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FINITE GROUPS

On connected components and perfect codes of proper order graphs of finite groups

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Abstract. Let G be a finite group with the identity element e. The *proper order graph* of G, denoted by $\mathscr{S}^*(G)$, is an undirected graph with a vertex set $G \setminus \{e\}$, where two distinct vertices x and y are adjacent whenever $o(x) \mid o(y)$ or $o(y) \mid o(x)$, where $o(x)$ and $o(y)$ are the orders of x and y, respectively. This paper studies the perfect codes of $\mathscr{S}^*(G)$. We characterize all connected components of a proper order graph and give a necessary and sufficient condition for a connected proper order graph. We also determine the perfect codes of the proper order graphs of a few classes of finite groups, including nilpotent groups, CP-groups, dihedral groups and generalized quaternion groups.

Keywords: perfect code, proper order graph, finite group.

1. INTRODUCTION

In the field of algebraic graph theory, a popular and interesting research topic is groups and graphs, which is the study of the graph representations of an algebraic structure, such as a group or a ring, for example, the celebrated Cayley graphs over groups. Furthermore, graphs associated with some algebraic structures have been actively investigated in the literature since they have valuable applications, see, for example, Cayley graphs in data mining [18]. They are also related to the automata theory [16].

Another well-known graph representation by a group is the power graph. Let G be a group. The *directed power graph* of G is a digraph with a vertex set G, and for distinct $x, y \in G$, there exists a directed edge from x to y if and only if y is a power of x. In 2000, Kelarev and Quinn [17] first introduced the directed power graph of a group. In 2009, Chakrabarty et al. [6] introduced the undirected power graph $\mathscr{P}(S)$ of a semigroup S, which is an undirected graph with a vertex set S, and two distinct vertices are adjacent if one is a power of the other. Since a group is also a semigroup, the definition of an undirected power graph $\mathscr{P}(G)$ of a group G is also introduced in [6]. In the past ten years, the study of power graphs has been growing. See, for example, the two survey papers [2,20] containing almost all the results and open questions on power graphs. In recent years, some authors generalized and modified the concept of a power graph in various ways, such

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as the enhanced power graphs [1] and the quotient power graphs [5]. Also, the distance Laplacian spectra of power graphs were studied in [24].

Given a finite group G, the *order graph* of G, denoted by $\mathscr{S}(G)$, is the graph with a vertex set G, where two distinct vertices x and y are adjacent whenever $o(x) \mid o(y)$ or $o(y) \mid o(x)$, where $o(x)$ and $o(y)$ are the orders of x and y, respectively. Notice that for any group G, $\mathscr{P}(G)$ is always a spanning subgraph of $\mathscr{S}(G)$. Hamzeh and Ashrafi [9] first introduced the definition of the order graph of a group and called $\mathscr{S}(G)$ as the main supergraph of the power graph of G. Also, they characterized the full automorphism group of the order graph of a finite group. In 2018, Hamzeh and Ashrafi [10] studied some properties of $\mathscr{S}(G)$ together with the relationship between $\mathscr{S}(G)$ and $\mathscr{P}(G)$. In [22], Ma and Su studied the independence number of an order graph. Hamiltonianity and Eulerianness of $\mathscr{S}(G)$ were investigated in [12] and spectrum and L-spectrum of $\mathscr{S}(G)$ were investigated in [11]. In [3], Asboei and Salehi studied the well-known Thompson's problem and recognized the projective special linear groups and the projective linear groups by their order graphs. Recently, Asboei and Salehi [4] identified many families of finite non-solvable groups by their order graphs, which is an important work for the Thompson's problem.

Perfect code has been an important object of study in coding theory ever since the beginning of information theory. Roughly speaking, a code is perfect if it can achieve the maximum possible error correction without ambiguity. In the classical setting, much work has been focused on perfect codes under the Hamming or Lee metric. Since the beginning of coding theory in the late 1940s, perfect code has been an important object of study in information theory (see the survey paper [13] on perfect codes and related definitions in the classical setting). Beginning with [19], perfect codes in general graphs have also attracted considerable attention in the community of graph theory (see [23]). In particular, perfect codes in Cayley graphs of groups are especially charming objects of study (see [14]).

In this paper, a graph always means an undirected graph without loops and multiple edges. Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of Γ, respectively. A subset C of $V(\Gamma)$ is called a *perfect code* if C is an independent set such that every vertex in $V(\Gamma) \setminus C$ is adjacent to exactly one vertex in C. In graph theory, a perfect code is also called an *efficient dominating set* [7] or *independent perfect dominating set* [21].

