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Transposed Poisson superalgebra

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Abstract. In this paper, we propose the notion of a transposed Poisson superalgebra. We prove that a transposed Poisson superalgebra can be constructed by means of a commutative associative superalgebra and an even degree derivation of this algebra. Making use of this, we construct two examples of the transposed Poisson superalgebra. One of them is the graded differential algebra of differential forms on a smooth finite dimensional manifold, where we use the Lie derivative as an even degree derivation. The second example is the commutative superalgebra of basic fields of the quantum Yang–Mills theory, where we use the BRST-supersymmetry as an even degree derivation to define a graded Lie bracket. We show that a transposed Poisson superalgebra has six identities that play an important role in the study of the structure of this algebra.

Keywords: commutative superalgebra, Lie superalgebra, Poisson algebra, transposed Poisson algebra, graded differential algebra.

1. INTRODUCTION

Poisson algebras play an important role in many branches of differential geometry, theoretical mechanics, and field theory. Thus, this area of research is actively developing, as evidenced by various generalizations of the concept of Poisson algebra. One direction of development in the theory of Poisson algebras is the extension of the concept of Poisson algebra to structures with an n-ary multiplication law. Initially, such a generalization was proposed by Nambu in his paper [7], and the model of quarks in the theory of elementary particles served as a stimulus for such a generalization. Later, this generalization called Nambu–Poisson algebra was studied in a number of papers, and an excellent survey of this direction in the development of the theory of Poisson algebras is the paper by Takhtajan [9]. A Nambu–Poisson algebra is an n-Lie algebra, where the n-Lie algebra is based on an extension of the binary Lie bracket to an n-ary Lie bracket. The concept of an n-Lie algebra was proposed by Filippov [5]. The elements of an n-Lie algebra satisfy the Filippov–Jacobi identity, which is a generalization of the Jacobi identity to an n-ary bracket. The classification of simple linearly compact n-Lie superalgebras is given in [4]. A class of ternary Lie superalgebras, constructed by means of the supertrace, and applications of these ternary Lie superalgebras in BRST-supersymmetries are proposed in [1,2].

Recently, the notion of a transposed Poisson algebra has been proposed. A transposed Poisson algebra can be considered as a structure dual to the concept of a Poisson algebra. It was shown in [3] that a transposed

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Poisson algebra in its properties is very similar to a Poisson algebra. For example, the class of transposed Poisson algebras is closed under the tensor product of such algebras. In [3], it is also indicated that there is an important connection between transposed Poisson algebras and Novikov–Poisson algebras. More exactly, if we equip a Novikov–Poisson algebra with the Lie commutator, constructed with the help of multiplication in the Novikov–Poisson algebra, then the Novikov–Poisson algebra becomes a transposed Poisson algebra. In [3], it is shown that a transposed Poisson algebra possesses a number of identities, which is an important property of a transposed Poisson algebra. In [6], all structures of complex transposed Poisson algebras on Galilean type Lie algebras and superalgebras are described.

In this paper, we propose the notion of a transposed Poisson superalgebra. In analogy with the transposed Poisson algebra, a transposed Poisson superalgebra is a vector space endowed with two structures, where one of them is a commutative associative superalgebra and the second is a Lie superalgebra. These two structures satisfy the graded compatibility condition. We prove that if $\mathfrak{G}$ is a commutative associative superalgebra and δ is an even derivation of $\mathfrak{G}$, then the graded Lie bracket

$$[u,v]_\delta = u\cdot\delta(v) - (-1)^{|u|\,|v|}\, v\cdot\delta(u), \tag{1}$$

where $u,v\in\mathfrak{G}$, $|u|,|v|$ are degrees of elements u,v, respectively, defines the structure of a transposed Poisson superalgebra on $\mathfrak{G}$. We propose the classification of transposed Poisson superalgebras in the low-dimensional case, i.e. in the case of (1,1)-superalgebras. We give two examples of the transposed Poisson superalgebra based on the graded Lie bracket (1). The first example is a transposed Poisson superalgebra constructed with the help of the graded differential algebra. Particularly, we show that the algebra of differential forms on a smooth finite-dimensional manifold M^n can be equipped with a structure of a transposed Poisson superalgebra if we use (1), where u,v are differential forms and δ is the Lie derivative $\mathscr{L}_{\mathtt{X}}$, $\mathtt{X}$ is a vector field on M^n. We prove that the graded Lie bracket of two differential forms induces a graded Lie bracket on the cohomology classes of differential forms. The second example of a transposed Poisson superalgebra is related to the quantum Yang–Mills theory. The BRST-supersymmetry of the quantum Yang–Mills theory can be considered as an even derivation of the superalgebra of basic fields of this theory. We construct the graded Lie bracket on this superalgebra using the operator of the BRST-supersymmetry. Finally, we prove that a transposed Poisson superalgebra possesses a rich class of identities that can be considered as graded versions of the identities proposed in [3].

2. TRANSPOSED POISSON SUPERALGEBRA

In this section, we propose a notion of a graded transposed Poisson algebra. We construct an example of a graded transposed Poisson algebra by means of a graded differential algebra.

