, we may write from (1):

$$ss_1^{-1} = ss_2^{-1}, \quad s_1^{-1}s = s_2^{-1}s. \tag{2}$$

Now, using (2), we calculate in $T_1 *_S T_2$:

$$s_1^{-1} = s_1^{-1}(ss_1^{-1}) = s_1^{-1}(ss_2^{-1}) = (s_1^{-1}s)s_2^{-1} = (s_2^{-1}s)s_2^{-1} = s_2^{-1}. \tag{3}$$

Because $S\phi_1$ and $S\phi_2$ are non-inverse, there exists $s \in S$ such that $s_i^{-1} \notin T_i$, $1 \leq i \leq 2$. The amalgam $(S; T_1, T_2)$, therefore, fails to embed by (3). □

Example 3.3. Let $\mathbb{L}$ denote the lattice of ample submonoids of the symmetric inverse semigroup $\mathcal{I}_n$ over a finite chain $C_n : 1 < 2 < \cdots < n$, given in [8] (Fig. 1). The chain $\mathbb{L}_{\text{INV}} : \{\iota\} \subseteq \mathcal{OI}_n \subset \mathcal{RI}_n \subset \mathcal{I}'_n \subset \mathcal{I}_n$ constitutes the sublattice of $\mathbb{L}$ comprising the inverse submonoids of $\mathcal{I}_n$. This gives a (finite) set

$$\{(S, T) : S \in \mathbb{L} \setminus \mathbb{L}_{\text{INV}},\ T \in \mathbb{L}_{\text{INV}} \text{ with } S \subseteq T\}$$

of antiamalgamation pairs for the class of inverse semigroups.

Theorem 3.4. *Let a non-inverse semigroup S be made into a right (respectively, left) ample semigroup by an associated inverse semigroup T_1 (respectively, T_2). Then the amalgam $(S; T_1, T_2)$ is not embeddable (in any semigroup).*

Proof. Let S, T_1 and T_2 be as given in the statement of the theorem. Then, as before, the identification of S with its isomorphic copies in T_1 and T_2 gives (1). Since S is right ample in T_1 and left ample in T_2, we have $s_1^{-1}s,\ ss_2^{-1} \in S$. Subsequently, ss_1^{-1}, ss_2^{-1} commute in T_1 and $s_1^{-1}s, s_2^{-1}s$ commute in T_2. Using the argument from the proof of Theorem 3.2, we can once more deduce (2) from (1). However, (2) gives $s_1^{-1} = s_2^{-1}$, implying (as in the said proof) that $(S; T_1, T_2)$ is non-embeddable. □

4. Weak amalgamation

Given a subsemigroup S of an inverse semigroup T, we define its *dual* to be the subsemigroup

$$S' = \{s^{-1} \in T : s \in S\}.$$

Defining $\alpha : S \longrightarrow S'$ by $s \mapsto s^{-1}$, we have:

$$(xy)\alpha = (xy)^{-1} = y^{-1}x^{-1} = (y)\alpha(x)\alpha, \forall\, x, y \in S,$$

whence S and S' are *anti-isomorphic*. Clearly, if non-empty, $S \cap S'$ is an inverse subsemigroup of S and S' with $E(S) = E(S') \subseteq S \cap S'$. Also, if S is right (respectively, left) ample in T, then S' is a left (respectively, right) ample subsemigroup of T.

Lemma 4.1. *Let T_1 and T_2 be inverse semigroups containing isomorphic copies, say S_1 and S_2, of a semigroup S. Then there exists a bijection $\psi : S_1 \cup S'_1 \longrightarrow S_2 \cup S'_2$ such that for all $x \in S_1 \cup S'_1$ one has:*

$$(x^{-1})\psi = (x\psi)^{-1}.$$

Proof. Let ϕ be the isomorphism from S_1 to S_2. Then $\phi' = \alpha_1^{-1} \circ \phi \circ \alpha_2$ is an isomorphism from S'_1 to S'_2, where $\alpha_i : S_i \longrightarrow S'_i$, $i = 1, 2$, are the anti-isomorphisms defined by $s_i \mapsto s_i^{-1}$, $s_i \in S_i$. Let $x \in S_1 \cap S'_1$. Then $x^{-1} \in S_1 \cap S'_1$ and, in particular, $x, x^{-1} \in S_1$. Now, using the assumption that ϕ is an isomorphism, we have

$$\begin{aligned} x\phi &= (xx^{-1}x)\phi = (x)\phi(x^{-1})\phi(x)\phi, \\ (x^{-1})\phi &= (x^{-1}xx^{-1})\phi = (x^{-1})\phi(x)\phi(x^{-1})\phi, \end{aligned} \tag{4}$$

whenever $x, x^{-1} \in S_1$. Using, next, the uniqueness of inverses in T_2, we have $(x^{-1})\phi = ((x)\phi)^{-1}$ for all $x \in S_1 \cap S'_1$. We may, therefore, calculate:

$$x\phi = ((x^{-1})^{-1})\phi = ((x^{-1})\phi)^{-1} = x\phi',\ \forall x \in S_1 \cap S'_1.$$

