Proceedings of the Estonian Academy of Sciences,
2023, **72**, 1, 61–76
https://doi.org/10.3176/proc.2023.1.07
Available online at www.eap.ee/proceedings

MATHEMATICS

Algebra with ternary cyclic relations, representations and quark model

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Received 7 September 2022, accepted 25 October 2022, available online 15 February 2023

Abstract. We propose a unital associative algebra, motivated by a generalization of the Pauli's exclusion principle proposed for the quark model. The generators of this algebra satisfy the following relations: The sum of squares of all generators is equal to zero (binary relation) and the sum of cyclic permutations of the factors in any triple product of generators is equal to zero (ternary relations). We study the structure of this algebra and calculate the dimensions of spaces spanned by homogeneous monomials. We establish a relation between our algebra and the irreducible representations of the rotation group. In particular, we show that the 10-dimensional space spanned by triple monomials is the space of a double irreducible unitary representation of the rotation group. We use ternary q- and $\bar{q}$-commutators, where $q,\bar{q}$ are primitive 3rd order roots of unity, to split the 10-dimensional space spanned by triple monomials into a direct sum of two 5-dimensional subspaces. We endow these subspaces with a Hermitian scalar product by means of an orthonormal basis of triple monomials. In each subspace, there is an irreducible unitary representation $\mathrm{so}(3) \to \mathrm{su}(5)$. We calculate the matrix of this representation. The structure of this matrix indicates a possible connection between our algebra and the Georgi–Glashow model.

Keywords: ternary algebras, irreducible representations, ternary extension of Lie algebra, ternary generalization of Pauli's exclusion principle, quark model.

1. INTRODUCTION

The idea of using algebras with n-ary law of multiplication in theoretical physics is becoming more and more popular. An n-Lie algebra, where $n \geq 3$, is an n-ary extension of the concept of a Lie algebra, introduced by Filippov [10], and the important examples of n-Lie algebra constructed by means of commuting vector fields were proposed by Dzhumadildaev [9]. n-Lie algebras with a totally skew-symmetric n-ary Lie bracket, satisfying the Filippov–Jacobi identity, were used in the theory of M2 and M5 branes to construct a generalization of the Nahm's equation [4,5,7]. Independently of Filippov, Nambu [17] developed an analog of Hamiltonian mechanics defined on spaces of odd dimensions by introducing into consideration an n-ary analog of the Poisson bracket. This totally skew-symmetric n-ary bracket is now called an n-ary Nambu–Poisson bracket. Later it was proved that an n-ary Nambu–Poisson bracket satisfies the Filippov–Jacobi identity, which means that the algebraic structure induced by an n-ary Nambu–Poisson bracket can be considered as an n-Lie algebra. A concept of n-ary Lie algebra can be extended to Lie superalgebras and the paper [6] (and the references therein) can be used as an introduction to this aspect of the theory of n-ary

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Lie algebras. It is worth mentioning that originally one of the motivations for Nambu to introduce an n-ary generalization of the Poisson bracket was the quark model.

The quark model poses questions to theoretical physics that still have no clear answers. Undoubtedly, one of the most important problems of the quark model is confinement. However, no less intriguing is the question of why quarks have three colors and whether the coincidence of the number of colors with the dimension of our space is accidental. As the quark model still contains unsolved problems, it serves as a source of new ideas and approaches for theoretical physicists. Kerner [15,16] proposed a generalization of the Pauli's exclusion principle within the framework of the quark model. This generalization can be formulated as follows: *Three quarks* having identical quantum characteristics cannot form a stable configuration perceived as a strongly interacting particle. Note that this generalization of the Pauli's exclusion principle allows for a coexistence of *two quarks* with the same isospin value. Considering a mathematical form of this generalization of the Pauli's exclusion principle, we obtain an algebra with ternary relations. Therefore, Kerner suggests calling this generalization of the Pauli's exclusion principle the ternary generalization of the Pauli's principle.