Let $\mathbb{K}$ be a field either of real or complex numbers. A transposed Poisson algebra $\mathfrak{P}$ [3] is a vector space over $\mathbb{K}$ with two algebraic operations. One of them will be denoted by $(x,y)\in\mathfrak{P}\times\mathfrak{P}\to x\cdot y\in\mathfrak{P}$, and it defines a structure of associative commutative algebra on $\mathfrak{P}$. The second is a Lie bracket $(x,y)\in\mathfrak{P}\times\mathfrak{P}\to[x,y]\in\mathfrak{P}$, i.e. a bilinear, skew-symmetric mapping that satisfies the Jacobi identity. These two structures, i.e. an associative commutative algebra and a Lie algebra, define the structure of a transposed Poisson algebra if they are compatible, and the condition of compatibility of these two structures has the following form:

$$2z\cdot[x,y] = [z\cdot x,y]+[x,z\cdot y],\quad x,y,z\in\mathfrak{P}. \tag{2}$$

An important example of a transposed Poisson algebra [3] is an associative commutative algebra $\mathscr{A}$ endowed with the Lie bracket $[u,v]_\delta = u\cdot\delta(v)-v\cdot\delta(u)$, where $u,v\in\mathscr{A}$, $(u,v)\to u\cdot v$ is a multiplication in $\mathscr{A}$, and $\delta:\mathscr{A}\to\mathscr{A}$ is a derivation. Particularly, if M^n is a smooth n-dimensional manifold, $\mathscr{F}(M^n)$ is the algebra of smooth functions on M^n and $\mathtt{X}$ is a vector field, then

$$[f,g]_{\mathtt{X}} = f\,\mathtt{X}(g)-g\,\mathtt{X}(f),\quad f,g\in\mathscr{F}(M^n) \tag{3}$$

defines the transposed Poisson algebra of smooth functions on a manifold M^n.

We extend the notion of a transposed Poisson algebra to a super case by the following definition.

Definition 2.1. *Let $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ be a super vector space over $\mathbb{K}$, where the degree of a homogeneous element $x \in \mathfrak{G}$ will be denoted by $|x|$, i.e. $|x| \in \mathbb{Z}_2$. A transposed Poisson superalgebra is a triple $(\mathfrak{G}, \cdot, [\,,\,])$, where $(\mathfrak{G}, \cdot)$ is an associative commutative superalgebra and $(\mathfrak{G}, [\,,\,])$ is a Lie superalgebra. Hence, we have the following properties:*

$$|x \cdot y| = |x| + |y|, \;\; x \cdot y = (-1)^{|x||y|}\, y \cdot x$$

and

$$\begin{aligned}
&|[x,y]| = |x| + |y|,\\
&[x,y] = -(-1)^{|x|\,|y|}[y,x],\\
&[x,[y,z]] = [[x,y],z] + (-1)^{|x|\,|y|}[y,[x,z]].
\end{aligned}$$

The compatibility condition for these two structures has the form

$$2z \cdot [x,y] = [z \cdot x, y] + (-1)^{|x|\,|z|}[x, z \cdot y]. \tag{4}$$

Obviously, $(\mathfrak{G}_0, \cdot)$ is an associative commutative algebra and $(\mathfrak{G}_0, [\,,\,])$ is a Lie algebra. In the case of even degree elements, i.e. elements of $\mathfrak{G}_0$, the compatibility condition (4) takes on the form (2). Hence, $(\mathfrak{G}_0, \cdot, [\,,\,])$ is a transposed Poisson algebra.

An important example of a transposed Poisson algebra is an algebra constructed by means of an associative commutative algebra and its derivation. The next theorem shows that this construction can be extended to the case of superalgebras.

Theorem 1. *Let $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ be an associative commutative superalgebra and $\delta : \mathscr{A} \to \mathscr{A}$ be an even degree derivation of $\mathscr{A}$. Define bracket by*

$$[u,v]_\delta = u \cdot \delta(v) - (-1)^{|u|\,|v|} v \cdot \delta(u). \tag{5}$$

Then (5) defines the structure of the Lie superalgebra on $\mathscr{A}$, and this structure satisfies the compatibility condition (4), i.e. the bracket (5) defines the structure of the transposed Poisson superalgebra on $\mathscr{A}$.

Proof. It is easy to see that the bracket (5) is graded skew-symmetric, i.e. $[u,v]_\delta = -(-1)^{|u|\,|v|}[v,u]_\delta$ and $|[u,v]| = |u| + |v|$. Then the double brackets can be expressed as follows:

$$\begin{aligned}
\big[w,[u,v]_\delta\big]_\delta &= w\,u\,\delta^2(v) - \delta(w)\,u\,\delta(v) - w\,\delta^2(u)\,v + \delta(w)\,\delta(u)\,v,\\
\big[[w,u]_\delta, v\big]_\delta &= -w\,\delta^2(u)\,v + w\,\delta(u)\,\delta(v) + \delta^2(w)\,u\,v - \delta(w)\,u\,\delta(v),\\
(-1)^{|u|\,|w|}\big[u,[w,v]_\delta\big]_\delta &= w\,u\,\delta^2(v) - w\,\delta(u)\,\delta(v) - \delta^2(w)\,u\,v + \delta(w)\delta(u)\,v.
\end{aligned}$$

From this it follows that the graded bracket (5) satisfies the graded Jacobi identity

$$[w,[u,v]_\delta]_\delta = [[w,u]_\delta, v]_\delta + (-1)^{|u|\,|w|}[u,[w,v]_\delta]_\delta.$$

