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RING
THEORY

About unital and non-unital duo rings

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Abstract. Several results about one-sided duo rings and duo rings are generalized from the case of unital rings to the case of arbitrary associative rings in this paper. For example, the characterization of one-sided duo rings and duo rings is given, and it is shown that all idempotents of a duo ring commute with all elements of the ring and that every prime ideal is completely prime.

Keywords: duo rings, left duo rings, right duo rings, principal left ideal, principal right ideal, principal ideal, idempotent, centre, prime ideal, completely prime ideal.

1. INTRODUCTION

We will consider all rings to be associative but not necessarily unital in this paper. The zero element of a ring R will be denoted by θ_R.

As far as we know, the term *duo ring* was first introduced in the context of unital rings in 1958 by Edmund H. Feller (see [4], Definition 1.4, p. 79). According to Feller, a unital ring R is called a duo ring if every right ideal of R is also a left ideal of R, and every left ideal of R is also a right ideal of R. As a generalization of this definition, one-sided duo rings were soon defined and studied. For example, according to R. C. Courter, a unital ring R is called a right duo ring if every right ideal of R is a two-sided ideal of R (see [2], p. 157). A left duo ring is defined analogously, demanding that every left ideal of a unital ring should be a two-sided ideal.

Actually, these definitions could be generalized to arbitrary rings (with or without a unit element) as follows.

Definition 1. *A ring R is called*

1) **a left duo ring** *if every left ideal of R is also a two-sided ideal of R;*
2) **a right duo ring** *if every right ideal of R is also a two-sided ideal of R;*
3) **a duo ring** *if every one-sided ideal of R is also a two-sided ideal of R.*

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We will use the term *one-sided duo ring* for a ring that is either a left duo ring or a right duo ring and use the term duo ring for the case, where a ring is both a left duo ring and a right duo ring at the same time, as it was originally used by Feller. Hence, every duo ring is both a left duo ring and a right duo ring, but a one-sided duo ring does not have to be a duo ring.

The properties of duo rings were also studied, for example, in [8], [1] and [6], to name a few.

Examples. 1. Every commutative ring is a duo ring ([3], Corollary 1.17, p. 7).

2. The ring $\mathbb{Z}$ of all integers is a commutative unital duo ring ([3], Example 1.18, p. 7).

3. The ring $2\mathbb{Z}$ of all even numbers is a commutative non-unital duo ring ([3], Example 1.19, p. 7).

4. The set of all three-dimensional vectors of a three-dimensional space (with addition of vectors and the cross product of vectors as ring operations) is a non-commutative non-unital duo ring ([3], Example 1.20, pp. 7–8).

It can be shown by constructing (usually quite technical) examples that the notions of a left duo ring, right duo ring and duo ring are indeed different ones. For example, as it was pointed out by Courter (see [2], p. 157), Theorem 1 of [5], p. 149 provides an example of a left duo ring, which is not a right duo ring; hence it cannot be a duo ring if a certain monomorphism in this example is not an epimorphism.

The present paper is based on the bachelor's thesis of the second author, supervised by the first author, where several results known for unital duo rings or unital one-sided duo rings were generalized to the case where the ring might not be unital. Therefore, we will not assume anything about the existence of a unit element, although the unital case will also be a special case of our results.

In case the results hold for left duo rings, right duo rings and duo rings, we present the proofs only for the left side case, unless there are some differences in the proofs for other cases. In [3], the proofs were mainly given for the right side case. The main goal of this paper is to introduce the results obtained in [3], which was written in the Estonian language as a bachelor's thesis in Tallinn University and might not be easily accessed by interested readers abroad. The present paper is structured a bit differently than [3], and some shorter or more elegant proofs have been included while a large part of [3] has been omitted because it is the overview of well-known definitions and results.

2. CHARACTERIZATION OF ONE-SIDED DUO RINGS AND DUO RINGS

By the definition, the principal left (right or two-sided) ideal, generated by an element r in a ring R, is the intersection of all left (right or two-sided, respectively) ideals of R that include r. Hence, the principal left (right or two-sided) ideal, generated by an element r, is contained in every left (right or two-sided) ideal of R that includes r.

