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OPTICAL SOLITONS

Highly dispersive optical soliton perturbation with Kerr law for complex Ginzburg–Landau equation

Ming-Yue Wang[a], Anjan Biswas[b,c,d,e*], Yakup Yıldırım[f,g], Maggie Aphane[e], Seithuti P. Moshokoa[h] and Abdulah A. Alghamdi[c]

[a] Department of Mathematics, Northeast Petroleum University, Daqing 163318, China
[b] Department of Mathematics and Physics, Grambling State University, Grambling, LA 71245, USA
[c] Mathematical Modeling and Applied Computation (MMAC) Research Group, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[d] Department of Applied Sciences, Cross-Border Faculty, Dunarea de Jos University of Galati, 111 Domneasca Street, Galati 800201, Romania
[e] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, South Africa
[f] Department of Computer Engineering, Biruni University, Istanbul 34010, Turkey
[g] Department of Mathematics, Near East University, Nicosia 99138, Cyprus
[h] Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria 0008, South Africa

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Abstract. In this paper, highly dispersive optical solitons are obtained with the perturbed complex Ginzburg–Landau equation, incorporating the Kerr law of nonlinearity, by the complete discriminant classification approach. A variety of solutions emerge from this scheme that include solitons, periodic solutions and doubly periodic solutions. The numerical sketches support the analytical findings.

Keywords: solitons, discriminant classification, dispersive.

1. INTRODUCTION

One of the lesser-known models that is studied in the context of soliton transmission through optical fibers across intercontinental distances is the complex Ginzburg–Landau equation (CGLE) apart from the frequently studied model, namely the nonlinear Schrödinger equation. The CGLE has been studied in detail [1–9]. The exact solutions to the perturbed CGLE with Kerr and cubic–quintic–septic law nonlinearities are obtained using the trial equation method and a complete discriminant system [1]. The exact bright and dark soliton solutions of the CGLE with parabolic and dual-power law nonlinearities are obtained by the usage of the solitary wave ansatz [2]. Cubic–quartic optical solitons with the perturbed CGLE, having six forms of self-phase modulation structures, are retrieved by the aid of the enhanced Kudryashov method [3]. The highly dispersive bright 1-soliton solution for the perturbed CGLE with three forms of nonlinear refractive

* Corresponding author, biswas.anjan@gmail.com

index structures is recovered by virtue of the semi-inverse variational principle [4]. The conservation law for the cubic–quartic CGLE with five nonlinear forms is exhibited by the aid of the Lie symmetry [5]. A spectrum of cubic–quartic optical solitons with the CGLE, having Hamiltonian-type perturbation terms, are secured by the aid of powerful and prolific integration structures [6]. The first integrals and exact solutions of the CGLE are found using traveling wave reduction [7]. Numerical solutions of the CGLE are also given to establish approximate solutions of the model, using a linearized element-free Galerkin method [8]. The dynamics of dissipative solitons in a fractional CGLE is addressed with the aid of variational approximation [9].

Very recently, CGLE has gained popularity with the emerging concept of highly dispersive (HD) optical solitons [10–15], where high-order nonlinear differential equations describing the propagation of pulses in an optical fiber are studied by using a method for finding HD solitary wave solutions [10]. HD optical solitons with a nonlinear sixth-order differential equation, having various polynomial nonlinearities, are handled in [11]. HD optical solitons for the generalized nonlinear eighth-order Schrödinger equation with the third, fifth, seventh and ninth power of nonlinearity are studied in [12]. HD optical solitons with the perturbed nonlinear Schrödinger equation, having dispersion terms of all orders and containing Kudryashov's sextic power-law of self-phase modulation, are secured using the trial equation method [13]. Ultrashort light pulse propagation through an inhomogeneous monomodal optical fiber exhibiting HD effects is addressed in [14]. Quartic and dipole solitons in an HD optical waveguide with self-steepening nonlinearity and varying parameters are reported in [15].

The concept of HD solitons was defined a couple of years ago when chromatic dispersion (CD) was supplemented with additional dispersion effects for its possible low count. These additional dispersive effects came from inter-modal dispersion (IMD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD) and sixth-order dispersion (6OD). These dispersive terms together with the pre-existing CD collectively produce HD solitons as modeled by the CGLE. The current paper is a study of this model in the presence of perturbation terms. The integration methodology is the complete discriminant classification approach [16–22]. The governing model is first transformed into an ordinary differential equation (ODE), which is subsequently integrated based on the structural classification of the corresponding discriminant. This yields a variety of soliton solutions to the model in addition to other solutions that are also listed. The details of the derivation are outlined in the rest of the paper after the model is introduced, followed by some mathematical preliminaries.