This implies that ϕ and ϕ' agree on $S_1 \cap S_1'$. Consequently, the map

$$\psi = \phi \cup \phi' : S_1 \cup S_1' \longrightarrow S_2 \cup S_2'$$

is well-defined. Using a dual argument, one may also construct

$$\psi^{-1} = \phi_1^{-1} \cup (\phi_1')^{-1} : S_2 \cup S_2' \longrightarrow S_1 \cup S_1',$$

such that $\psi \circ \psi^{-1}$ and $\psi^{-1} \circ \psi$ are both identity functions. This implies that ψ is a bijection, as required.

It remains to show that $(x^{-1})\psi = (x\psi)^{-1}$. If x (and hence x^{-1}) belong to $S_1 \cap S_2$, then $(x^{-1})\psi = (x^{-1})\phi = (x\phi)^{-1} = (x\psi)^{-1}$. On the other hand, when $x \in S_1 \setminus S_1'$ (and consequently $x^{-1} \in S_1' \setminus S_1$), then

$$(x^{-1})\psi = (x^{-1})\phi' = (x^{-1})\alpha_1^{-1} \circ \phi \circ \alpha_2 = (x)\phi \circ \alpha_2 = (x\phi)^{-1} = (x\psi)^{-1}.$$

That $(x^{-1})\psi = (x\psi)^{-1}$ when $x \in S_1' \setminus S_1$ follows by symmetry. □

Proposition 4.2. *Let S be any semigroup and T_i, $1 \leq i \leq 2$, be inverse semigroups admitting monomorphisms $\phi_i : S \longrightarrow T_i$. Then the amalgam $(S, T_1, T_2; \phi_1, \phi_2)$ is weakly embeddable in an inverse semigroup if and only if $(S; V_1, V_2)$ constitutes a special amalgam, where V_i is the inverse hull of $S\phi_i$ in T_i.*

Proof. ($\Longrightarrow$) Let S, T_1, T_2, V_1, V_2 and ϕ_1, ϕ_2 be as described in the statement of the theorem. We shall denote $S\phi_i$, $1 \leq i \leq 2$, by S_i. Assume that $(S; T_1, T_2)$ is weakly embeddable in an inverse semigroup W via monomorphisms $\mu_1 : T_1 \longrightarrow W$ and $\mu_2 : T_2 \longrightarrow W$.

Observe that any element of V_1 may be written in the form $x_1x_2 \cdots x_n$, where $x_1, x_2, \ldots, x_n \in S_1 \cup S_1'$, and, for all $1 \leq i \leq n-1$, the elements x_i, x_{i+1} are not both in S_1 or $S_1' \setminus S_1$. Similarly, the elements of V_2 can be written as $y_1y_2 \cdots y_m$, where $y_1, y_2, \ldots, y_m \in S_2 \cup S_2'$, and, for all $1 \leq i \leq m-1$, the elements y_i, y_{i+1} do not both belong to S_2 or $S_2' \setminus S_2$. Also, for each $i \in \{1, 2\}$ and $x \in S_i$, we have:

$$(x^{-1})\mu_i = (x\mu_i)^{-1}, \quad \text{where } x^{-1} \in S_i'.$$

We define $\theta : V_1 \longrightarrow V_2$ by

$$(x_1x_2 \cdots x_n)\theta = (x_1x_2 \cdots x_n)\mu_1\mu_2^{-1}.$$

Then θ is clearly an isomorphism from V_1 to V_2. Moreover, for every $x\phi_1 \in S_1$, we have:

$$(x\phi_1)\theta = (x\phi_1)\mu_1\mu_2^{-1} = (x\phi_1\mu_1)\mu_2^{-1} = (x\phi_2\mu_2)\mu_2^{-1} = (x\phi_2)\mu_2\mu_2^{-1} = x\phi_2.$$

Thus, (S, V_1, V_2) is a special amalgam.