Hence, the graded bracket (5) defines the Lie superalgebra on a commutative superalgebra $\mathscr{A}$. We check the compatibility condition (4) by straightforward calculations. Indeed we find that

$$\begin{aligned}
2w \cdot [u,v]_\delta &= 2w \cdot u \cdot \delta(v) - 2w \cdot \delta(u) \cdot v,\\
[w \cdot u, v]_\delta &= w \cdot u \cdot \delta(v) - \delta(w) \cdot u \cdot v - w \cdot \delta(u) \cdot v,\\
(-1)^{|u|\,|w|}[u, w \cdot v]_\delta &= -w \cdot \delta(u) \cdot v + \delta(w) \cdot u \cdot v + w \cdot u \cdot \delta(v).
\end{aligned}$$

Thus, a commutative superalgebra $\mathscr{A}$ endowed with the graded Lie bracket (5) is a transposed Poisson superalgebra. $\square$

3. CLASSIFICATION OF TRANSPOSED POISSON SUPERALGEBRAS IN DIMENSION (1,1)

In this section, we consider a low-dimensional case of a transposed Poisson superalgebra over the field of complex numbers and give a classification of transposed Poisson superalgebras in this case. We consider transposed Poisson superalgebras $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$, where $\dim \mathfrak{G}_0 = \dim \mathfrak{G}_1 = 1$. Let e_0, e_1 be a basis for a super vector space $\mathfrak{G}$, such that e_0 is an even degree element that spans the even subspace $\mathfrak{G}_0$ and e_1 is an odd degree element that spans the odd subspace $\mathfrak{G}_1$. We assume that there is a commutative associative multiplication on a super vector space $\mathfrak{G}$, which will be denoted by $(x,y) \to x\cdot y, x,y \in \mathfrak{G}$. From the structure of the superalgebra and commutativity of a multiplication it follows that

$$e_0 \cdot e_0 = a\,e_0,\;\; e_0 \cdot e_1 = e_1 \cdot e_0 = b\,e_1,\;\; e_1 \cdot e_1 = 0, \tag{6}$$

where a,b are complex numbers. In this simple case there is only one combination of three generators, which leads to a non-trivial condition derived from associativity. This combination is e_0, e_0, e_1. We have

$$e_0 \cdot (e_0 \cdot e_1) = (e_0 \cdot e_0) \cdot e_1 \;\Rightarrow\; b\,e_0 \cdot e_1 = a\,e_0 \cdot e_1 \;\Rightarrow\; b^2 e_1 = ab\,e_1. \tag{7}$$

Hence, a multiplication is associative in three cases: $a = b = 0$, $a = b \neq 0$, and $a \neq 0, b = 0$.

Now, we consider the second structure of a transposed Poisson superalgebra, i.e. a Lie superalgebra. It is known that in the case of Lie superalgebras of (1,1)-type, there are three non-isomorphic Lie superalgebras:

I) $[e_0,e_0] = 0,\;\; [e_0,e_1] = 0,\;\; [e_1,e_1] = 0,$

II) $[e_0,e_0] = 0,\;\; [e_0,e_1] = 0,\;\; [e_1,e_1] = e_0,$

III) $[e_0,e_0] = 0,\;\; [e_0,e_1] = e_1,\;\; [e_1,e_1] = 0.$

In the case of the Abelian Lie superalgebra I, we have two transposed Poisson superalgebras:

$$\begin{gathered} e_0 \cdot e_0 = e_0,\; e_0 \cdot e_1 = e_1 \cdot e_0 = 0,\; e_1 \cdot e_1 = 0, \\ [e_0,e_0] = 0,\; [e_0,e_1] = 0,\; [e_1,e_1] = 0 \end{gathered}$$

and

$$\begin{gathered} e_0 \cdot e_0 = e_0,\; e_0 \cdot e_1 = e_1 \cdot e_0 = e_1,\; e_1 \cdot e_1 = 0, \\ [e_0,e_0] = 0,\; [e_0,e_1] = 0,\; [e_1,e_1] = 0. \end{gathered}$$

In the case of the Lie superalgebra II, we have $a = b = 0$. Hence, we have only one transposed Poisson superalgebra

$$\begin{gathered} e_0 \cdot e_0 = e_0,\; e_0 \cdot e_1 = 0 \cdot e_0 = 0,\; e_1 \cdot e_1 = 0, \\ [e_0,e_0] = 0,\; [e_0,e_1] = 0,\; [e_1,e_1] = e_0. \end{gathered}$$

In the case of the Lie superalgebra III, the condition $a = b$ arises from compatibility, and we get one more transposed Poisson superalgebra

$$\begin{gathered} e_0 \cdot e_0 = e_0,\; e_0 \cdot e_1 = e_1 \cdot e_0 = e_1,\; e_1 \cdot e_1 = 0, \\ [e_0,e_0] = 0,\; [e_0,e_1] = e_1,\; [e_1,e_1] = 0. \end{gathered}$$

4. INFINITE DIMENSIONAL TRANSPOSED POISSON SUPERALGEBRAS

In this section, we apply Theorem 1 to construct a transposed Poisson superalgebra by means of differential forms on a smooth n-dimensional manifold M^n.