1.1. Governing model

The complex Ginzburg–Landau equation with additional dispersion effects is presented as below:

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx}$$

$$+\frac{1}{|q|^2 q^*}\left[\alpha |q|^2 \left(|q|^2\right)_{xx} - \beta \left\{\left(|q|^2\right)_x\right\}^2\right] + F\left(|q|^2\right) q = i\left[\lambda \left(|q|^{2m} q\right)_x + \theta \left(|q|^{2m}\right)_x q + \sigma |q|^{2m} q_x\right], \quad (1)$$

where a_1, a_2, a_3, a_4, a_5 and a_6 give IMD, CD, 3OD, 4OD, 5OD and 6OD, in sequence. α and β come from nonlinear effects. λ, θ and σ stem from the self-steepening effect, self-frequency shift and nonlinear dispersion, in sequence. x depicts spatial variable, whereas $q(x,t)$ denotes the wave profile. t implies to temporal variable, while m depicts full nonlinearity. The first term also stems from temporal evolution, where $i = \sqrt{-1}$, while F comes from self-phase modulation. The function $F\left(|q|^2\right) q$ is a real-valued algebraic function and is k times continuously differentiable, so that

$$F\left(|q|^2\right) q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n) \times (-m,m); R^2\right).$$

The function $F\left(|q|^2\right)q$ for the Kerr law of nonlinear form turns out to be

$$F\left(|q|^2\right)q = b_0 |q|^2 q,$$

where b_0 is the arbitrary constant. The model with the Kerr law of nonlinear form is therefore structured as below:

$$\begin{aligned} &iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} \\ &+\frac{1}{|q|^2 q^*}\left[\alpha |q|^2 \left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + b_0 |q|^2 q = i\left[\lambda\left(|q|^6 q\right)_x + \theta\left(|q|^6\right)_x q + \sigma |q|^6 q_x\right], \end{aligned} \tag{2}$$

where b_0 stems from the Kerr law of nonlinearity and the nonlinear parameter appears with $m = 3$.

2. MATHEMATICAL START-UP

The starting hypothesis is given by

$$s = x - vt, \quad q(x,t) = g(s)e^{i(-kx+\omega t+\theta_0)}. \tag{3}$$

Here, $g(s)$ comes from the amplitude component, where s is the wave variable and v is the velocity. Also, from the phase component, θ_0 is the phase constant, ω is the wave number and κ is the frequency.

Inserting Eq. (3) into Eq. (2) leaves us with the simplest equations

$$P_1 g^2 + P_2 gg'' + P_3 gg^{(iv)} + a_6 gg^{(vi)} + 2(\alpha - 2\beta)(g')^2 + b_0 g^4 - k(\lambda + \sigma)g^8 = 0 \tag{4}$$

and

$$\begin{aligned} &\{7\lambda + 6\theta + \sigma\}g^6 g' + \left(v - a_1 + 2a_2 k + 3a_3 k^2 - 4a_4 k^3 - 5a_5 k^4 + 6a_6 k^5\right) g' \\ &- \left(a_3 - 4a_4 k - 10a_5 k^2 + 20a_6 k^3\right) g''' - \left(a_5 - 6a_6 k\right) g^{(v)} = 0, \end{aligned} \tag{5}$$

where

$$P_1 = -a_6 k^6 + a_4 k^4 + a_5 k^5 - a_3 k^3 - a_2 k^2 + a_1 k - \omega,$$
$$P_2 = a_2 + 2\alpha + 3a_3 k - 6a_4 k^2 - 10a_5 k^3 + 15a_6 k^4$$

and

$$P_3 = a_4 + 5a_5 k - 15a_6 k^2.$$

Eq. (5) provides us the velocity

$$v = a_1 - 2a_2 k - 3a_3 k^2 + 4a_4 k^3 + 5a_5 k^4 - 6a_6 k^5, \tag{6}$$

by the aid of the constraints

$$\begin{aligned} &7\lambda + 6\theta + \sigma = 0, \\ &a_3 - 4a_4 k - 10a_5 k^2 + 20a_6 k^3 = 0, \\ &a_5 = 6a_6 k. \end{aligned} \tag{7}$$

Consider the trial equation

$$(g')^2 = \sum_{i=0}^{n} c_i g^i. \tag{8}$$

Substituting Eq. (8) into Eq. (4) and then balancing $-k(\lambda+\sigma)g^8$ and $a_6 gg^{(vi)}$ simplifies Eq. (8) to

$$(g')^2 = c_4 g^4 + c_3 g^3 + c_2 g^2 + c_1 g + c_0, \tag{9}$$

where

$$\begin{aligned}
c_4 &= (\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{3}},\\
c_3 &= 0,\\
c_2 &= -\frac{P_3}{70a_6},\\
c_1 &= -\frac{2}{105a_6},\\
c_0 &= \frac{P_3}{14700a_6^2(\alpha-2\beta)}
\end{aligned} \tag{10}$$