Every group considered in our paper is finite. We always use G to denote a finite group and use e to denote the identity element of G. Note that in $\mathscr{S}(G)$, e is always adjacent to any non-trivial element. Therefore, by the definition of a perfect code, every perfect code of the order graph of a group has size one, which is $\{e\}$. Thus, in this paper, we consider the subgraph of $\mathscr{S}(G)$ induced by $G \setminus \{e\}$, which is denoted by $\mathscr{S}^*(G)$ and is called the *proper order graph* of G.

In this paper, we study the perfect codes of the proper order graph of a finite group. Specifically, we characterize all connected components of a proper order graph and give a necessary and sufficient condition for the connected proper order graph of a group. We also determine the perfect codes of the proper order graphs of a few classes of finite groups, including nilpotent groups, CP-groups, dihedral groups and generalized quaternion groups.

2. CONNECTED COMPONENTS

In this section, we characterize all connected components of a proper order graph (see Proposition 2.1) and give a necessary and sufficient condition for the connected proper order graph of a group (see Corollary 2.3). As applications, we show that for $n \geq 3$, $\mathscr{S}^*(\mathbf{S}_n)$ is connected if and only if $n \neq p$ or $p+1$, where p is a prime (see Theorem 2.5).

Throughout this paper, $\mathbb{Z}_n$ denotes the cyclic group of order n. For an element g of G, the *order* of g is the size of the subgroup $\langle g \rangle$ generated by g, which is denoted by $o(g)$. If $o(x) = 2$ for some $x \in G$, then x is called an *involution*. The *symmetric group* of degree n, denoted by $\mathbf{S}_n$, is the group of all permutations on n letters. Remark that if $n \geq 3$, then $\mathbf{S}_n$ is a non-nilpotent group. Let $\langle x \rangle$ be a cyclic subgroup of G. If $\langle x \rangle \nsubseteq \langle y \rangle$ for any cyclic subgroup $\langle y \rangle$ of G, then $\langle x \rangle$ is called a *maximal cyclic subgroup* of G. The set of all maximal cyclic subgroups of G is denoted by $\mathscr{M}(G)$. It is easy to see that $|\mathscr{M}(G)| = 1$ if and only if

G is cyclic. Note that if $|G| \geq 2$, then $\langle e\rangle \notin \mathscr{M}(G)$. Let Γ be a graph. If $S \subseteq V(\Gamma)$, then the subgraph of Γ induced by S is denoted by $\Gamma[S]$. A *connected component* of a graph Γ is a subgraph of Γ in which any two vertices are connected to each other by paths and which is connected to no additional vertices in Γ. In particular, if Γ is connected, then Γ has a unique connected component which is itself.

Recall that G is a finite group. In the following, write

$$\mathscr{M}(G) = \{M_1, M_2, \ldots, M_t\}. \tag{1}$$

Define a binary relation $\approx$ over $\mathscr{M}(G)$ as follows:

$$M_i \approx M_j \Leftrightarrow (|M_i|, |M_j|) \neq 1, \quad i, j \in \{1, 2, \ldots, t\}.$$

Observe that $\approx$ is reflexive and symmetric. However, $\approx$ does not satisfy transitivity, so it is not an equivalence relation over $\mathscr{M}(G)$. Next, we give another binary relation $\equiv$ over $\mathscr{M}(G)$ as follows:

$$\begin{aligned} M_i \equiv M_j \Leftrightarrow & \text{ either } M_i \approx M_j \text{ or there exist } M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_l} \text{ in } \mathscr{M}(G) \text{ such that} \\ & M_i \approx M_{\alpha_1} \approx M_{\alpha_2} \approx \cdots \approx M_{\alpha_l} \approx M_j, \end{aligned}$$

where $i, j \in \{1, 2, \ldots, t\}$. It is readily seen that the relation $\equiv$ is an equivalence relation on $\mathscr{M}(G)$. Let $\overline{M_i}$ denote the equivalence $\equiv$-class containing M_i for each $1 \leq i \leq t$. Write

$$\widehat{M_i} = \bigcup_{M \in \overline{M_i}} (M \setminus \{e\}).$$

Refer to (1), we denote by

$$\{\overline{M_{\alpha_1}}, \overline{M_{\alpha_2}}, \ldots, \overline{M_{\alpha_k}}\} \tag{2}$$

the set of all equivalence $\equiv$-classes on $\mathscr{M}(G)$, where k is a positive integer. We next give all connected components of $\mathscr{S}^*(G)$.