Let us consider a graded commutative differential algebra $(\mathfrak{A}, d)$, where $\mathfrak{A} = \oplus_{p\in\mathbb{Z}}\mathfrak{A}^p$, and $d : \mathfrak{A}^i \to \mathfrak{A}^{i+1}$ is a differential of $\mathfrak{A}$, i.e. an anti-derivation of degree 1. Let us denote a multiplication in $\mathfrak{A}$ by $(u,v) \to u\cdot v$, where $u, v \in \mathfrak{A}$, and the degree of a homogeneous element u by $|u|$. Then, for $u \in \mathfrak{A}^p$ we have $|u| = p$ and

$$|u\cdot v| = |u| + |v|,\;\; u\cdot v = (-1)^{|u|\,|v|}v\cdot u,\;\; d(u\cdot v) = (du)\cdot v + (-1)^{|u|}u\cdot dv.$$

Assume that δ is a degree –1 anti-derivation of $\mathfrak{A}$, i.e. $\delta : \mathfrak{A}^p \to \mathfrak{A}^{p-1}$ and $\delta(u\cdot v) = (\delta u)\cdot v + (-1)^{|u|}u\cdot\delta v$. Then $D = d\circ\delta + \delta\circ d$ is the 0-degree derivation of $\mathfrak{A}$. We can consider $\mathfrak{A}$ as a commutative superalgebra if we define the parity of a homogeneous element u as $|u| (\mathrm{mod}\, 2)$. Then d, δ are odd derivations and D is the even derivation of $\mathfrak{A}$. Define the bracket by

$$[u,v]_D = u\cdot D(v) - (-1)^{|u|\,|v|}\, v\cdot D(u). \tag{8}$$

Then, according to Theorem 1, the triple $(\mathfrak{A}, \cdot, [\,,\,]_D)$ is a transposed Poisson superalgebra.

As an example of the structure described above, we consider the algebra of differential forms on a smooth n-dimensional manifold M^n. Let $\Omega(M^n) = \oplus_p\Omega^p(M^n)$ be the graded algebra of differential forms on M^n. If we denote by $\mathscr{F}(M^n)$ the commutative algebra of smooth functions on a manifold M^n, then $\Omega^0(M^n)$ is identified with the algebra of functions $\mathscr{F}(M^n)$, i.e. $\Omega^0(M^n) = \mathscr{F}(M^n)$. The graded algebra of differential forms $\Omega(M^n)$ can be considered as a superalgebra if we define the degree of p-form ω by $|\omega| = p\ (\mathrm{mod}\, 2)$. Then it follows from the property of the wedge product of the two forms $\omega\wedge\theta = (-1)^{pq}\theta\wedge\omega$, where $\omega\in\Omega^p(M^n), \theta\in\Omega^q(M^n)$, that $\Omega(M^n)$ is a commutative superalgebra. Let $\mathtt{X}$ be a smooth vector field on a manifold M^n. Then the operator of contraction of a differential form with a vector field $i_{\mathtt{X}} : \Omega^p \to \Omega^{p-1}$ and the exterior differential $d : \Omega^p \to \Omega^{p+1}$ are odd derivations of the commutative superalgebra of differential forms. Then the Lie derivative

$$\mathscr{L}_{\mathtt{X}} = d\circ i_{\mathtt{X}} + i_{\mathtt{X}}\circ d$$

is an even degree derivation of the commutative superalgebra of differential forms, i.e. $\mathscr{L}_{\mathtt{X}} : \Omega^p(M^n) \to \Omega^p(M^n)$ and $\mathscr{L}_{\mathtt{X}}(\omega\wedge\theta) = \mathscr{L}_{\mathtt{X}}(\omega)\wedge\theta + \omega\wedge\mathscr{L}_{\mathtt{X}}(\theta)$. Then the bracket (8) takes on the form

$$[\omega,\theta]_{\mathtt{X}} = \omega\wedge\mathscr{L}_{\mathtt{X}}(\theta) - (-1)^{|\omega|\,|\theta|}\theta\wedge\mathscr{L}_{\mathtt{X}}(\omega), \tag{9}$$

and the commutative superalgebra of differential forms $\Omega(M^n)$ endowed with the bracket (9) is a transposed Poisson superalgebra. Particularly, if we consider two 0-forms, i.e. two functions $f, g \in \mathscr{F}(M^n)$, then the graded Lie bracket (9) takes on the form

$$[f,g]_{\mathtt{X}} = f\,\mathscr{L}_{\mathtt{X}}(g) - g\,\mathscr{L}_{\mathtt{X}}(f) = f\,\mathtt{X}(g) - g\,\mathtt{X}(f).$$

Hence, the graded bracket (9) restricted to the subalgebra of smooth functions $\mathscr{F}(M^n)$ makes it a transposed Poisson algebra (3). Let $\Omega^p_{\mathsf{c}}(M^n)$ be the vector space of closed p-differential forms and $\Omega^p_{\mathsf{e}}(M^n) \subset \Omega^p_{\mathsf{c}}(M^n)$ be the vector space of exact p-differential forms.

Proposition 1. *If $\omega\in\Omega^p_{\mathsf{c}}(M^n), \theta\in\Omega^q_{\mathsf{c}}(M^n)$, then $[\omega,\theta]_{\mathtt{X}} \in \Omega^{p+q}_{\mathsf{c}}(M^n)$, i.e. the graded Lie bracket (9) of two closed forms is a closed form. If $\omega\in\Omega^p_{\mathsf{e}}(M^n)$ and $\theta\in\Omega^q_{\mathsf{c}}(M^n)$, then $[\omega,\theta]_{\mathtt{X}} \in \Omega^{p+q}_{\mathsf{e}}(M^n)$, i.e. if one of two closed differential forms is exact, then the graded Lie bracket (9) of these two forms is an exact differential form.*