and c_4, c_2, c_1 and c_0 satisfy the restrictions

$$\begin{aligned}
&b_0+2(\alpha-2\beta)c_4+182a_6c_2^2c_4+504a_6c_0c_4^2+2c_4P_2=0,\\
&8c_2c_4+210a_6c_1c_2c_4+12c_1c_4P_3=0,\\
&2(\alpha-2\beta)c_2+a_6c_2^3+3c_1c_4+45a_6c_1^2c_4+132a_6c_0c_2c_4+c_2P_2+12c_0c_4P_3+P_1=0,\\
&2(\alpha-2\beta)c_1+c_2^2+\frac{1}{2}a_6c_1c_2^2+36a_6c_0c_1c_4+\frac{1}{2}c_1P_2=0.
\end{aligned} \tag{11}$$

Setting

$$h=(c_4)^{\frac{1}{4}}g, \quad s_1=(c_4)^{\frac{1}{4}}s, \tag{12}$$

Eq. (9) comes out as

$$(h_{s_1})^2=h^4+d_2h^2+d_1h+d_0, \tag{13}$$

where

$$d_2=c_2(c_4)^{-\frac{1}{2}},\ d_1=c_1(c_4)^{-\frac{1}{4}},\ d_0=c_0.$$

Rewrite Eq. (13) as

$$\pm(s_1-s_0)=\int\frac{dh}{\sqrt{F(h)}}, \tag{14}$$

where

$$F(h)=h^4+d_2h^2+d_1h+d_0.$$

Next, we give the discriminant system [16–22]:

$$\begin{aligned}
D_1 &= 1,\\
D_2 &= -d_1,\\
D_3 &= -2d_1^3+8d_1d_3-9d_2^2,\\
D_4 &= -d_1^3d_2^2+4d_1^4d_3+36d_1d_2^2d_3-32d_1^2d_3^2-\frac{27}{4}d_2^4+64d_3^3,\\
E_2 &= 9d_2^2-32d_1d_3.
\end{aligned} \tag{15}$$

By classifying the roots of $F(h)$, we arrive at:
(1) $D_4>0\&((D_2>0\&D_3\le 0)\|D_2\le 0)$, then $F(h)=[(h-\varepsilon_1)^2+\varepsilon_2^2][(h-\varepsilon_3)^2+\varepsilon_4^2]$,
(2) $D_4<0\&((D_2<0\&D_3<0)\|(D_2=0\&D_3\le 0)\|D_2>0)$, then $F(h)=(h-\varepsilon_1)(h-\varepsilon_2)[(h-\varepsilon_3)^2+\varepsilon_4^2]$,
(3) $D_4>0$, $D_3>0$, $D_2>0$, then $F(h)=(h-\varepsilon_1)(h-\varepsilon_2)(h-\varepsilon_3)(h-\varepsilon_4)$,
(4) $D_4=0$, $D_3<0$, then $F(h)=(h-\varepsilon_1)^2[(h-\varepsilon_2)^2+\varepsilon_3^2]$,
(5) $E_2=D_4=D_3=0$, $D_2>0$, then $F(h)=(h-\varepsilon_1)^3(h-\varepsilon_2)$,
(6) $E_2<0$, $D_4=D_3=0$, $D_2<0$, then $F(h)=[(h-\varepsilon_1)^2+\varepsilon_2^2]^2$,
(7) $D_4=0$, $D_3>0$, $D_2>0$, then $F(h)=(h-\varepsilon_1)^2(h-\varepsilon_2)(h-\varepsilon_3)$,
(8) $E_2>0$, $D_4=D_3=0$, $D_2>0$, then $F(h)=(h-\varepsilon_1)^2(h-\varepsilon_2)^2$,
(9) $D_4=0$, $D_3=0$, $D_2=0$, then $F(h)=h^4$,
where $\varepsilon_i(i\le i\le 4)$ are constants.

3. THE OPTICAL WAVE PATTERNS

Case 1. $D_4 = 0, D_3 = 0, D_2 = 0$, then

$$\pm(s_1 - s_0) = \int \frac{dh}{h^2}. \tag{16}$$

In this case, a singular rational pattern comes out as

$$q_1 = \left\{ (\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}[-((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s - s_0)]^{-1} \right\} e^{i(-kx+\omega t+\theta_0)}. \tag{17}$$

Case 2. $E_2 > 0, D_4 = D_3 = 0, D_2 > 0$, then

$$\pm(s_1 - s_0) = \int \frac{dh}{(h-\varepsilon_1)(h-\varepsilon_2)}. \tag{18}$$