Proposition 2.1. *With reference to* (2)*, the connected components of* $\mathscr{S}^*(G)$ *are*

$$\mathscr{S}^*(G)[\widehat{M_{\alpha_1}}], \mathscr{S}^*(G)[\widehat{M_{\alpha_2}}], \ldots, \mathscr{S}^*(G)[\widehat{M_{\alpha_k}}]. \tag{3}$$

Proof. By (2), we have $\{\widehat{M_{\alpha_1}}, \widehat{M_{\alpha_2}}, \ldots, \widehat{M_{\alpha_k}}\}$ as a partition of G. Thus, it suffices to prove that $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ is a connected component of $\mathscr{S}^*(G)$ for any $1 \leq i \leq k$. We first prove that $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ is connected. Now let a, b be two distinct vertices of $\widehat{M_{\alpha_i}}$. If $a, b \in \langle c\rangle$, where $\langle c\rangle \in \overline{M_{\alpha_i}}$, then it is easy to see that either $a \sim b$ or $a \sim c \sim b$, as desired. In the following, we assume that $a \in \langle x\rangle$ and $b \in \langle y\rangle$ for distinct $\langle x\rangle, \langle y\rangle \in \overline{M_{\alpha_i}}$. It follows that either $\langle x\rangle \approx \langle y\rangle$ or there are $\langle g_1\rangle, \ldots, \langle g_m\rangle \in \mathscr{M}(G)$ such that $\langle x\rangle \approx \langle g_1\rangle \approx \cdots \approx \langle g_m\rangle \approx \langle y\rangle$. Suppose that $\langle x\rangle \approx \langle y\rangle$. Take a prime $p \mid (o(x), o(y))$ and let $c \in G$ with $o(c) = p$. Then it is easy to see that $a \sim x \sim c \sim y \sim b$ (here a may be equal to x or c), and so a and b are connected. It is similar to the latter case, where we also can obtain that a and b are connected. Thus, $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ is connected, as desired.

We then prove that $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ is a connected component. Namely, for any $g \in V(\mathscr{S}^*(G)) \setminus \widehat{M_{\alpha_i}}$, we have that $N(g) \cap \widehat{M_{\alpha_i}} = \emptyset$. Assume, to the contrary, that there exists $h \in V(\mathscr{S}^*(G)) \setminus \widehat{M_{\alpha_i}}$ such that h is adjacent to some vertex $w \in \widehat{M_{\alpha_i}}$. Let $h \in \langle h'\rangle$ and $w \in \langle w'\rangle$ with $\langle h'\rangle \in \mathscr{M}(G)$ and $\langle w'\rangle \in \overline{M_{\alpha_i}}$. Note that $o(h) \mid o(w)$ or $o(w) \mid o(h)$. It follows that $o(h) \mid (|\langle h'\rangle|, |\langle w'\rangle|)$ or $o(w) \mid (|\langle h'\rangle|, |\langle w'\rangle|)$. As a result, we have that $\langle w'\rangle \approx \langle h'\rangle$, which implies that $\langle h'\rangle \in \overline{M_{\alpha_i}}$. It follows that $h \in \widehat{M_{\alpha_i}}$, a contradiction. □

For a finite group G, denote by $\pi_e(G)$ the set of the orders of all non-trivial elements of G. We next use the following example to illustrate Proposition 2.1.

Example 2.2. *It is easy to see that* $\pi_e(\mathbf{S}_5) = \{2,3,4,5,6\}$. *Thus, every maximal cyclic subgroup of* $\mathbf{S}_5$ *has order* $4,5$ *or* 6. *Let* $M_1 \in \mathscr{M}(\mathbf{S}_5)$ *with* $|M_1| = 4$ *and* $M_2 \in \mathscr{M}(\mathbf{S}_5)$ *with* $|M_2| = 5$. *It follows that there are precisely two equivalence* $\equiv$*-classes on* $\mathscr{M}(\mathbf{S}_5)$, *that is,* $\overline{M_1}$ *and* $\overline{M_2}$. *Hence,* $\mathscr{S}^*(\mathbf{S}_5)$ *has two connected components. In fact, we have*

$$\overline{M_2} = \{x \in \mathbf{S}_5 : o(x) = 5\}, \ \overline{M_1} = \{x \in \mathbf{S}_5 \setminus \{e\} : o(x) \neq 5\}.$$

As an application of Proposition 2.1, we obtain a necessary and sufficient condition for which $\mathscr{S}^*(G)$ is connected.