Proof. Let $\omega\in\Omega^p_{\mathsf{c}}(M^n), \theta\in\Omega^q_{\mathsf{c}}(M^n)$, i.e. $d\omega = d\theta = 0$. We have

$$\begin{aligned}
[\omega,\theta]_{\mathtt{X}} &= \omega\wedge\mathscr{L}_{\mathtt{X}}(\theta) - (-1)^{|\omega|\,|\theta|}\theta\wedge\mathscr{L}_{\mathtt{X}}(\omega)\\
&= \omega\wedge i_{\mathtt{X}}\circ d(\theta) + \omega\wedge d\circ i_{\mathtt{X}}(\theta) - (-1)^{|\omega|\,|\theta|}(\theta\wedge i_{\mathtt{X}}\circ d(\omega) + \theta\wedge d\circ i_{\mathtt{X}}(\omega))\\
&= \omega\wedge d\circ i_{\mathtt{X}}(\theta) - (-1)^{|\omega|\,|\theta|}\theta\wedge d\circ i_{\mathtt{X}}(\omega).
\end{aligned}$$

Now, differentiating the graded Lie bracket, we obtain

$$\begin{aligned} d([\omega,\theta]_{\mathtt{X}}) &= d(\omega)\wedge d\circ i_{\mathtt{X}}(\theta)+(-1)^{|\omega|}\omega\wedge d^2\circ i_{\mathtt{X}}(\theta) \\ &\quad -(-1)^{|\omega||\theta|}d(\theta)\wedge d\circ i_{\mathtt{X}}(\omega)-(-1)^{(|\omega|+1)|\theta|}\theta\wedge d^2\circ i_{\mathtt{X}}(\omega)=0. \end{aligned}$$

If ω is an exact form, i.e. $\omega\in\Omega^p_{\mathrm{e}}(M^n),\omega=d\zeta,\zeta\in\Omega^{p-1}(M^n)$, then making use of $d\circ\mathscr{L}_{\mathtt{X}}=\mathscr{L}_{\mathtt{X}}\circ d$, we get

$$\begin{aligned} [\omega,\theta]_{\mathtt{X}}=[d\zeta,\theta]_{\mathtt{X}} &= d\zeta\wedge\mathscr{L}_{\mathtt{X}}(\theta)-(-1)^{|\omega||\theta|}\theta\wedge\mathscr{L}_{\mathtt{X}}(d\zeta) \\ &= d\zeta\wedge\mathscr{L}_{\mathtt{X}}(\theta)-(-1)^{|\omega||\theta|}\theta\wedge d\circ\mathscr{L}_{\mathtt{X}}(\zeta) \\ &= d\big(\zeta\wedge\mathscr{L}_{\mathtt{X}}(\theta)-(-1)^{|\zeta||\theta|}\theta\wedge\mathscr{L}_{\mathtt{X}}(\zeta)\big). \end{aligned}$$

□

Thus, the proved proposition shows that the graded Lie bracket (9) induces the graded Lie bracket on the graded algebra of de Rham cohomologies, which defines the structure of a transposed Poisson superalgebra.

Our next example of a transposed Poisson superalgebra is related to the quantum theory of Yang–Mills fields [8]. The gauge group of the Yang–Mills theory is $\mathrm{SU}(2)$. Let $\mathfrak{su}(2)$ be the Lie algebra of $\mathrm{SU}(2)$, t_a be a basis for $\mathfrak{su}(2)$, and $[t_a,t_b]=K^c_{ab}t_c$, i.e. K^c_{ab} are structure constants of $\mathfrak{su}(2)$. The basic fields of the quantum Yang–Mills theory are $\mathfrak{su}(2)$-valued functions $A_\mu,c,\bar{c}$ defined on a space-time, where A_μ $(\mu=0,1,2,3)$ are Yang–Mills fields and $c,\bar{c}$ are ghost fields. Since the basic fields are $\mathfrak{su}(2)$-valued functions, we can express them in the terms of a basis as follows: $A_\mu=A^a_\mu\,t_a, c=c^a\,t_a,\bar{c}=\bar{c}^a\,t_a$. We consider the algebra of fields of the quantum Yang–Mills theory as an algebra generated by $A^a_\mu,b^a,c^a,\bar{c}^a$, their all orders being partial derivatives with respect to the coordinates of space-time and Grassmann variables $\xi,\eta,\ldots$, which do not depend on a point of space-time. We assume that the gauge fields A^a_μ, an auxiliary field b^a, and all their partial derivatives are commuting generators of algebra, which also commute with ghost fields, their partial derivatives and Grassmann variables $\xi,\eta,\ldots$. The ghost fields $c^a,\bar{c}^a$, their partial derivatives and $\xi,\eta,\ldots$ are Grassmann-type generators, i.e. they anticommute with each other. Now, attributing degree zero to the gauge fields A^a_μ and an auxiliary field b^a (and their all partial derivatives), degree one to ghost fields (and their partial derivatives) and Grassmann variables $\xi,\eta,\ldots$, we turn the algebra of the fields of quantum Yang–Mills theory into a commutative superalgebra. An important role in the quantum theory of Yang–Mills fields is played by the BRST-supersymmetry, which has the form [8]

$$A^a_\mu\;\to\;A^a_\mu+\delta A^a_\mu,\;c^a\;\to\;c^a+\delta c^a,\;\bar{c}^a\;\to\;\bar{c}^a+\delta\bar{c}^a, \tag{10}$$

where

$$\delta A^a_\mu=\xi\,D_\mu c^a,\;\;\delta c^a=-\frac{1}{2}K^a_{bd}\,\xi\,c^b\,c^d\;,\;\delta\bar{c}^a=\xi\,b^a, \tag{11}$$