As a result, optical singular and dark soliton patterns read as

$$q_2 = \left\{ (\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}[\frac{\varepsilon_2-\varepsilon_1}{2}(\coth\frac{(\varepsilon_1-\varepsilon_2)[(\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s - s_0]}{2} - 1) + \varepsilon_2] \right\} e^{i(-kx+\omega t+\theta_0)} \tag{19}$$

and

$$q_3 = \left\{ (\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}[\frac{\varepsilon_2-\varepsilon_1}{2}(\tanh\frac{(\varepsilon_1-\varepsilon_2)[(\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s - s_0]}{2} - 1) + \varepsilon_2] \right\} e^{i(-kx+\omega t+\theta_0)}. \tag{20}$$

Case 3. $D_4 = 0, D_3 > 0, D_2 > 0$, then

$$\pm(s_1 - s_0) = \int \frac{dh}{(h-\varepsilon_1)\sqrt{(h-\varepsilon_2)(h-\varepsilon_3)}}. \tag{21}$$

An optical bright soliton pattern is thus defined as

$$q_4 = \left\{ \frac{2(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)}{(\varepsilon_2-\varepsilon_3)\cosh[\sqrt{(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s - s_0)] - (2\varepsilon_1-\varepsilon_2-\varepsilon_3)} \right\} e^{i(-kx+\omega t+\theta_0)}, \tag{22}$$

while a singular periodic pattern is therefore introduced as below:

$$q_5 = \left\{ \frac{2(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)}{\pm(\varepsilon_2-\varepsilon_3)\sin[\sqrt{-(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s - s_0)] - (2\varepsilon_1-\varepsilon_2-\varepsilon_3)} \right\} e^{i(-kx+\omega t+\theta_0)}. \tag{23}$$

Case 4. $E_2 < 0, D_4 = D_3 = 0, D_2 < 0$, then

$$\pm(s_1 - s_0) = \int \frac{dh}{(h-\varepsilon_1)^2 + \varepsilon_2^2}. \tag{24}$$

Hence, a singular periodic pattern evolves as

$$q_6 = \left\{ (\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}[\varepsilon_2\sin(\varepsilon_2(\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s - s_0) + \varepsilon_1] \right\} e^{i(-kx+\omega t+\theta_0)}. \tag{25}$$

Case 5. $E_2 = D_4 = D_3 = 0, D_2 > 0$, then

$$\pm(s_1 - s_0) = \int \frac{dh}{\sqrt{(h-\varepsilon_1)^3(h-\varepsilon_2)}}. \tag{26}$$

As a result, a rational singular pattern stands as

$$q_7=\left\{(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}[\varepsilon_1+\frac{4(\varepsilon_1-\varepsilon_2)}{(\varepsilon_2-\varepsilon_1)^2[(\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0]^2-4}]\right\}e^{i(-kx+\omega t+\theta_0)}. \tag{27}$$

Case 6. $D_4=0$, $D_3<0$, then

$$\pm(s_1-s_0)=\int\frac{dh}{(h-\varepsilon_1)\sqrt{(h-\varepsilon_2)^2+\varepsilon_3^2}}. \tag{28}$$

Consequently, an exponential pattern sticks out as

$$q_8=\left\{(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}\frac{e^{\pm\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0)}-\frac{\varepsilon_1-2\varepsilon_2}{\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}}+2\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}-(\varepsilon_1-2\varepsilon_2)}{(e^{\pm\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}((-\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s_1-s_0)}-\frac{\varepsilon_1-2\varepsilon_2}{\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}})^2-1}\right\}$$

$$\times e^{i(-kx+\omega t+\theta_0)}. \tag{29}$$

Case 7. $D_4>0$, $D_3>0$, $D_2>0$, then

$$\pm(s_1-s_0)=\int\frac{dh}{\sqrt{(h-\varepsilon_1)(h-\varepsilon_2)(h-\varepsilon_3)(h-\varepsilon_4)}}. \tag{30}$$

In this case, two double periodic patterns shape up as

$$q_9=\left\{(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}\frac{\varepsilon_2(\varepsilon_1-\varepsilon_4)sn^2(\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0),m)-\varepsilon_1(\varepsilon_2-\varepsilon_4)}{(\varepsilon_1-\varepsilon_4)sn^2(\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0),m)-(\varepsilon_2-\varepsilon_4)}\right\}e^{i(-kx+\omega t+\theta_0)}, \tag{31}$$

$$q_{10}=\left\{(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}\frac{\varepsilon_4(\varepsilon_2-\varepsilon_3)sn^2(\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0),m)-\varepsilon_3(\varepsilon_2-\varepsilon_4)}{(\varepsilon_2-\varepsilon_3)sn^2(\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0),m)-(\varepsilon_2-\varepsilon_4)}\right\}e^{i(-kx+\omega t+\theta_0)}, \tag{32}$$

where

$$m^2=\frac{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)}{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}.$$