($\Longleftarrow$) Let $(S; V_1, V_2; \phi_1, \phi_2)$ be made into a special amalgam by the isomorphism $\nu : V_1 \longrightarrow V_2$. Then

$$\phi_1 \circ \nu = \phi_2. \tag{5}$$

Consider a semigroup V admitting isomorphisms $\gamma_i : V \longrightarrow V_i$, for each $1 \leq i \leq 2$, with $V \cap V_i = \emptyset$ and

$$\gamma_1 \circ \nu = \gamma_2; \tag{6}$$

that is $(V; V_1, V_2)$ is a special amalgam. Then, being an inverse semigroup amalgam, $(V; T_1, T_2; \gamma_1, \gamma_2)$ is embeddable in an inverse semigroup, say W, via monomorphisms, say $\mu_i : T_i \longrightarrow W$. This implies that

$$\gamma_1 \circ \mu_1 = \gamma_2 \circ \mu_2. \tag{7}$$

Now, using (5) and (6), we have:

$$\phi_1 \circ \gamma_1^{-1} = \phi_1 \circ (\nu \circ \gamma_2^{-1}) = (\phi_1 \circ \nu) \circ \gamma_2^{-1} = \phi_2 \circ \gamma_2^{-1}. \tag{8}$$

Finally, using (7) and (8), we may calculate:

$$\phi_1 \circ \mu_1 = \phi_1 \circ \gamma_1^{-1} \circ \gamma_1 \circ \mu_1 = \phi_2 \circ \gamma_2^{-1} \circ \gamma_2 \circ \mu_2 = \phi_2 \circ \mu_2.$$

Hence, $(S; T_1, T_2)$ is weakly embeddable. □

4.1. Weak amalgamation over rich ample semigroups

In this subsection, we introduce the notion of rich (right, left) ample semigroups. Given inverse semigroups T_1 and T_2, we show that an amalgam $(S; T_1, T_2)$ is weakly embeddable in an inverse semigroup if S is rich ample in T_1 and T_2. It follows that $(S; T_1, T_2)$ is weakly embeddable in a group if T_1 and T_2 are both groups. We begin by recalling that any inverse semigroup S comes equipped with the *natural partial order*:

$$\forall x, y \in S,\ x \leq y \ \text{ iff } \ x = ey, \ \text{ for some } \ e \in E(S).$$

Remark 4.3. Let U be an inverse semigroup. Then uu^{-1} is the minimum idempotent with respect to the natural partial order such that $(uu^{-1})u = u$. To see this, let $eu = u$ for some idempotent $e \in U$. Then $u^{-1}e = u^{-1}$, and we have $uu^{-1} = uu^{-1}e$. This implies that $uu^{-1} \leq e$, and hence the assertion.

Definition 4.4. A subsemigroup S of an inverse semigroup T is called *rich right ample* in T if

$$\forall x, y \in S,\ x^{-1}y \in S \cup S',$$

where $S' = \{z^{-1} \in T : z \in S\}$ is the dual of S. We say that S is *rich left ample* in T if

$$\forall x, y \in S,\ xy^{-1} \in S \cup S'.$$

A subsemigroup of T is called *rich ample* in T if it is both rich right and rich left ample in T. A submonoid S of an inverse semigroup T is rich (right, left) ample in T if it is such as a subsemigroup. By saying that S is a rich (left, right) ample subsemigroup of T, we mean that S is rich (right, left) ample in T.

Lemma 4.5. *Let S, S' and T be as defined above. Then*

1. *S is rich right (respectively, left) ample in T if and only if (its dual) S' is rich left (respectively, right) ample in T,*
2. *S is (rich) ample in T if and only if S' is (rich) ample in T.*

Proof. The proof is straightforward. □

Proposition 4.6. *Let S be a subsemigroup of an inverse semigroup T. Then S is rich ample in T if and only if the inverse hull of S in T equals $S \cup S'$.*

Proof. The proof is straightforward. □

Lemma 4.7. *Let S be a rich ample subsemigroup of an inverse semigroup T. Then S and S' are the down-closed subsemigroups of $S \cup S'$ with respect to the natural partial order.*

Proof. We show that S is down-closed in $S \cup S'$. It will follow from the symmetry that S' is also such. Let $\leq$ denote the natural partial order on the inverse semigroup $S \cup S'$. Then, clearly, it suffices to prove that

$$(\exists s' \in S')(\exists s \in S)(s' \leq s) \implies s' \in S.$$

Assuming the premise of the above implication, we have $s' = es$ for some idempotent $e \in E(S \cup S')$. Now, because $e \in S \cap S'$, it follows (in particular, from $e \in S$) that $s' = es \in S$. □

Proposition 4.8. *A subsemigroup S of an inverse semigroup T is rich right ample if and only if*

1. *S is right ample in T, and*
2. *for all $x, y \in S$, $x^{-1}y = x^{-1}xa$, where $a \in S \cup S'$.*

Proof. ($\implies$) Let S be rich right ample in T. Then for all $x \in U$, the element $x^{-1}x$ belongs to $S \cup S'$. Because $x^{-1}x$ is an idempotent, we in fact have $x^{-1}x \in S \cap S' \subseteq S$, meaning that S is right ample in T. Next, let $x, y \in S$, and observe that

$$x^{-1}y = x^{-1}xx^{-1}y = x^{-1}xa,$$

where $a = x^{-1}y \in S \cup S'$. Hence, the second condition is also satisfied.