and $D_\mu c^a=\partial_\mu c^a+K^a_{bd}A^b_\mu\,c^d$ is the covariant derivative. The BRST-supersymmetry can be considered as an even derivation of the commutative superalgebra of the fields of the quantum Yang–Mills theory. Indeed, let us define the mapping s on the basic fields as follows:

$$sA^a_\mu=(D_\mu c)^a\;,\;sc^a=-\frac{1}{2}K^a_{bd}\,c^b\,c^d\;,\;s\bar{c}^a=b^a\;,\;sb^a=0, \tag{12}$$

and extend this mapping to the odd derivation of the commutative superalgebra of fields by assuming that it acts on products of fields according to the graded Leibniz rule. We also assume that the derivation s commutes with partial derivatives and Grassmann variables $\xi,\eta,\ldots$, i.e. $s\circ\partial_\mu=\partial_\mu\circ s, s\xi=\xi s$. It can be verified that the odd derivation s has a very important property $s^2=0$. Then $\delta=\xi\,s$ is the even derivation of the commutative superalgebra of fields. Indeed, for any product of basic fields ϕ,ψ, we have

$$\begin{aligned} \delta(\phi\,\psi) &= \xi\,s(\phi\,\psi)=\xi\,\big(\delta(\phi)\,\psi+(-1)^{|\phi|}\phi\,\delta(\psi)\big) \\ &= \xi\,\delta(\phi)\,\psi+(-1)^{|\phi|}\xi\,\phi\,\delta(\psi) \\ &= \xi\,\delta(\phi)\,\psi+(-1)^{2|\phi|}\phi\,\xi\,\delta(\psi)=\delta(\phi)\,\psi+\phi\,\delta(\psi). \end{aligned}$$

Let Φ,Ψ be two elements of the commutative superalgebra of the fields of quantum Yang–Mills theory, i.e. finite polynomials on the basic fields and their partial derivatives. We define the graded Lie bracket of these two polynomials by

$$[\Phi,\Psi]_\delta = \Phi\,\delta(\Psi) - (-1)^{|\Phi|\,|\Psi|}\,\Psi\,\delta(\Phi). \tag{13}$$

Then the algebra of the fields of quantum Yang–Mills theory endowed with the graded Lie bracket (13) is a transposed Poisson superalgebra.

5. IDENTITIES IN TRANSPOSED POISSON SUPERALGEBRA

The aim of this section is to show that the identities that hold in the case of a transposed Poisson algebra [3] can be extended to a transposed Poisson superalgebra if we modify the identities proposed in [3] with the help of the rule of signs.

Theorem 2. *Let $(\mathfrak{G},\cdot,[\,,\,])$ be a transposed Poisson superalgebra. Then, for any $h,x,y,z\in\mathfrak{G}$, we have the following identities:*

$$\begin{aligned}
(-1)^{|x|\,|z|}\,x\cdot[y,z]+(-1)^{|x|\,|y|}\,y\cdot[z,x]+(-1)^{|y|\,|z|}\,z\cdot[x,y] &= 0, &(14)\\
(-1)^{|x|\,|z|}\,[h\cdot[x,y],z]+(-1)^{|x|\,|y|}\,[h\cdot[y,z],x]+(-1)^{|y|\,|z|}\,[h\cdot[z,x],y] &= 0, &(15)\\
(-1)^{|x|\,|z|}\,[h\cdot x,[y,z]]+(-1)^{|x|\,|y|}\,[h\cdot y,[z,x]]+(-1)^{|y|\,|z|}\,[h\cdot z,[x,y]] &= 0, &(16)\\
(-1)^{|x|\,|z|}\,[h,x]\,[y,z]+(-1)^{|x|\,|y|}\,[h,y]\,[z,x]+(-1)^{|y|\,|z|}\,[h,z]\,[x,y] &= 0, &(17)\\
2\,u\cdot v\cdot[x,y]=(-1)^{|x|\,|v|}[u\cdot x,v\cdot y]+(-1)^{|u|\,(|x|+|v|)}[v\cdot x,u\cdot y],& &(18)\\
(-1)^{|u|\,|yv|}\,x\cdot[u,y\cdot v]+(-1)^{|v|\,|xy|}\,v\cdot[x\cdot y,u]+(-1)^{|x|\,|yv|}y\cdot[v,x]\cdot u &= 0. &(19)
\end{aligned}$$

Proof. In order to prove identity (14), we make use of the compatibility condition (4) as follows:

$$\begin{aligned}
(-1)^{|x|\,|z|}\,2x\cdot[y,z] &= (-1)^{|x|\,|z|}[x\cdot y,z]+(-1)^{|x|\,|z|+|x|\,|y|}[y,x\cdot z],\\
(-1)^{|x|\,|y|}\,2y\cdot[z,x] &= (-1)^{|x|\,|y|}[y\cdot z,x]+(-1)^{|x|\,|y|+|y|\,|z|}[x,y\cdot x],\\
(-1)^{|y|\,|z|}\,2z\cdot[x,y] &= (-1)^{|y|\,|z|}[z\cdot x,y]+(-1)^{|y|\,|z|+|x|\,|z|}[x,z\cdot y].
\end{aligned}$$