Corollary 2.3. $\mathscr{S}^*(G)$ *is connected if and only if* $M \equiv N$ *for each two* $M,N \in \mathscr{M}(G)$.

Let n be a positive integer of at least 3. The dihedral group of order $2n$, denoted by D_{2n}, has the following presentation:

$$D_{2n} = \langle a,b : a^n = b^2 = e, bab = a^{-1}\rangle. \tag{4}$$

Note that D_{2n} is non-abelian and $D_{2n} = \langle a\rangle \cup \{ab, a^2b, \ldots, a^{n-1}b, b\}$, where a^ib is an involution for every $0 \leq i \leq n-1$. It is clear that

$$\mathscr{M}(D_{2n}) = \{\langle a\rangle, \langle ab\rangle, \langle a^2b\rangle, \ldots, \langle b\rangle\}. \tag{5}$$

Suppose that $m \geq 2$ is a positive integer. Johnson [15] defined the generalized quaternion group Q_{4n} of order $4n$, which has a presentation as follows:

$$Q_{4n} = \langle x,y : x^n = y^2, y^4 = x^{2n} = e, y^{-1}xy = x^{-1}\rangle. \tag{6}$$

It is clear that Q_{4n} is a non-abelian group for each $n \geq 2$. Furthermore, it is easy to see that Q_{4n} has a unique involution x^n. Note that $o(x^iy) = 4$ for any $1 \leq i \leq 2n$. It is easy to check that

$$Q_{4n} = \langle x\rangle \cup \{x^iy : 1 \leq i \leq 2n\} \tag{7}$$

and

$$\mathscr{M}(Q_{4n}) = \{\langle x\rangle, \langle xy\rangle, \langle x^2y\rangle, \ldots, \langle x^ny\rangle\}. \tag{8}$$

As applications of Proposition 2.1 and Corollary 2.3, we determine the connected components of $\mathscr{S}^*(D_{2n})$ and $\mathscr{S}^*(Q_{4n})$.

Proposition 2.4. *Let* D_{2n} *be the dihedral group as presented in* (4), *and let* Q_{4n} *be the generalized quaternion group as presented in* (6). *If* n *is even, then* $\mathscr{S}^*(D_{2n})$ *is connected; otherwise,* $\mathscr{S}^*(D_{2n})$ *has two connected components* $\mathscr{S}^*(D_{2n})[\langle a\rangle \setminus \{e\}]$ *and* $\mathscr{S}^*(D_{2n})[D_{2n} \setminus \langle a\rangle]$. *Moreover,* $\mathscr{S}^*(Q_{4n})$ *is connected.*

Proof. The desired results trivially follow from Proposition 2.1, Corollary 2.3, equalities (5) and (8). □

We conclude the section by an application of Corollary 2.3, which determines all positive integers $n \geq 3$ such that $\mathscr{S}^*(\mathbf{S}_n)$ is connected. In fact, the following result has been obtained (see [10, Theorem 2.32]).

Theorem 2.5. *For* $n \geq 3$, $\mathscr{S}^*(\mathbf{S}_n)$ *is connected if and only if* $n \neq p$ *or* $p+1$, *where* p *is a prime.*

Proof. We first show that if $n = p$ or $p+1$ for some prime p, then $\mathscr{S}^*(\mathbf{S}_n)$ is not connected. Suppose now that $n = p$ or $p+1$, where p is a prime. Let

$$A = \{x \in \mathbf{S}_n : o(x) = p\}.$$

Suppose for a contradiction that there exists a nontrivial element y in $\mathbf{S}_n \setminus A$ such that y is adjacent to a vertex $x \in A$. Then it must be $p \mid o(y)$ and $o(y) \neq p$. Let $o(y) = pk$ for some $k \geq 2$. Note that y is a product of disjoint cycles of lengths of at least 2 and the largest cycle of $\mathbf{S}_n$ is $(1,2,\cdots,p)$. Thus, such y does not exist, a contradiction. We conclude that $\mathscr{S}^*(\mathbf{S}_n)[A]$ is a connected component of $\mathscr{S}^*(\mathbf{S}_n)$ and is complete. As a result, $\mathscr{S}^*(\mathbf{S}_n)$ is not connected, as desired.