As it is known, for a non-unital ring R, the principal ideals generated by an element $r \in R$ have the following forms:

the principal left ideal, generated by r, is $L(r) = \{sr + mr : s \in R, m \in \mathbb{Z}\}$;

the principal right ideal, generated by r, is $R(r) = \{rs + mr : s \in R, m \in \mathbb{Z}\}$;

the principal two-sided ideal, generated by r, is

$$T(r) = \Big\{sr + rt + mr + \sum_{i=1}^{n} a_i r b_i : m \in \mathbb{Z}, n \in \mathbb{Z}^+, a_1, \ldots, a_n, b_1, \ldots, b_n \in R\Big\}.$$

Remark 1. *In the case R is a unital ring, these descriptions of principal ideals coincide with the well-known descriptions $L(r) = \{sr : s \in R\}, R(r) = \{rs : s \in R\}$ and*

$$T(r) = \Big\{\sum_{i=1}^{n} a_i r b_i : n \in \mathbb{Z}^+, a_1, \ldots, a_n, b_1, \ldots, b_n \in R\Big\}$$

of principal ideals of unital rings. Therefore, we will use the descriptions of principal ideals in the form they were given in the non-unital case in the following proofs in order to prove the results for both unital and non-unital cases at the same time.

For unital duo rings there is a classical result which states that a unital ring R is a duo ring if and only if for all $r,s \in R$ there exist $u,v \in R$ such that $rs = ur = sv$. In order to prove a similar result for the non-unital case and also for one-sided duo rings, we first need a simple lemma (which is known to be true in the unital case).

Lemma 1. *Let R be a left duo ring (a right duo ring or a duo ring) and $r \in R$. Then the principal two-sided ideal $T(r)$, generated by r, is a subset of $L(r)$ (of $R(r)$ or of both $L(r)$ and $R(r)$, respectively).*

Proof. Suppose that R is a left duo ring and $r \in R$. Notice that $r = \theta_R r + 1r \in L(r)$. As $L(r)$ is a left ideal of R and R is a left duo ring, then $L(r)$ is also a two-sided ideal.

Take any element $k \in T(r)$. Then there exist $m \in \mathbb{Z}, n \in \mathbb{Z}^+, a_1,\dots,a_n,b_1,\dots,b_n \in R$ such that

$$k = sr + rt + mr + \sum_{i=1}^{n} a_i r b_i.$$

Now, $sr, rt, a_1rb_1,\dots,a_nrb_n \in L(r)$ because $L(r)$ is a two-sided ideal of R. Moreover, as $mr = \theta_R r + mr \in L(r)$ and since every ideal is closed with respect to addition, we obtain that $k \in L(r)$. Hence, $T(r) \subseteq L(r)$.

The proofs for the cases of a right duo ring and a duo ring are similar. □

Now we are ready to give the characterization of one-sided duo rings and duo rings in the non-unital case.

Theorem 1. *A ring R is a left duo ring (a right duo ring or a duo ring) if and only if for every $r,s \in R$ there exist $u \in R$ and $m \in \mathbb{Z}$ such that $rs = ur + mr$ (there exist $v \in R$ and $n \in \mathbb{Z}$ such that $rs = sv + ns$ or there exist $u,v \in R$ and $m,n \in \mathbb{Z}$ such that $rs = ur + mr = sv + ns$, respectively).*

Proof. Let R be a ring and $r,s \in R$. Then

$$T(r) \ni \theta_R r + rs + 0r + \theta_R r \theta_R = rs = rs + \theta_R s + 0s + \theta_R s \theta_R \in T(s).$$

Suppose that R is a left duo ring. Then $rs \in T(r) \subseteq L(r)$, which means that there exist $u \in R$ and $m \in \mathbb{Z}$ such that $rs = ur + mr$.

Similarly, in the case of a right duo ring, $rs \in R(s)$, and there exist $v \in R$ and $n \in \mathbb{Z}$ such that $rs = sv + ns$, and in the case of a duo ring, we obtain that $rs = ur + mr = sv + ns$ for some $u,v \in R$ and $m,n \in \mathbb{Z}$.

Suppose that I is an arbitrary left ideal of R and that for every $r,s \in R$ there exist $u \in R$ and $m \in \mathbb{Z}$ such that $rs = ur + mr$.

Let us show that then I is also a right ideal, i.e., that R is a left duo ring. For that, take any $x \in I$ and consider the principal left ideal $L(x)$ in R. Then $x \in L(x) \subseteq I$ (by the definition of a left principal ideal).

Take any $y \in R$. By the assumption, there exist $u \in R$ and $m \in \mathbb{Z}$ such that $xy = ux + mx$. Now, $ux = ux + 0x \in L(x) \subseteq I$ and $mx = \theta_R x + mx \in L(x) \subseteq I$. As the ideal I is closed with respect to addition, we obtain that $xy \in I$. Hence, I is a right ideal.