($\Longleftarrow$) Let S be a subsemigroup of an inverse semigroup T satisfying both conditions of the proposition. Let $x, y \in S$. Then, by the second condition, $x^{-1}y = x^{-1}xa$, where $a \in S \cup S'$. Now, $x^{-1}x \in S \cap S'$ because, by the first condition, S is right ample. This implies that

$$x^{-1}y = x^{-1}xa \in S \cup S',$$

whence S is rich right ample in T. □

Proposition 4.9. *A subsemigroup S of an inverse semigroup T is rich left ample if and only if*

1. *S is left ample in T, and*
2. *for all $x, y \in S$, $xy^{-1} = byy^{-1}$, where $b \in S \cup S'$.*

Proof. Similar to the proof of the above proposition. □

Remark 4.10. If $\phi : T_1 \longrightarrow T_2$ is a homomorphism of inverse semigroups and S is rich right (left) ample in T_1, then it can be easily verified that such is $S\phi$ in T_2.

Theorem 4.11. *Let T_1 and T_2 be inverse semigroups admitting monomorphisms ϕ_1 and ϕ_2 from a semigroup S, respectively, such that $S\phi_1$ is rich right ample in T_1, but $S\phi_2$ is not such in T_2. Then the amalgam $(S; T_1, T_2)$ fails to embed weakly in any inverse semigroup.*

Proof. Let S, T_1, T_2, ϕ_1 and ϕ_2 be as described in the statement of the theorem. Assume, on the contrary, that $(S; T_1, T_2)$ is weakly embeddable in an inverse semigroup. Let $(S; V_1, V_2)$ be the special amalgam and $\theta : V_1 \longrightarrow V_2$ be the isomorphism given by Proposition 4.2. Clearly, $S_1 = S\phi_1$ is rich right ample in V_1. But then its image $S_1\theta = S\phi_2$ must be such in V_2 and, hence, in T_2, a contradiction. □

Remark 4.12. The dual statement obtained by replacing ‘rich right ample’ with ‘rich left ample’ in Theorem 4.11 can be proved on similar lines.

Lemma 4.13. *Let T_1 and T_2 be inverse semigroups containing isomorphic copies S_1 and S_2 of a semigroup S, respectively, that are rich ample in the respective oversemigroups. Then the posets $S_1 \cup S_1'$ and $S_2 \cup S_2'$ are order-isomorphic.*

Proof. Let the map

$$\psi = \phi \cup \phi' : S_1 \cup S_1' \longrightarrow S_2 \cup S_2'$$

be as defined in Lemma 4.1. Then it follows from Lemma 4.7 that ψ is indeed an order-embedding. □

Theorem 4.14. *Let T_1 and T_2 be inverse semigroups containing isomorphic copies, say $S\phi_1$ and $S\phi_2$, respectively, of a semigroup S. Assume also that $S\phi_i$ is rich ample in T_i for each $i \in \{1, 2\}$. If S is not inverse, then the amalgam $(S; T_1, T_2; \phi_1, \phi_2)$ is weakly (but not strongly) embeddable in an inverse semigroup.*

Proof. Let $S_1 = S\phi_1$ and $S_2 = S\phi_2$. Then, by Proposition 4.6, the inverse hull of S_i in T_i, $1 \le i \le 2$, is $S_i \cup S_i'$. The main objective is to prove that $S_1 \cup S_1'$ and $S_2 \cup S_2'$ are isomorphic. We shall prove, to this end, that the poset order-isomorphism $\psi : S_1 \cup S_1' \longrightarrow S_2 \cup S_2'$, considered in Lemma 4.13, is a homomorphism of semigroups.

Clearly, if $x, y \in S_1$ (equivalently, S_1'), then $(xy)\psi = (x)\psi(y)\psi$. We prove that $(xz)\psi = (x)\psi(z)\psi$, for all $x \in S_1$ and $z \in S_1'$, and that $(zx)\psi = (z)\psi(x)\psi$ will follow from the symmetry. If $x \in S_1$ and $z \in S_1'$, then $z = y^{-1}$ for some $y \in S_1$, and we observe by Proposition 4.9 that

$$(xz)\psi = (xy^{-1})\psi = (byy^{-1})\psi = (b)\psi(yy^{-1})\psi, \text{ where } b \in S \cup S'.$$

We first show that $(yy^{-1})\psi = (y)\psi(y^{-1})\psi$. To this end, let us first recall from Remark 4.3 that

$$yy^{-1} = \min\{e \in E_1 : ey = y\}, \tag{9}$$

where $E_1 = E(S_1 \cup S_1')$. Then we note that ψ maps E_1 bijectively to $E_2 = E(S_2 \cup S_2')$, and $ey = y$ for $e \in E_1$ if and only if $y\psi = (ey)\psi = (e\psi)(y\psi)$. Thus,

$$(\{e \in E_1 : ey = y\})\psi = \{e\psi \in E_1\psi : (e)\psi(y)\psi = y\psi\}. \tag{10}$$