Making use of the commutativity of a superalgebra $\mathfrak{G}$ and the graded skew-symmetry of the bracket, we can represent the right-hand sides of the above identities as follows:

$$\begin{aligned}
(-1)^{|x|\,|z|}\,2x\cdot[y,z] &= (-1)^{|x|\,|z|}[x\cdot y,z]-(-1)^{|y|\,|z|}[z\cdot x,y],\\
(-1)^{|x|\,|y|}\,2y\cdot[z,x] &= (-1)^{|x|\,|y|}[y\cdot z,x]-(-1)^{|x|\,|z|}[x\cdot y,z],\\
(-1)^{|y|\,|z|}\,2z\cdot[x,y] &= (-1)^{|y|\,|z|}[z\cdot x,y]-(-1)^{|x|\,|y|}[y\cdot z,x].
\end{aligned}$$

Summing up the left-hand sides and right-hand sides of the obtained identities, we get identity (14). The proof of identity (15) is similar to the proof of identity (14). First, we use the compatibility condition (4) to obtain the following three identities:

$$\begin{aligned}
(-1)^{|x|\,|z|}\,2\,h\cdot[[x,y],z] &= (-1)^{|x|\,|z|}\,[h\cdot[x,y],z]+(-1)^{|x|\,|z|+(xy,h)}[[x,y],h\cdot z],\\
(-1)^{|x|\,|y|}\,2\,h\cdot[[y,z],x] &= (-1)^{|x|\,|y|}\,[h\cdot[y,z],x]+(-1)^{|x|\,|y|+(yz,h)}[[y,z],h\cdot x],\\
(-1)^{|y|\,|z|}\,2\,h\cdot[[z,x],y] &= (-1)^{|y|\,|z|}\,[h\cdot[z,x],y]+(-1)^{|y|\,|z|+(xz,h)}[[z,x],h\cdot y],
\end{aligned}$$

where $(xy,h)=|x|\,|h|+|y|\,|h|,(yz,h)=|y|\,|h|+|z|\,|h|,(xz,h)=|x|\,|h|+|z|\,|h|$. Due to the super Jacobi identity, the sum of the left-hand sides of the above identities is equal to zero. Hence, we get

$$\begin{aligned}(-1)^{|x|\,|z|}\,[h\cdot[x,y],z]+(-1)^{|x|\,|y|}\,[h\cdot[y,z],x]+(-1)^{|y|\,|z|}\,[h\cdot[z,x],y]\\ +(-1)^{|x|\,|z|+(xy,h)}[[x,y],h\cdot z]+(-1)^{|x|\,|y|+(yz,h)}[[y,z],h\cdot x]\\ +(-1)^{|y|\,|z|+(xz,h)}[[z,x],h\cdot y]=0.\end{aligned} \tag{20}$$

The sum of the first three terms in the above identity is the left-hand side of identity (15). Thus, in order to prove identity (15), we need to show that the sum of the last three terms in the above identity is zero. For the first one of them, i.e. $[[x,y],h\cdot z]$ (we temporarily omit the coefficient $(-1)^{|x|\,|z|+(xy,h)}$, which will be taken into account later), we have the super Jacobi identity

$$(-1)^{|x|\,|h|+|x|\,|z|}[[x,y],h\cdot z]+(-1)^{|x|\,|y|}[[y,h\cdot z],x]+(-1)^{|y|\,|h|+|y|\,|z|}[[h\cdot z,x],y]=0. \tag{21}$$

Writing the compatibility condition in the form

$$2\,h\cdot[y,z]=[h\cdot y,z]+(-1)^{|y|\,|h|}[y,h\cdot z]\ \Rightarrow\ [y,h\cdot z]=(-1)^{|y|\,|h|}(2\,h\cdot[y,z]-[h\cdot y,z]),$$

substituting the obtained expression into the middle term of the super Jacobi identity (21) and then multiplying both sides of identity (21) by $(-1)^{|x|\,|z|+(xy,h)}$, we get

$$\begin{aligned}(-1)^{|y|\,|h|}[[x,y],h\cdot z]+(-1)^{(yzh,x)}([2\,h\cdot[y,z],x]\ -\ [[h\cdot y,z],x])\\ +\ (-1)^{|y|\,|z|+(zh,x)}[[h\cdot z,x],y]=0,\end{aligned} \tag{22}$$

where $(yzh,x)=|y|\,|x|+|z|\,|x|+|h|\,|x|$ and $(zh,x)=|z|\,|x|+|h|\,|x|$. Applying the same calculations to the next two terms in (20), we get two more identities:

$$\begin{aligned}(-1)^{|z|\,|h|}[[y,z],h\cdot x]+(-1)^{(xzh,y)}([2\,h\cdot[z,x],y]\ -\ [[h\cdot z,x],y])\\ +\ (-1)^{|x|\,|z|+(xh,y)}[[h\cdot x,y],z]=0,\end{aligned} \tag{23}$$

$$\begin{aligned}(-1)^{|x|\,|h|}[[y,z],h\cdot x]+(-1)^{(xyh,z)}([2\,h\cdot[z,x],y]\ -\ [[h\cdot z,x],y])\\ +\ (-1)^{|x|\,|y|+(yh,z)}[[h\cdot x,y],z]=0.\end{aligned} \tag{24}$$

Summing up identities (22), (23), (24), multiplied by $(-1)^{|x|\,(|z|+|h|)},(-1)^{|y|\,(|x|+|h|)}$, and $(-1)^{|z|\,(|y|+|h|)}$, respectively, we obtain

$$\begin{aligned}(-1)^{|x|\,|z|+(xy,h)}[[x,y],h\cdot z]+(-1)^{|x|\,|y|+(yz,h)}[[y,z],h\cdot x]+(-1)^{|y|\,|z|+(xz,h)}[[z,x],h\cdot y]\\ 2\left((-1)^{|x|\,|z|}\,[h\cdot[x,y],z]+(-1)^{|x|\,|y|}\,[h\cdot[y,z],x]+(-1)^{|y|\,|z|}\,[h\cdot[z,x],y]\right)=0.\end{aligned} \tag{25}$$

Subtracting identity (20) from (25), we get identity (15). By virtue of the proven identity (15), the sum of the first three terms in (20) will be equal to zero. The remaining three terms, after a suitable permutation of the arguments, give identity (16).