Case 8. $D_4<0\&((D_2<0\&D_3<0)\|(D_2=0\&D_3\leq 0)\|D_2>0)$, then

$$\pm(s_1-s_0)=\int\frac{dh}{\sqrt{(h-\varepsilon_1)(h-\varepsilon_2)[(h-\varepsilon_3)^2+\varepsilon_4^2]}}. \tag{33}$$

A double periodic pattern is thus introduced as below:

$$q_{11}=\left\{(\frac{k(\lambda+\sigma)}{720a_6})^{-\frac{1}{12}}\frac{\varepsilon_1 cn^2(\frac{\sqrt{\mp 2\varepsilon_4 e_1(\varepsilon_1-\varepsilon_2)}}{2e_1 e}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0),e)+\varepsilon_2}{\varepsilon_3 cn^2(\frac{\sqrt{\mp 2\varepsilon_4 e_1(\varepsilon_1-\varepsilon_2)}}{2e_1 e}((\frac{k(\lambda+\sigma)}{720a_6})^{\frac{1}{12}}s-s_0),e)+\varepsilon_4}\right\}e^{i(-kx+\omega t+\theta_0)}, \tag{34}$$

where

$$\begin{aligned}
&\varepsilon_1 = \frac{1}{2}(\varepsilon_1+\varepsilon_2)\varepsilon_3 - \frac{1}{2}(\varepsilon_1-\varepsilon_2)\varepsilon_4, \\
&\varepsilon_2 = \frac{1}{2}(\varepsilon_1+\varepsilon_2)\varepsilon_4 - \frac{1}{2}(\varepsilon_1-\varepsilon_2)\varepsilon_3, \\
&\varepsilon_3 = \varepsilon_1 - \varepsilon_3 - \frac{\varepsilon_4}{e_1}, \\
&\varepsilon_4 = \varepsilon_1 - \varepsilon_3 - \varepsilon_4 e_1, \\
&E = \frac{\varepsilon_4^2 + (\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)}{\varepsilon_4(\varepsilon_1-\varepsilon_2)}, \\
&e_1 = E \pm \sqrt{E^2+1}, \\
&e^2 = \frac{1}{1+e_1^2}.
\end{aligned} \tag{35}$$

Case 9. $D_4 > 0\&((D_2 > 0\&D_3 \leq 0)\|D_2 \leq 0)$, then

$$\pm(s_1 - s_0) = \int \frac{dh}{\sqrt{[(h-\varepsilon_1)^2+\varepsilon_2^2][(h-\varepsilon_3)^2+\varepsilon_4^2]}}. \tag{36}$$

A double periodic pattern is therefore recovered as

$$q_{12} = \left\{ \left(\frac{k(\lambda+\sigma)}{720a_6}\right)^{-\frac{1}{12}} \frac{\begin{pmatrix} \varepsilon_1 sn\left(\frac{\varepsilon_2\sqrt{(\varepsilon_3^2+\varepsilon_4^2)(e_1^2\varepsilon_3^2+\varepsilon_4^2)}}{\varepsilon_3^2+\varepsilon_4^2}\left(\left(\frac{k(\lambda+\sigma)}{720a_6}\right)^{\frac{1}{12}} s - s_0\right), e\right) \\ +\varepsilon_2 cn\left(\frac{\varepsilon_2\sqrt{(\varepsilon_3^2+\varepsilon_4^2)(e_1^2\varepsilon_3^2+\varepsilon_4^2)}}{\varepsilon_3^2+\varepsilon_4^2}\left(\left(-\frac{k(\lambda+\sigma)}{720a_6}\right)^{\frac{1}{12}} s - s_0\right), e\right) \end{pmatrix}}{\begin{pmatrix} \varepsilon_3 sn\left(\frac{\varepsilon_2\sqrt{(\varepsilon_3^2+\varepsilon_4^2)(e_1^2\varepsilon_3^2+\varepsilon_4^2)}}{\varepsilon_3^2+\varepsilon_4^2}\left(\left(-\frac{k(\lambda+\sigma)}{720a_6}\right)^{\frac{1}{12}} s - s_0\right), e\right) \\ +\varepsilon_4 sn\left(\frac{\varepsilon_2\sqrt{(\varepsilon_3^2+\varepsilon_4^2)(e_1^2\varepsilon_3^2+\varepsilon_4^2)}}{\varepsilon_3^2+\varepsilon_4^2}\left(\left(-\frac{k(\lambda+\sigma)}{720a_6}\right)^{\frac{1}{12}} s - s_0\right), e\right) \end{pmatrix}} \right\} \tag{37}$$

$$\times e^{i(-kx+\omega t+\theta_0)},$$

where

$$\begin{aligned}
&\varepsilon_1 = \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4, \\
&\varepsilon_2 = \varepsilon_1\varepsilon_4 - \varepsilon_2\varepsilon_3, \\
&\varepsilon_3 = -\varepsilon_2 - \frac{\varepsilon_4}{e_1}, \\
&\varepsilon_4 = \varepsilon_1 - \varepsilon_3, \\
&E = \frac{(\varepsilon_1-\varepsilon_3)^2+\varepsilon_2^2+\varepsilon_4^2}{2\varepsilon_2\varepsilon_4}, \\
&e_1 = E + \sqrt{E^2-1}, \\
&e = \sqrt{\frac{e_1^2-1}{e_1^2}}.
\end{aligned} \tag{38}$$

4. PHYSICAL REALIZATIONS OF SOLUTIONS

The physical realization under specific parameters is obtained, and the 3D diagrams of the solution intensity $I = |q_i|^2 = q_j q_j^*$ are shown in this section.