Now, recall from Lemma 4.13 that the map $\psi : S_1 \cup S_1' \longrightarrow S_2 \cup S_2'$ is an order-isomorphism of posets, whence we may write from (9) and (10):

$$\begin{aligned}(yy^{-1})\psi &= (\min\{e \in E_1 : ey = y\})\psi \\ &= \min(\{e \in E_1 : ey = y\})\psi \\ &= \min\{e\psi \in E_1\psi : (e)\psi(y)\psi = y\psi\} \\ &= (y)\psi(y\psi)^{-1}, \quad \text{since } E_1\psi = E(S_2 \cup S_2'), \\ &= (y)\psi(y^{-1})\psi, \quad \text{by Lemma 4.1.}\end{aligned}$$

Coming back to proving that $(xz)\psi = (x)\psi(z)\psi$, we consider, in view of Definition 4.4, two cases.

If $xy^{-1} = b \in S_1$ (cf. $a = x^{-1}y$ in the proof of Proposition 4.8), then we have:

$$\begin{aligned}(xz)\psi &= (xy^{-1})\psi = (byy^{-1})\psi = (b)\psi(yy^{-1})\psi \\ &= (b)\psi(y)\psi(y^{-1})\psi = (by)\psi(y^{-1})\psi \\ &= (byy^{-1}y)\psi(y^{-1})\psi = (xy^{-1}y)\psi(y^{-1})\psi \\ &= (x)\psi(y^{-1}yy^{-1})\psi = (x)\psi(y^{-1})\psi = (x)\psi(z)\psi.\end{aligned}$$

On the other hand, if $xy^{-1} \in S_1' \setminus S_1$, then, using the rich right ampleness of S_1', we may write:

$$xz = xy^{-1} = (x^{-1})^{-1}y^{-1} = (x^{-1})^{-1}x^{-1}a, \text{ where } a = (x^{-1})^{-1}y^{-1} = xy^{-1}.$$

Note that $a \in S_1' \setminus S_1$, for otherwise we get $xy^{-1} \in S_1$, a contradiction. Now, one may calculate:

$$\begin{aligned}(xz)\psi &= (xy^{-1})\psi = ((x^{-1})^{-1}y^{-1})\psi \\ &= ((x^{-1})^{-1}x^{-1}a)\psi = ((x^{-1})^{-1}x^{-1})\psi(a)\psi \\ &= (xx^{-1})\psi(a)\psi = (x)\psi(x^{-1})\psi(a)\psi \\ &= (x)\psi(x^{-1}a)\psi = (x)\psi(x^{-1}xx^{-1}a)\psi \\ &= (x)\psi(x^{-1}xy^{-1})\psi = (x)\psi(x^{-1}x)\psi(y^{-1})\psi \\ &= (xx^{-1}x)\psi(y^{-1})\psi = (x)\psi(y^{-1})\psi \\ &= (x)\psi(z)\psi.\end{aligned}$$

This completes the proof that $\psi : S_1 \cup S_1' \longrightarrow S_2 \cup S_2'$ is an isomorphism of semigroups. That $(S; T_1, T_2)$ is weakly embeddable follows from Proposition 4.2. The amalgam $(S; T_1, T_2)$, however, fails to embed strongly by Propositions 4.8, 4.9 and Theorem 3.2 (alternatively, Theorem 3.4). □

Corollary 4.15. *Consider an amalgam $\mathcal{A} = (S; G_1, G_2; \phi_1, \phi_2)$ in which G_1 and G_2 are groups and S is not a group. If $S\phi_i$ is rich ample in G_i for each $i \in \{1, 2\}$, then $\mathcal{A}$ is weakly embeddable in a group.*

Proof. By the above theorem, $(S; G_1, G_2)$ weakly embeds in an inverse semigroup, say W. Let $\psi_1 : G_1 \longrightarrow W$ and $\psi_1 : G_1 \longrightarrow W$ be the embedding monomorphisms. Then, by the properties of a pushout, there exists a unique homomorphism

$$\psi : G_1 *_S G_2 \longrightarrow W,$$

such that $\eta_i\psi = \psi_i$ for $1 \le i \le 2$, where η_i are as given in Fig. 1. This implies that $(S; G_1, G_2)$ weakly embeds in $(G_1 *_S G_2)\psi$. The proof will be accomplished if we show that $(G_1 *_S G_2)\psi$ is a group.