As I was an arbitrary left ideal of R, we obtain that R is a left duo ring.

The proofs for a right duo ring and a duo ring are similar. □

3. IDEMPOTENTS AND THE CENTRE OF DUO RINGS

In [8], p. 167, there is a result claiming that every idempotent of a unital duo ring belongs to the centre of the duo ring. We will generalize this result for duo rings for which the existence of a unit element is not needed. In order to prove this result, we will first state and prove two simpler lemmas.

In [6], p. 1425, the authors show that in a unital left (right) duo ring one has $T(r) = L(r)$ and $R(r) \subseteq L(r)$ ($T(r) = R(r)$ and $L(r) \subseteq R(r)$, respectively) for all $r \in R$. We will prove an analogous result for the general case, when the existence of the unit element is not needed.

Lemma 2. *Let R be a left (a right) duo ring and $r \in R$. Then $T(r) = L(r)$ and $R(r) \subseteq L(r)$ ($T(r) = R(r)$ and $L(r) \subseteq R(r)$, respectively).*

Proof. We will give the proof for a left duo ring R. For a right duo ring the proof is similar.

Take any $r \in R$. By Lemma 1, we already know that $T(r) \subseteq L(r)$.

Take any $p \in L(r)$. Then there exist $s \in R$ and $m \in \mathbb{Z}$ such that $p = sr + mr$. Notice that

$$p = sr + mr = sr + r\theta_R + mr + \theta_R r \theta_R \subseteq T(r).$$

Hence, $L(r) \subseteq T(r)$ and we have proved that $T(r) = L(r)$.

Take any $q \in R(r)$. Then there exist $s \in R$ and $m \in \mathbb{Z}$ such that $q = rs + mr$. Similarly as above, we see that

$$q = rs + mr = \theta_R r + rs + mr + \theta_R r \theta_R \in T(r) = L(r).$$

Hence, $R(r) \subseteq L(r)$. □

In [8], p. 167, there is a result (without the proof) stating that for a unital duo ring R and any $a, b, c \in R$ there exist $u, v \in R$ such that $abc = ub = bv$. We will provide the generalization of that result for the case when the existence of the unit element is not assumed.

Lemma 3. *Let R be a left duo ring (a right duo ring or a duo ring) and $a, b, r \in R$. Then there exist $u \in R$ and $m \in \mathbb{Z}$ such that $arb = ur + mr$ (there exist $v \in R$ and $n \in \mathbb{Z}$ such that $arb = rv + nr$ or there exist $u, v \in R$ and $m, n \in \mathbb{Z}$ such that $arb = ur + mr = rv + nr$, respectively).*

Proof. Let R be a ring and $a, b, r \in R$. Then $arb = \theta_R r + r\theta_R + 0r + arb \in T(r)$.

By Lemma 1, we know that for a left duo ring R we have $T(r) \subseteq L(r)$, for a right duo ring R we have $T(r) \subseteq R(r)$ and for a duo ring R we have $T(r) \subseteq L(r) \cap R(r)$.

Hence, for a left duo ring R there exist $u \in R$ and $m \in \mathbb{Z}$ such that $arb = ur + mr$, for a right duo ring R there exist $v \in R$ and $n \in \mathbb{Z}$ such that $arb = rv + nr$ and for a duo ring R there exist $u, v \in R$ and $m, n \in \mathbb{Z}$ such that $arb = ur + mr = rv + nr$. □

Now we are ready for the main result of this section.

Proposition 1. *All idempotents of a duo ring R belong to the centre of R.*

Proof. Let g be an idempotent of a duo ring R and k an arbitrary element of R. Then $kgg = kg$ and $ggk = gk$. Our task is to show that $kg = gk$.

By Lemma 3, there exist $u, v \in R$ and $m, n \in \mathbb{Z}$ such that $ggk = ug + mg$ and $kgg = gv + ng$. Hence,

$$kg = kgg = gv + ng = gv + ngg = ggv + gng = g(gv + ng) = g(kgg) = g(kg) =$$

$$= (gk)g = (ggk)g = (ug + mg)g = ugg + mgg = ug + mg = ggk = gk.$$

As k was an arbitrary element of R, then g commutes with all elements of R, which means that g belongs to the centre of R. Therefore, all idempotents of a duo ring belong to the centre of R. □

We will end this section with an open problem.

Open problem. Does an analogue of Proposition 1 hold also for a left duo ring, which is not a right duo ring, or for a right duo ring, which is not a left duo ring?