Now it suffices to show that if $n \neq p$ or $p+1$ for some prime p, then $\mathscr{S}^*(\mathbf{S}_n)$ is connected. Suppose that $n \neq p$ or $p+1$, where p is a prime. Let q be the maximum prime with $q < n$. Then we must have $q \geq 7$ and $n-q \geq 2$ since $n \neq p$ or $p+1$. Note that $|\mathscr{M}(\mathbf{S}_n)| \geq 2$ since $\mathbf{S}_n$ is not cyclic. Now take distinct $M_1, M_2 \in \mathscr{M}(\mathbf{S}_n)$. By Corollary 2.3, it is enough to prove $M_1 \equiv M_2$. Let $q_1 \mid |M_1|$ and $q_2 \mid |M_2|$ be two primes. If $q_1 = q_2$, then it is clear that $M_1 \equiv M_2$, as desired. Thus, in the following we may assume $q_1 \neq q_2$.

Suppose that one of q_1 and q_2 is equal to q. Without a loss of generality, let $q_1 = q$. Since $q \geq 7$ and $n-q \geq 2$, we have that $\mathbf{S}_n$ has an element of order $2q$. Let $M_3 \in \mathscr{M}(\mathbf{S}_n)$ with $2q \mid |M_3|$ (here M_3 may be M_1). If $q_2 = 2$, then clearly, $M_1 \approx M_3 \approx M_2$, and so $M_1 \equiv M_2$, as desired. Now we may assume that $q_2 \neq 2$. It follows that $\mathbf{S}_n$ has an element of order $2q_2$. Let $M_4 \in \mathscr{M}(\mathbf{S}_n)$ with $2q_2 \mid |M_4|$ (here M_4 may be M_2). As a result, we have $M_1 \approx M_3 \approx M_4 \approx M_2$, and so $M_1 \equiv M_2$, as desired. Similarly, the desired result holds for this case $q_1 \neq q$ and $q_2 \neq q$. The proof is now complete. □

3. PERFECT CODES

In this section, we determine the perfect codes of the proper order graphs of a few classes of finite groups, including nilpotent groups (see Theorem 3.4), CP-groups (see Theorem 3.5), dihedral groups (see Theorem 3.7) and generalized quaternion groups (see Theorem 3.8).

For a finite group G, let $S \subseteq G$. We use $\pi(G)$ to denote the set of all prime divisors of $|G|$, and let

$$\pi(S) = \{p : \text{there exists an element } x \in S \text{ such that } o(x) = p \text{ is a prime}\}.$$

Clearly, $\pi(S) \subseteq \pi(G)$. The *exponent* of S, denoted by $\exp(S)$, is the least common multiple of the orders of elements in S. If S has an element such that its order is equal to $\exp(S)$, then S is called a *full exponent*; in particular, if S is a subgroup, then S is called a *full exponent group*. For example, abelian groups are full exponent groups.

Refer to (3), let $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ be a connected component of $\mathscr{S}^*(G)$. In the following, we study the perfect codes of $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$. For convenience, we assume that

$$\overline{M_{\alpha_i}} = \{M_1, M_2, \ldots, M_s\} \subseteq \mathscr{M}(G). \tag{9}$$

Lemma 3.1. *$\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ admits a perfect code of size one if and only if $\widehat{M_{\alpha_i}}$ is a full exponent.*

Proof. Suppose that $\widehat{M_{\alpha_i}}$ is a full exponent. Let $g \in \widehat{M_{\alpha_i}}$ with $o(g) = \exp(\widehat{M_{\alpha_i}})$. It is clear that $\{g\}$ is a perfect code of $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$. For the converse, suppose that $\mathscr{S}^*(G)[\widehat{M_{\alpha_i}}]$ has a perfect code of size one, say $\{a\}$. Let $\pi(\widehat{M_{\alpha_i}}) = \{p_1, p_2, \ldots, p_t\}$. If $t = 1$, then $\widehat{M_{\alpha_i}}$ is a full exponent since the order of any element of $\widehat{M_{\alpha_i}}$ is a power of p_1, as desired.