Example 1. Singular solutions

When $v = s_0 = \theta_0 = \omega = a_6 = 1$, $\sigma = 4$, $\lambda = 5$, $k = 10$, one arrives at

$$q_1 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} (1 - (\frac{1}{8})^{\frac{1}{12}} (x-t))^{-1} \right\} e^{i(-10x+t+1)}. \tag{39}$$

Setting $v = \varepsilon_2 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_1 = 2$, $\varepsilon_3 = 3$, $\sigma = 4$, $\lambda = 5$, $k = 10$ provides us with

$$q_5 = \left\{ \frac{(\frac{1}{8})^{-\frac{1}{12}}}{\sin[(\frac{1}{8})^{\frac{1}{12}} (x-t) - 1]} \right\} e^{i(-10x+t+1)}. \tag{40}$$

Taking $v = \varepsilon_1 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_2 = 2$, $\sigma = 4$, $\lambda = 5$, $k = 10$ paves way to

$$q_6 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} [2\tan(2(\frac{1}{8})^{\frac{1}{12}} (x-t) - 1) + 1] \right\} e^{i(-10x+t+1)}. \tag{41}$$

If $v = \varepsilon_2 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_1 = 2$, $\sigma = 4$, $\lambda = 5$, $k = 10$, one extracts

$$q_7 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} [2 + \frac{4}{((\frac{1}{8})^{\frac{1}{12}} (x-t) - 1)^2 - 4}] \right\} e^{i(-10x+t+1)}. \tag{42}$$

When $v = \varepsilon_3 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_2 = 2$, $\varepsilon_1 = 3$, $\sigma = 4$, $\lambda = 5$, $k = 10$, we acquire

$$q_8 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} \frac{e^{\sqrt{2}((\frac{1}{8})^{\frac{1}{12}} (x-t)-1)} + \frac{\sqrt{2}}{2} + 2\sqrt{2} + 1}{(e^{\sqrt{2}((\frac{1}{8})^{\frac{1}{12}} (x-t)-1)} + \frac{\sqrt{2}}{2})^2 - 1} \right\} e^{i(-10x+t+1)}. \tag{43}$$

Figures 1 and 2 display the 3D diagrams of $|q_5|^2$ and $|q_8|^2$.

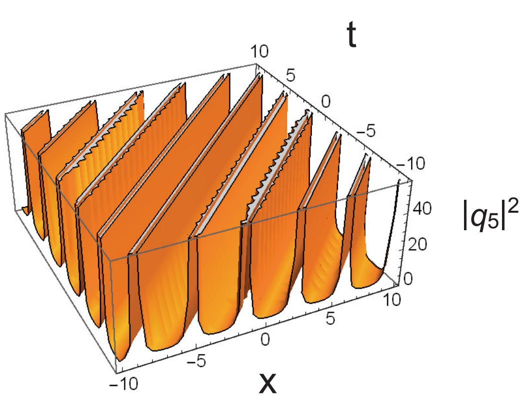

Fig. 1. $|q_5|^2$

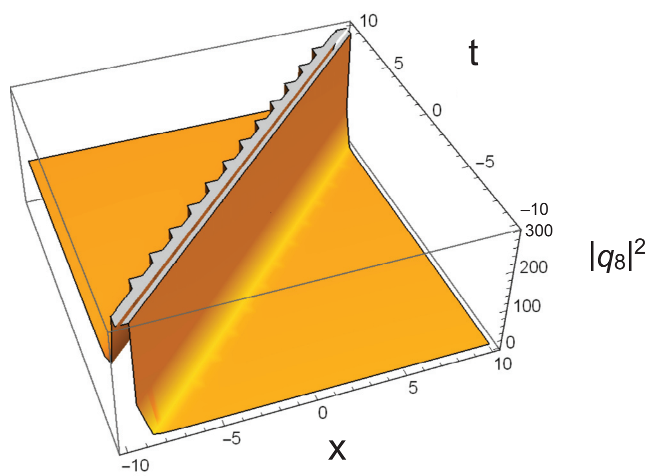

Fig. 2. $|q_8|^2$

Example 2. Optical solitons

Setting $v = \varepsilon_2 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_1 = 2$, $\sigma = 4$, $\lambda = 5$, $k = 10$ recovers the singular and dark solitons

$$q_2 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} [\frac{1}{2}\coth\frac{-(\frac{1}{8})^{\frac{1}{12}}(x-t)+1}{2} + \frac{5}{2}] \right\} e^{i(-10x+t+1)} \tag{44}$$

and

$$q_3 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} [\frac{1}{2}\tanh\frac{-(\frac{1}{8})^{\frac{1}{12}}(x-t)+1}{2} + \frac{5}{2}] \right\} e^{i(-10x+t+1)}. \tag{45}$$