Let 1_{G_1} and 1_{G_2} be the identities of G_1 and G_2, respectively. Then, because $S\phi_i$ is rich ample in G_i for each $i \in \{1, 2\}$, we must have $1_{G_i} \in S\phi_i$. Thus, S is, in fact, a monoid. That $(G_1 *_S G_2)\psi$ is a group follows from the observation that $G_1 *_S G_2$, being an amalgamated coproduct of groups over a monoid, is a group. □

Theorem 4.16. *Any amalgam $(S; G_1, G_2; \phi_1, \phi_2)$ in which G_1 and G_2 are groups is weakly embeddable in a group.*

Proof. If S is a monoid, then $S\phi_i$ is rich ample in G_i for each $i \in \{1, 2\}$, and the theorem follows from the above corollary. If S is a semigroup, then, by a similar token, the amalgam $\mathcal{A}' = (S^1; G_1, G_2; \phi'_1, \phi'_2)$ is weakly embeddable in a group, where S^1 is the monoid obtained from S in the standard way (see for instance [4]), and for each $i \in \{1, 2\}$, ϕ'_i is the obvious extension of ϕ_i. Now, the weak embedding of $(S; G_1, G_2; \phi_1, \phi_2)$ in a group may be obtained by restricting the embedding monomorphisms of $\mathcal{A}'$ to $S\phi_i$. □

4.2. Connection with dominions

Let U be a subsemigroup of a semigroup S. Then, recall, for instance from [4], that an element $d \in S$ is said to be dominated by U if for all homomorphisms $f, g : S \longrightarrow T$ with $f|_U = g|_U$ we have $(d)f = (d)g$. The set $\text{Dom}_S U$ of all elements of S dominated by U is a subsemigroup of S, called the *dominion* of U in S.

Proposition 4.17. *An ample subsemigroup S of an inverse semigroup T is rich ample in T if and only if $Dom_T S = S \cup S'$.*

Proof. ($\Longrightarrow$) Let S be rich ample in T. Then, by Proposition 4.6, the inverse hull of S in T is $S \cup S'$. Now, it follows from Proposition 1 of [8] that $\text{Dom}_T S = S \cup S'$.

($\Longleftarrow$) Assume that $\text{Dom}_T S = S \cup S'$, with S being an ample in T. Then, by Proposition 1 of [8], the inverse hull of S in T is $S \cup S'$. Consequently, S is rich ample in T by Proposition 4.6. □

Remark 4.18. It follows from the above proposition and zigzag theorem (see for instance [4], Theorem 8.3.3) that S is rich ample in T if and only if for all $t \in T$; the equality $t \otimes 1 = 1 \otimes t$ in the tensor product $T \otimes_S T$ implies that $t \in S \cup S'$.

5. Ultra-rich ample semigroups

In this section, we introduce a special class of rich ample semigroups, namely the ultra-rich ample semigroups; its idea stems from our earlier considerations. The reader may refer to Section 6 for some natural examples of rich and ultra-rich ample semigroups.

Definition 5.1. Let a subsemigroup S be rich right (respectively, left) ample in an inverse semigroup T such that the elements a (respectively, b) given in Proposition 4.8 (respectively, Proposition 4.9) are uniquely determined. Then we say that S is *ultra-rich right* (respectively, *left*) *ample* in T. We call S *ultra-rich ample* in T if it is both ultra-rich right and ultra-rich left ample in T. We also recall from [1] that a semigroup is called *unipotent* if it contains precisely one idempotent.

Lemma 5.2. *A subsemigroup S of an inverse semigroup T is ultra-rich right (respectively, left) ample if and only if S' is an ultra-rich left (respectively, right) ample subsemigroup of T.*

Proof. Straightforward. □

Proposition 5.3. *If S is an ultra-rich right (left) ample subsemigroup of an inverse semigroup T, then S and S' are unipotent.*

Proof. Let S be an ultra-rich right ample subsemigroup of an inverse semigroup T. Let $e_1, e_2 \in E(S)$. Then, by the ultra-rich right ampleness of S, we have

$$e_1^{-1}e_2 = e_1^{-1}e_1a$$

for a unique $a \in S \cup S'$. Because

$$e_1^{-1}e_2 = e_1^{-1}e_1(e_1^{-1}e_2), \text{ and } e_1^{-1}e_2 = e_1^{-1}e_1(e_2),$$

we get from the uniqueness of a:

$$a = e_2 = e_1^{-1}e_2.$$

Thus, $e_1e_2 = e_2$, for every idempotent is the inverse of itself. Similarly, we may calculate $e_2e_1 = e_1$, whence, by the commutativity of idempotents, we have $e_1 = e_2$. Thus, $E(S)$ is indeed a singleton.