In order to prove the last identity (17), we apply the compatibility condition to the following products:

$$\begin{aligned}2\,[x,y]\cdot[h,z]&=[[x,y]\cdot h,z]+(-1)^{|h|(|x|+|y|)}[h,[x,y]\cdot z],\\ 2\,[y,z]\cdot[h,x]&=[[y,z]\cdot h,x]+(-1)^{|h|(|y|+|z|)}[h,[y,z]\cdot x],\\ 2\,[z,x]\cdot[h,y]&=[[z,x]\cdot h,y]+(-1)^{|h|(|x|+|z|)}[h,[z,x]\cdot y].\end{aligned}$$

In the first brackets on the right-hand sides of these relations, we rearrange h and $[x,y]$, h and $[y,z]$, h and $[z,x]$. Next, we multiply the first relation by $(-1)^{|x|\,|h|+|y|\,|h|+|x|\,|z|}$, the second by $(-1)^{|y|\,|h|+|z|\,|h|+|x|\,|y|}$, and the third by $(-1)^{|x|\,|h|+|z|\,|h|+|y|\,|z|}$. Finally, we rearrange the brackets at the left-hand side of every relation. We obtain

$$\begin{aligned}(-1)^{|y|\,|z|}\,2\,[h,z]\cdot[x,y] &= (-1)^{|x|\,|z|}[h\cdot[x,y],z]+[h,(-1)^{|x|\,|z|}[x,y]\cdot z],\\ (-1)^{|x|\,|z|}\,2\,[h,x]\cdot[y,z] &= (-1)^{|x|\,|y|}[h\cdot[y,z],x]+[h,(-1)^{|x|\,|y|}[y,z]\cdot x],\\ (-1)^{|x|\,|y|}2\,[h,y]\cdot[z,x] &= (-1)^{|y|\,|z|}[h\cdot[z,x],y]+[h,(-1)^{|y|\,|z|}[z,x]\cdot y].\end{aligned}$$

Summing up the left-hand sides and the right-hand sides of these relations, we get the relation whose left-hand side (multiplied by 1/2) coincides with the left-hand side of identity (17), but the right-hand side is equal to zero. Indeed, the sum of the first terms at the right-hand side of every relation vanishes because of identity (15). The sum of the second terms vanishes because of identity (14). Identity (18) is proved by means of the compatibility condition. In order to prove identity (19), we take the sum of the following three relations: the compatibility condition for the products $2x\cdot[u,y\cdot v]$, $2v\cdot[x\cdot y,u]$ and identity (18) multiplied by 2. This ends the proof of Theorem 2. □

6. CONCLUSIONS

The impetus for writing this article was the paper [3], which introduced the concept of a transposed Poisson algebra, described its structure, and gave important examples and relations to other algebraic structures. Our goal in this article is to show that the notion of a transposed Poisson algebra can be extended to superalgebras in substance merely by means of the Kozul sign rule, and, furthermore, to show that important structures of a transposed Poisson algebra (example constructed with the help of derivation and identities) also hold in a transposed Poisson superalgebra in a graded form. We think that the example of a transposed Poisson superalgebra constructed by means of differential forms and the Lie derivative reveals a geometric meaning of the notion of a transposed Poisson superalgebra. In the paper [3], the authors show that the identities in a transposed Poisson algebra play an important role in establishing a relation with Hom-Lie algebras. We hope that the graded versions of these identities proved in this paper play the same important role in establishing connections between a transposed Poisson superalgebra and Hom-Lie superalgebras. This, and a notion of a transposed Poisson 3-Lie superalgebra, is our next goal in developing this direction.

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Transponeeritud Poissoni superalgebra

Viktor Abramov ja Olga Liivapuu

Artiklis transponeeritud Poissoni algebra mõistet laiendatakse superalgebratele ja seda laiendit nimetatakse transponeeritud Poissoni superalgebraks. Supervektorruumi nimetatakse transponeeritud Poissoni superalgebraks, kui supervektorruumil on kaks struktuuri – kommutatiivne (assotsiatiivne) superalgebra ja Lie superalgebra –, kusjuures peab olema täidetud kooskõla tingimus. Transponeeritud Poissoni algebra konstrueerimiseks assotsiatiivsel kommutatiivsel algebral kasutatakse Lie kommutaatorit defineeritud algebra derivatsiooni abil. Artiklis näidatakse, et mainitud meetodit saab laiendada kommutatiivsetele superalgebratele transponeeritud Poissoni superalgebra konstrueerimiseks, kasutades gradueeritud Lie kommutaatorit ja superalgebra paarisderivatsiooni. Leitud on transponeeritud Poissoni superalgebrate klassifikatsioon dimensioonis (1,1). Artiklis on toodud transponeeritud Poissoni superalgebra kaks näidet: diferentsiaalvormide gradueeritud algebra, kus paarisderivatsioon on Lie tuletis, ja Yang–Millsi kvantväljateooria väljade superalgebra, kus paarisderivatsioon on BRST-operaator. Näidatud on, et transponeeritud Poissoni superalgebras kehtib kuus samasust.