4. ON COMPLETELY PRIME IDEALS IN ONE-SIDED DUO RINGS AND DUO RINGS

For arbitrary subsets X,Y of a ring R and arbitrary elements $a,b \in R$, we denote

$$XY = \Big\{\sum_{i=1}^{k} x_i y_i : k \in \mathbb{Z}^+, x_1,\dots,x_k \in X, y_1,\dots,y_k \in Y\Big\},$$

$$aX = \{ax : x \in X\}, \quad aXb = \{axb : x \in X\}.$$

An ideal I in a ring R is said to be *prime* if from $X,Y \subseteq R, XY \subseteq I$ it follows that either $X \subseteq I$ or $Y \subseteq I$. Moreover, an ideal I of a ring R is said to be *completely prime* if from $x,y \in R, xy \in I$ it follows that either $x \in I$ or $y \in I$.

In Theorem 1 in [7], p. 825, McCoy proved that an ideal I of a ring R is prime if and only if from the conditions $x,y \in R, xRy \subseteq I$ it follows that either $x \in I$ or $y \in I$. Using this result, Thierrin proved in [8], p. 168 that every prime ideal of a unital duo ring is completely prime. If one examines the proof more carefully, one sees that Thierrin uses there the equivalent condition of McCoy instead of the original definition of a prime ideal in order to complete the proof. Actually, the same ideology works also for the non-unital case and even for one-sided duo rings.

Hereby we prove the generalization of Proposition 3 from [8], p. 168.

Proposition 2. *Let R be a ring and I a two-sided ideal of R such that from the conditions $x,y \in R$, $xRy \subseteq I$ it follows that $x \in I$ or $y \in I$. If R is either a left duo ring, a right duo ring or a duo ring, then I is completely prime.*

Proof. Let R be a left duo ring and I a two-sided ideal of R, which satisfies the conditions of Proposition 2. Fix any $x,y \in R$ with $xy \in I$ and consider the set $K = \{k \in R : ky \in I\}$.

First, we will show that K is a left ideal of R. Notice that $K \subseteq R$ and $x \in K$, which means that $K \neq \emptyset$. Moreover, for all $f,g \in K, h \in R, m \in \mathbb{Z}$, we have

$$(f \pm g)y = fy \pm gy \in I, (mf)y = m(fy) \subseteq mI \subseteq I \text{ and } (hf)y = h(fy) \in hI \subseteq RI \subseteq I$$

because I, as an ideal, is closed with respect to addition, subtraction and multiplication. Hence, $f \pm g, mh, hf \in K$, which means that K is a subring and a left ideal of R.

As R is a left duo ring and K is a left ideal of R, then K is a two-sided ideal of R. Hence, as $x \in K$, then $xr \in K$ for each $r \in R$, which means that $xry = (xr)y \in I$ for each $r \in R$. From this, we obtain that $xRy = \{xry : r \in R\} \subseteq I$. By the assumption, we get that either $x \in I$ or $y \in I$. Hence, I is a completely prime ideal.

In the case R is a right duo ring, we just have to consider the set $L = \{l \in R : xl \in I\}$ instead of K, and everything follows similarily to the case of a left duo ring.

As every duo ring is also a left duo ring, the result for a duo ring follows from the first part of the proof. □

We finish the paper with a corollary.

Corollary 1. *If R is either a left duo ring, a right duo ring or a duo ring, then every prime ideal of R is completely prime.*

Proof. By Theorem 1 (i,iii), p. 825 of [8], every prime ideal I satisfies the following condition:

from $x,y \in R$ and $xRy \subseteq I$ it follows that $x \in I$ or $y \in I$.

Hence, the claim is true by Proposition 2. □

5. CONCLUSION

In the present paper, we gave a characterization of one-sided duo rings and duo rings with or without a unit. We showed that each idempotent in a duo ring commutes with all elements of the ring and that each prime ideal of a one-sided duo ring or a duo ring is completely prime.

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Ühikuga ja ühikuta duo ringidest

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Artiklis uuritakse vasakpoolseid duo ringe, parempoolseid duo ringe ja duo ringe ilma eelduseta, et ring peaks omama ühikelementi. Töö käigus leitakse nende kolme tüübi duo ringide klassifikatsioon, näidatakse, et duo ringi iga idempotent kuulub selle ringi tsentrisse ning tõestatakse, et iga elementaarideaal vasakpoolses duo ringis, parempoolses duo ringis või duo ringis on täielikult elementaarne.