Similarly, one can show that any ultra-rich left ample subsemigroup of an inverse semigroup is unipotent. By Lemma 5.2, the theorem also holds for S'. □

Corollary 5.4. *Let S be an inverse semigroup. Then S is ultra-rich right (left) ample if and only if it is a group.*

Proof. Straightforward. □

Proposition 5.5. *Let S be a subsemigroup of an inverse semigroup T. Then S is ultra-rich ample in T if and only if it is unipotent and rich ample in T.*

Proof. ($\Longrightarrow$) A proof of the direct part follows from Proposition 5.3 and the fact that any ultra-rich ample subsemigroup of T is rich ample in T.
($\Longleftarrow$) Let S be a rich ample subsemigroup of an inverse semigroup T such that $E(S) = \{e\}$. This implies that $x^{-1}x = xx^{-1} = e$ for all $x \in S \cup S'$. Now, for any $y \in S$, there exists, by Condition (2) of Proposition 4.8, an element $a \in S \cup S'$, such that

$$x^{-1}y = x^{-1}xa \ (= ea).$$

To prove the uniqueness of a, let

$$x^{-1}y = x^{-1}xc \ (= ec),$$

for some $c \in S \cup S'$. Then

$$a = (aa^{-1})a = ea = x^{-1}xa = x^{-1}y = x^{-1}xc = ec = (cc^{-1})c = c.$$

This proves that U is ultra-rich right ample. Similarly, one can show that U is ultra-rich left ample. □

Corollary 5.6. *Let S be an ultra-rich ample subsemigroup of an inverse semigroup T. Then the inverse hull $S \cup S'$ of S is a subgroup of T.*

Proof. It follows from Proposition 4.6 that $S \cup S'$ is an inverse subsemigroup of T. Because S and S' are ultra-rich ample, such is $S \cup S'$. The proof now follows from Corollary 5.4. □

Corollary 5.7. *Let U be an ultra-rich ample subsemigroup of an inverse semigroup S and $\phi : S \longrightarrow T$ be a homomorphism of (inverse) semigroups. Then $U\phi$ is an ultra-rich ample subsemigroup of T.*

Proof. Note that $U\phi$ is rich ample by Remark 4.10. On the other hand, being the image of a group, $(U \cup U')\phi$ is a subgroup of T. So, $(U \cup U')\phi$ and, consequently, $U\phi$ must contain a unique idempotent. The corollary now follows by Proposition 5.5. □

It is an easy exercise to show that every subsemigroup of a finite group G is a subgroup of G. Also, recall from Corollary 5.6 that the inverse hull of an ultra-rich ample subsemigroup of an inverse semigroup T is a subgroup of T. Consequently, we make the following observation.

Remark 5.8. Assume that S is an ultra-rich ample subsemigroup of an inverse semigroup T. If T (or S) is finite, then S is a subgroup of T.

6. Examples

Clearly, every group is an inverse as well as an ultra-rich ample semigroup, whereas every inverse semigroup is a rich ample and hence an ample semigroup. The first six examples consider infinite inverse semigroups (cf. Remark 5.8). Examples 7–9 concern the finite case.

1. It is straightforward to observe that the (non-inverse) multiplicative monoid $\mathbb{N}$ of natural numbers is ample but not rich (right, left) ample in $\mathbb{Q}$.
2. On the other hand,
$$\mathbb{Q}_{\geq 1} = \{x \in \mathbb{Q} : x \geq 1\}$$
is a non-inverse ultra-rich ample submonoid of $\mathbb{Q}$, where $\geq$ is the usual partial order.
3. The non-inverse monoid $\mathbb{Q}_{\geq 1} \cup \{0\}$ is rich ample but not ultra-rich (right, left) ample in $\mathbb{Q}$.
4. The inverse monoid $\mathbb{Q}$ is not ultra-rich ample in itself.

5. The subgroup $\mathbb{Q} \setminus \{0\}$ is an inverse as well as an ultra-rich ample submonoid of $\mathbb{Q}$.

6. If S is a group, then it is straightforward to observe that every monogenic subsemigroup of S is ultra-rich ample. Thus, the monogenic subsemigroups of the symmetric group $S_{\mathbb{N}}$ are ultra-rich ample (in $S_{\mathbb{N}}$ and $\mathcal{I}_{\mathbb{N}}$).

7. Let $\mathcal{I}_n$, where $n \geq 2$, denote the symmetric inverse monoid over a finite chain C_n: $1 < \cdots < n$ of natural numbers. Let $\mathcal{ODI}_n$ and $\mathcal{ODI}_n^+$ denote the (non-inverse, non-commutative) ample submonoids of $\mathcal{I}_n$, comprising, respectively, the order-decreasing and order-increasing partial bijections that also preserve the order [8]. Let us also use the convention that the empty map belongs to $\mathcal{ODI}_n^+ \cap \mathcal{ODI}_n$. Now, one may verify, for instance, by brute force that $\mathcal{ODI}_n$ and $\mathcal{ODI}_n^+$ are rich (right, left) ample in $\mathcal{I}_n$ for $n \leq 3$. We show that $\mathcal{ODI}_n$ are not rich (right, left) ample in $\mathcal{I}_n$ for all $n \geq 4$. To this end, consider the following elements of $\mathcal{ODI}_4$:

$$u = \{(4,3),(3,2)\},$$
$$v = \{(4,4),(3,1)\}.$$

Then
$$u^{-1}v = \{(3,4),(2,1)\} \notin U \cup U',$$
implying that $\mathcal{ODI}_4$ is not rich right ample in $\mathcal{I}_n$. It can be shown similarly that $\mathcal{ODI}_4$ is not rich left ample in $\mathcal{I}_n$. Because $\mathcal{ODI}_m$ is contained, $\mathcal{ODI}_n$ for all $m \leq n$, it follows that $\mathcal{ODI}_n$ are not rich (right, left) ample in $\mathcal{I}_n$ for all $n \geq 4$. It follows from Lemma 4.5 that $\mathcal{ODI}_n^+$ are also not rich (left, right) ample in $\mathcal{I}_n$ for all $n \geq 4$.

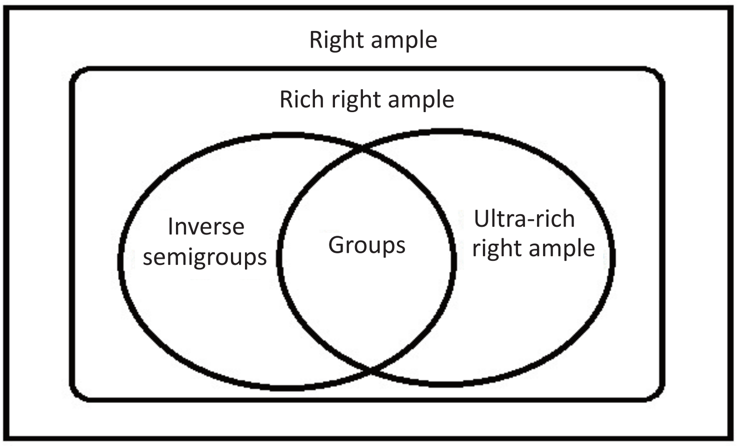
8. Let $\mathcal{I}_n$ ($n \geq 2$) denote the inverse monoid considered in the previous example. As every inverse subsemigroup of an inverse semigroup is rich ample, the members of the chain

$$\mathbb{L}_{\text{INV}} : \{\iota\} \subseteq \mathcal{OI}_n \subset \mathcal{RI}_n \subset \mathcal{I}_n' \subset \mathcal{I}_n$$

from Example 3.3 are rich ample in $\mathcal{I}_n$. However, because $\mathcal{OI}_n$, $\mathcal{RI}_n$, $\mathcal{I}_n'$ and $\mathcal{I}_n$ are not unipotent, by Proposition 5.3, none of them is ultra-rich (right, left) ample in $\mathcal{I}_n$.

9. The symmetric group S_n is ultra-rich ample in $\mathcal{I}_n$.

Based on the above examples, we get the following containment diagram for various classes of right ample subsemigroups of an inverse semigroup, considered in earlier sections. Clearly, there exist similar diagrams for the corresponding classes of left ample and (two-sided) ample subsemigroups of an inverse semigroup. (The oval representing ultra-rich ample semigroups disappears in the finite case.)



7. Conclusion

Though we have partially answered the question posed in [7], determining completely the embedding of $(T_1, T_2; S)$, where T_1 and T_2 are inverse and S is an arbitrary semigroup, is still an open problem. Let S be made into an (ultra-) rich right and an (ultra-) rich left ample semigroup by two different inverse semigroups. Then we also wonder if finding a single inverse semigroup that makes it a right as well as a left (ultra-) rich ample semigroup is decidable. As mentioned earlier, this problem is undecidable for (right, left) ample semigroups.

Data availability statement

All data are available in the article.

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Inverssete poolrühmade amalgaamimine üle külluslike poolrühmade

Nasir Sohail

Artiklis uuritakse poolrühmade amalgaame $(S; T_1, T_2)$, kus T_1 ja T_2 on inverssed poolrühmad, aga S ei ole inversne. On teada, et sellist amalgaami ei saa suuremasse poolrühma sisestada, kui T_1 ja T_2 on rühmad, aga S ei ole rühm. Artiklis tõestatakse, et amalgaami $(S; T_1, T_2)$ ei saa sisestada, kui S on külluslik poolrühm, mis ei ole inversne. Tuues sisse rikkaliku külluslikkuse mõiste, leitakse tarvilikud ja piisavad tingimused selleks, et amalgaami $(S; T_1, T_2)$ saaks nõrgalt sisestada inverssesse poolrühma. Muu hulgas tuleb välja, et amalgaami $(S; T_1, T_2)$ saab nõrgalt sisestada rühma, kui T_1 ja T_2 on rühmad. Lisaks uuritakse rikkalikult külluslike poolrühmade klassi uusi alamklasse.