Taking $v = \varepsilon_2 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_2 = 2$, $\varepsilon_1 = 3$, $\sigma = 4$, $\lambda = 5$, $k = 10$ presents the bright soliton

$$q_4 = \left\{ \frac{4(\frac{1}{8})^{-\frac{1}{12}}}{\cosh[\sqrt{2}((\frac{1}{8})^{\frac{1}{12}}(x-t)-1)]-3} \right\} e^{i(-10x+t+1)}. \tag{46}$$

Figures 3 and 4 exhibit the 3D diagrams of $|q_3|^2$ and $|q_4|^2$.

Example 3. Elliptic function double periodic solutions

If $v = \varepsilon_4 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_3 = 2$, $\varepsilon_2 = 3$, $\varepsilon_1 = \sigma = 4$, $\lambda = 5$, $k = 10$, one secures

$$q_9 = \left\{ (\frac{1}{8})^{-\frac{1}{12}} \frac{9sn^2((\frac{1}{8})^{\frac{1}{12}}(x-t)-1, \frac{\sqrt{3}}{2})-8}{3sn^2((\frac{1}{8})^{\frac{1}{12}}(x-t)-1, \frac{\sqrt{3}}{2})-2} \right\} e^{i(-10x+t+1)}, \tag{47}$$

$$q_{10} = \left\{ (\frac{1}{8})^{-\frac{1}{12}} \frac{sn^2((\frac{1}{8})^{\frac{1}{12}}(x-t)-1, \frac{\sqrt{3}}{2})-4}{sn^2((\frac{1}{8})^{\frac{1}{12}}(x-t)-1, \frac{\sqrt{3}}{2})-2} \right\} e^{i(-10x+t+1)}. \tag{48}$$

When $v = \varepsilon_4 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_3 = 2$, $\varepsilon_2 = 3$, $\varepsilon_1 = \sigma = 4$, $\lambda = 5$, $k = 10$, we retrieve

$$q_{11} = \left\{ (\frac{1}{8})^{-\frac{1}{12}} \frac{(18-3\sqrt{10})cn^2(10^{\frac{1}{4}}((\frac{1}{8})^{\frac{1}{12}}(x-t)-1), \sqrt{\frac{10-3\sqrt{10}}{20}})-6-3\sqrt{10}}{(5-\sqrt{10})cn^2(10^{\frac{1}{4}}((\frac{1}{8})^{\frac{1}{12}}(x-t)-1), \sqrt{\frac{10-3\sqrt{10}}{20}})-1-\sqrt{10}} \right\} e^{i(-10x+t+1)}. \tag{49}$$