In the following, let $t \geq 2$. Clearly, $o(a)$ is not a prime power. Let x be an element of $\widehat{M_{\alpha_i}}$ with order p_i^n, where $1 \leq i \leq t$ and n is a positive integer. It follows that $p_i^n \mid o(a)$, and, as a result, we have $o(a) = \exp(\widehat{M_{\alpha_i}})$, as desired. □

Corollary 3.2. *For a group G, $\mathscr{S}^*(G)$ admits a perfect code of size one if and only if G is a full exponent. In particular, if G is a full exponent, then $\{a\}$ is a perfect code of $\mathscr{S}^*(G)$, where $o(a) = \exp(G)$.*

For some $M \in \mathscr{M}(G)$, if there is no maximal cyclic subgroup M' such that $|M|$ is a proper divisor of $|M'|$, then M is called a *maximal order cyclic subgroup* of G. Denote by $\mathscr{M}_o(G)$ the set of all maximal order cyclic subgroups of G.

Lemma 3.3. *If G satisfies the following two conditions:*

(a) *there exist M_1, M_2 in $\mathscr{M}_o(G)$ such that $|M_1| = p^n, |M_2| = p^a q^b r^c h$, where p, q, r are pairwise distinct primes, $n \geq 2$, $1 \leq a < n$, $b \geq 1$, $c \geq 1$, $h \geq 1$, and $(h, pqr) = 1$;*

(b) *if G has no elements with order $q^{b'} r^{c'} l$, where l is a positive integer, and one of $b' > b, c' \geq c$ and $b' \geq b, c' > c$ occurs, then $\mathscr{S}^*(G)$ does not admit perfect codes.*

Proof. Suppose for a contradiction that $\mathscr{S}^*(G)$ admits a perfect code, say C. Let $M_1 = \langle x \rangle$ and $M_2 = \langle y \rangle$. Note that $o(x) = p^n$ and $\langle x \rangle \in \mathscr{M}_o(G)$. Since C is a perfect code and $p^n \geq 3$, we must have that one of x and x^{-1} does not belong to C. Without loss of generality, assume $x \notin C$, and similarly, let $y \notin C$. It follows that there exists $z \in C$ such that $o(z) = p^m$, where $1 \leq m \leq n$. In the following, we consider two cases.

Case 1. $m = 1$.

Since $q^b r^c \in \pi_e(G)$ and $q^b r^c \geq 3$, we have that there exists $w \in C$ such that $o(w) \mid q^b r^c$ or $q^b r^c \mid o(w)$. If $o(w) \mid q^b r^c$, then it is easy to see that there exists an element y' of order $p \cdot o(w)$ in $V(\mathscr{S}^*(G)) \setminus C$, which is impossible because both z and w are adjacent to y'. It follows that $q^b r^c \mid o(w)$. In view of this condition (b), it must be that $o(w) = q^b r^c$. However, $y \notin C$, $o(z) \mid o(y)$ and $o(w) \mid o(y)$: this contradicts that C is a perfect code.

Case 2. $m \geq 2$.

Note that in this case, every element of order p must not belong to C. Let $u \in G$ with $o(u) = p$. Then $u \notin C$. Since $o(pq) \geq 3$, we have that there exists an element of order pq in $V(\mathscr{S}^*(G)) \setminus C$. It follows that there exists $d \in C$ such that $o(d) \mid pq$ or $pq \mid o(d)$. If $pq \mid o(d)$, then both z and d are adjacent to u, a contradiction. We conclude that $o(d) \mid pq$ is a prime, and it must be $o(d) = q$. Similarly, we consider the elements of order pr, and we deduce that C has an element v of order r. However, as $qr \geq 3$, $V(\mathscr{S}^*(G)) \setminus C$ has an element of order qr, which is adjacent to both d and v. This contradicts that C is a perfect code. □

As we all know, a finite group is a nilpotent group if and only if this group is the direct product of its Sylow subgroups. In particular, in a nilpotent group, two elements of different prime orders can commute. Thus, a nilpotent group is a full exponent. The following result follows from Corollary 3.2.

Theorem 3.4. *Let G be a nilpotent group. Then $\mathscr{S}^*(G)$ admits perfect codes. In particular, if $a \in G$ with $o(a) = \exp(G)$, then $\{a\}$ is a perfect code of $\mathscr{S}^*(G)$.*

If every nontrivial element of a finite group has prime power order, then this finite group is called a *CP-group* [8]. For example, the alternating group of degree five is a CP-group. Moreover, for some prime p, any p-group is a CP-group. All finite CP-groups were characterized in [8, Theorem 4].