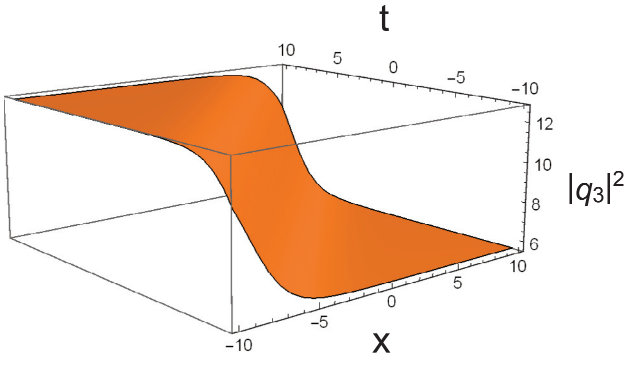

Fig. 3. $|q_3|^2$

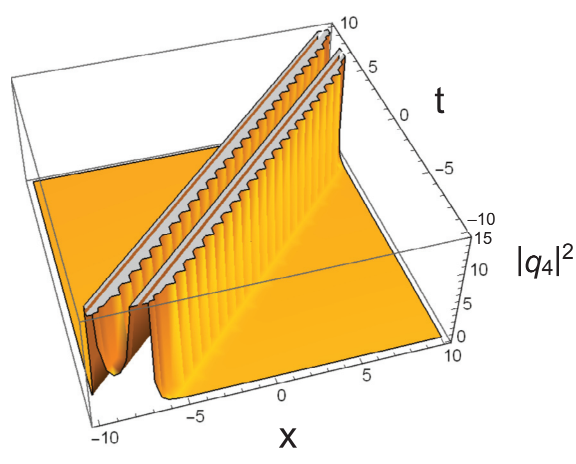

Fig. 4. $|q_4|^2$

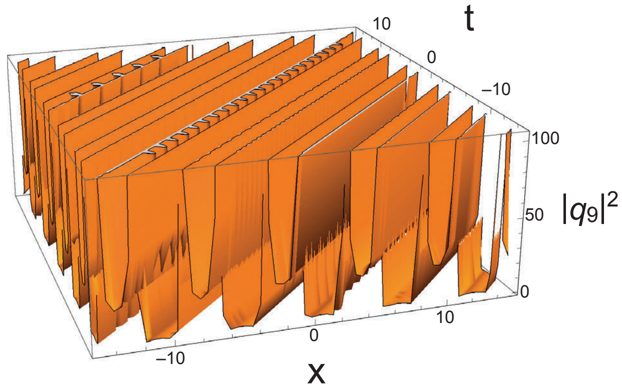

Fig. 5. $|q_9|^2$

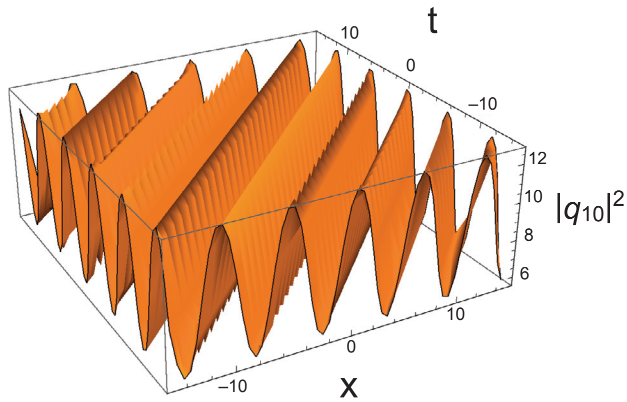

Fig. 6. $|q_{10}|^2$

Setting $v = \varepsilon_4 = \varepsilon_3 = s_0 = \theta_0 = \omega = a_6 = 1$, $\varepsilon_2 = \varepsilon_1 = 2$, $\sigma = 4$, $\lambda = 5$, $k = 10$ derives

$$q_{12} = \left\{ \left(\frac{1}{8}\right)^{-\frac{1}{12}} \frac{\left(\begin{array}{c} (9+\sqrt{5})sn^2(Y((\frac{1}{8})^{\frac{1}{12}}(x-t)-1), \frac{\sqrt{6\sqrt{5}-10}}{2}) \\ +(-5-\sqrt{5})cn^2(Y((\frac{1}{8})^{\frac{1}{12}}(x-t)-1), \frac{\sqrt{6\sqrt{5}-10}}{2}) \end{array} \right)}{\left(\begin{array}{c} \frac{7+\sqrt{5}}{2} sn^2(Y((\frac{1}{8})^{\frac{1}{12}}(x-t)-1), \frac{\sqrt{6\sqrt{5}-10}}{2}) \\ +cn^2(Y((\frac{1}{8})^{\frac{1}{12}}(x-t)-1), \frac{\sqrt{6\sqrt{5}-10}}{2}) \end{array} \right)} \right\}$$

$$\times e^{i(-10x+t+1)}, \tag{50}$$

where

$$Y = \frac{2\sqrt{6016+109\sqrt{5}}}{29+7\sqrt{5}}.$$

Figures 5 and 6 visualize the 3D diagrams of $|q_9|^2$ and $|q_{10}|^2$.

5. CONCLUSIONS

The current paper derives and enlists HD soliton solutions to the CGLE that is studied in the context of soliton transmission through optical fibers across intercontinental distances. The model is considered with the Kerr law of nonlinearity and a few Hamiltonian-type perturbation terms. HD solitons are derived by the complete discriminant classification approach. Such solitons are employed when CD is supplemented with additional dispersion effects due to the low count of CD. In addition to soliton solutions, a different variety of solutions naturally emerged based on the structure and sign of the discriminant. This led to a wide spectrum of solutions that include solitons, periodic solutions and doubly periodic solutions. The numerical sketches support the analytical findings.

The derived soliton solutions are going to lay a strong footing for further studies with the model. An immediate study would involve computing the conservation laws that would lead to the study of quasi-stationary soliton solutions in the presence of weak perturbations, which would be both of Hamiltonian as well as non-Hamiltonian type. Also, additional form(s) of self-phase modulation sources have not been examined yet. This is, thus, an open problem and will be later investigated. The results are yet to be released and are currently awaited. This would subsequently lead to a very interesting structure of the results that would give a plethora of physical insight into the governing model. Moreover, the model will be later further extended with the effects of birefringent fibers and DWDM systems that would give a truly broader and novel perspective on HD solitons [1].

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