Theorem 3.5. *Let G be a CP-group with $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. Then $\mathscr{S}^*(G)$ has perfect codes. In particular, every perfect code of $\mathscr{S}^*(G)$ has the following form:*

$$\{x_1, x_2, \ldots, x_k\}, \tag{10}$$

where $\alpha_i \geq 1$ and $o(x_i) = p_i^{\alpha_i}$ for all $1 \leq i \leq k$.

Proof. Note that G is a CP-group with $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. By Proposition 2.1, it is easy to see that $\mathscr{S}^*(G)$ has k connected components. Moreover, for a fixed integer i ($1 \leq i \leq k$), the set of all elements of order p_i^l induces a connected component of $\mathscr{S}^*(G)$, where $l \geq 1$. Clearly, any connected component of $\mathscr{S}^*(G)$ is complete. As a result, $\mathscr{S}^*(G)$ has perfect codes. □

In the following, we use the following example to illustrate Theorem 3.5.

Example 3.6. *For the alternating group $\mathbf{A}_5$, we have $\pi_e(\mathbf{A}_5) = \{2,3,5\}$, so $\mathbf{A}_5$ is a CP-group. It is easy to see that $\mathscr{S}^*(\mathbf{A}_5)$ has 3 connected components, and the set*

$$\{(1,2)(3,4),(1,2,3),(1,2,3,4,5)\}$$

is a perfect code of $\mathscr{S}^*(\mathbf{A}_5)$.

For the dihedral group D_{2n}, as presented in (4), we have

$$\mathscr{M}_o(D_{2n}) = \begin{cases} \mathscr{M}(D_{2n}), & \text{if } n \text{ is odd;} \\ \{\langle a \rangle\}, & \text{if } n \text{ is even.} \end{cases} \tag{11}$$

Theorem 3.7. *Suppose that D_{2n} is the dihedral group as presented in* (4)*.*
(a) *If n is even, then $\mathscr{S}^*(D_{2n})$ has perfect codes, and, particularly, $\{a\}$ is a perfect code of $\mathscr{S}^*(D_{2n})$;*
(b) *if n is odd, then $\mathscr{S}^*(D_{2n})$ has perfect codes, and, particularly, $\{a,b\}$ is a perfect code of $\mathscr{S}^*(D_{2n})$.*

Proof. (a) Suppose that n is an even number. Then, by (11), we have that D_{2n} is a full exponent and a is an element with order $\exp(D_{2n})$. Thus, the required result holds by Corollary 3.2.

(b) Suppose that n is an odd number. Then (11) implies that $\mathscr{S}^*(D_{2n})$ has two connected components, where a connected component consisting of all involutions is complete, and another connected component consisting of $\langle a \rangle$ is a full exponent. As a result, every perfect code has a form $\{x,y\}$, where $o(x) = n$ and $o(y) = 2$. In particular, $\{a,b\}$ is a perfect code of $\mathscr{S}^*(D_{2n})$, as desired. □

In the following, we determine the generalized quaternion groups for which $\mathscr{S}^*(Q_{4n})$ has perfect codes.

Theorem 3.8. *Let Q_{4n} be the generalized quaternion group as presented in* (6)*.*
(a) *If n is even, then $\mathscr{S}^*(Q_{4n})$ has perfect codes, and, particularly, $\{x\}$ is a perfect code of $\mathscr{S}^*(Q_{4n})$;*
(b) *if $n = p^m$, where p is an odd prime and $m \geq 1$, then $\mathscr{S}^*(Q_{4n})$ has perfect codes, and every perfect code has a form $\{a,b\}$ with $o(a) = 4$ and $o(b) = p$. Particularly, $\{x^{2p^{m-1}},y\}$ is a perfect code of $\mathscr{S}^*(Q_{4n})$;*
(c) *if $n = p^m q^t l$, where p,q are two distinct odd primes and l is an odd integer with $(l,pq) = 1$, then $\mathscr{S}^*(Q_{4n})$ has no perfect codes.*

Proof. (a) Suppose that n is even. Note that if $n = 2$, then $\mathscr{S}^*(Q_{4n})$ is complete, and the required result follows. Now assume that $n \geq 3$. By (7) and (8), it is easy to see that $\mathscr{M}_o(Q_{4n}) = \{\langle x \rangle\}$. It follows that $\mathscr{S}^*(Q_{4n})$ is connected and a full exponent. Moreover, x is an element with order $\exp(Q_{4n})$. Thus, the required result follows from Corollary 3.2.

(b) Suppose that $n = p^m$, where p is an odd prime and $m \geq 1$. Then (7) and (8) imply that $\mathscr{M}_o(Q_{4n}) = \mathscr{M}(Q_{4n})$ and

$$\pi_e(Q_{4n}) = \{2,4,p,p^2,\ldots,p^m,2p,2p^2,\ldots,2p^m\}. \tag{12}$$

Suppose that $\mathscr{S}^*(Q_{4n})$ has a perfect code, say C. Since $\langle y \rangle \in \mathscr{M}_o(Q_{4n})$ and $o(y) = 4$, we have that C has an element a with $o(a) = 4$ or 2. Assume, to the contrary, that $o(a) = 2$. Then there exists an element $b \in C$ such that b is adjacent to an element of order p^m in $\mathscr{S}^*(Q_{4n})$. By (12), it must be $o(b) = p^k$ for some $1 \leq k \leq m$. On the other hand, $V(\mathscr{S}^*(Q_{4n})) \setminus C$ must contain an element c of order $2p^k$, which is a contradiction because both a and b are adjacent to c.

We conclude that $o(a) = 4$. Next we consider the elements of order $2p$. It follows that $V(\mathscr{S}^*(Q_{4n})) \setminus C$ must contain an element u of order $2p$. Thus, there exists an element $b \in C$ such that b is adjacent to u. Clearly, C has no involutions. If C has an element c of order $2p^k$ for some $1 \leq k \leq m$, then the unique involution is adjacent to both a and c, which is impossible. As a result, it follows that b must have order p. Now it is easy to check that $\{a,b\}$ is a perfect code of $\mathscr{S}^*(Q_{4n})$ by (12), as desired.

(c) Suppose that $n = p^m q^t l$, where p,q are two distinct odd primes and l is an odd integer with $(l,pq) = 1$. (7) and (8) also imply that $\mathscr{M}_o(Q_{4n}) = \mathscr{M}(Q_{4n})$. Now $\langle x \rangle, \langle y \rangle \in \mathscr{M}_o(Q_{4n})$, $o(x) = 4$, $o(y) = 2p^m q^t l$, and

$(l, 2pq) = 1$. Moreover, by (7), we have that Q_{4n} has no elements of order $p^{m'}q^{t'}l'$, where l' is a positive integer, and one of $m' > m, t' \geq t$ and $m' \geq m, t' > t$ occurs. It follows from Lemma 3.3 that $\mathscr{S}^*(Q_{4n})$ has no perfect codes. □

4. CONCLUSION

Hamzeh and Ashrafi first introduced the order graph of a group. Since the identity element e of a group G is always adjacent to any non-trivial element, the subgraph of $\mathscr{S}(G)$ induced by $G \setminus \{e\}$, which is denoted by $\mathscr{S}^*(G)$ and called the proper order graph of G, is the notion to be considered.

This article mainly studied the perfect codes of the proper order graph of a finite group. We characterized all connected components of a proper order graph and gave a necessary and sufficient condition for the connected proper order graph of a group. In particular, applying the results on symmetric groups, we proved that $\mathscr{S}^*(S_n)$ is connected if and only if $n \neq p$ or $p+1$, where p is a prime. We, then, determined the perfect codes of the proper order graphs of a few classes of finite groups including nilpotent groups, CP-groups, dihedral groups and generalized quaternion groups.

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Lõplike rühmade järkude pärisgraafide sidusad komponendid ja perfektsed koodid

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Olgu G lõplik rühm ühikelemendiga e. Rühma G järkude pärisgraaf, mida tähistatakse $S^*(G)$, on graaf tippudega $G \setminus \{e\}$ ning selle tipud x ja y on kaastipud, kui $o(x)|o(y)$ või $o(y)|o(x)$; $o(x)$ ja $o(y)$ on vastavalt elementide x ja y järgud. Selles artiklis uuritakse graafi $S^*(G)$ perfektseid koode. Leitakse graafi kõik sidusad komponendid ning tarvilikud ja piisavad tingimused graafi sidususeks. Samuti leitakse järkude pärisgraafide perfektsed koodid mõningate lõplike rühmade klasside puhul (nilpotentsed rühmad, CP-rühmad, dieedri rühmad ja üldistatud kvaternioonide rühmad).