Consider a wave function $\psi(u,u,u)$ that represents the tensor product of three identical quantum states $|u\rangle$. According to the ternary generalization of the Pauli's exclusion principle, this function must vanish, $\psi(u,u,u)=0$. Suppose now that we have a superposition of three different quantum states $|v\rangle = \lambda\,|u_1\rangle+\mu\,|u_2\rangle+\nu\,|u_3\rangle$. According to the ternary generalization of the Pauli's exclusion principle, we have $\psi(v,v,v)=0$. Assume the indices a,b,c run the values 1, 2, 3. Using the linearity of a wave function ψ, we obtain that $\psi(v,v,v)$ is equal to zero if and only if for any pair of integers $a\neq b$ one has

$$\psi(u_a,u_b,u_b)+\psi(u_b,u_b,u_a)+\psi(u_b,u_a,u_b)=0, \tag{1}$$

and for any triple of integers $a\neq b\neq c$ one has

$$\begin{aligned}\psi(u_a,u_b,u_c) \;&+\; \psi(u_b,u_c,u_a)+\psi(u_c,u_a,u_b)\\ &+\; \psi(u_c,u_b,u_a)+\psi(u_b,u_a,u_c)+\psi(u_a,u_c,u_b)=0.\end{aligned} \tag{2}$$

Obviously, the above conditions are satisfied if we assume that a wave function ψ is skew-symmetric in all arguments. However, this corresponds to the classical Pauli's exclusion principle, which excludes the coexistence of two quarks with the same isospin, i.e. in the case of skew-symmetry of a wave function each term in (1) is separately equal to zero, $\psi(u_a,u_b,u_b)=\psi(u_b,u_b,u_a)=\psi(u_b,u_a,u_b)=0$.

Still, there is another possibility to solve the conditions (1) and (2), a possibility that does not use the skew-symmetry of a wave function ψ and is consistent with the ternary generalization of the Pauli's exclusion principle. Indeed, if we assume that for any integers a,b,c (we admit a possibility of equal values) a wave function ψ has the property

$$\psi(u_a,u_b,u_c)+\psi(u_b,u_c,u_a)+\psi(u_c,u_a,u_b)=0, \tag{3}$$

then this wave function will satisfy (1) and (2).

Mathematically, the classical Pauli's principle can be expressed by means of the skew symmetry of a wave function of fermions, which in turn leads to a Grassmann algebra, i.e. to an algebra generated by anticommuting generators. Arguing in a similar way, we conclude that an algebra motivated by the ternary generalization of the Pauli's exclusion principle should be an algebra generated by θ^a, and it follows from the equation (3) that these generators should obey the ternary relations

$$\theta^a\theta^b\theta^c+\theta^b\theta^c\theta^a+\theta^c\theta^a\theta^b=0. \tag{4}$$

It should be noted that if we assume that generators θ^a satisfy the relations

$$\theta^a\theta^b\theta^c=q\,\theta^b\theta^c\theta^a=\bar{q}\,\theta^c\theta^a\theta^b,\; q=\exp(2\pi i/3), \tag{5}$$

which are evidently stronger than the relations (4), then the relations (4) will follow from (5). The algebra with relations (5) was introduced and studied in [1,2,11,13,14]. In this paper, we propose an approach to an algebra motivated by the ternary generalization of the Pauli's exclusion principle, which is based on the relations (4), which are more general than the relations (5). Note that an algebra with the relations (4) can be endowed with an involution by means of adding to the generators $\theta^1,\theta^2,\theta^3$ the conjugate generators $\bar\theta^1,\bar\theta^2,\bar\theta^3$, and the relations of an algebra with involution are given in [3].

In the present paper, we consider an associative unital algebra over the complex numbers $\mathbb{C}$ generated by $\theta^1,\theta^2,\ldots,\theta^N$, which obey the following relations:

$$\begin{aligned}\left(\theta^1\right)^2+\left(\theta^2\right)^2+\ldots+\left(\theta^N\right)^2 &= 0 \text{ (binary relation)}, &(6)\\ \theta^a\theta^b\theta^c+\theta^b\theta^c\theta^a+\theta^c\theta^a\theta^b &= 0 \text{ (ternary relations)}, &(7)\end{aligned}$$

where a,b,c is any triple of integers $1,2,\ldots,N$. We denote this algebra by $\mathfrak{R}$. This definition shows that in algebra $\mathfrak{R}$, in addition to the ternary relations (7) (which are due to the ternary generalization of the Pauli's exclusion principle), there is one quadratic relation (6). It will be shown that this relation is necessary in order to have a double irreducible representation of the rotation group in the space spanned by triple monomials of algebra $\mathfrak{R}$.

In this paper, we study the structure of algebra $\mathfrak{R}$ in the case when it is generated by three generators $\theta^1,\theta^2,\theta^3$. We show that $\mathfrak{R}$ is a finite-dimensional algebra, and we find the dimensions of subspaces spanned by homogeneous monomials. Let $\mathfrak{R}^p\subset\mathfrak{R}$ be a subspace spanned by homogeneous monomials of degree p. We show that the algebra $\mathfrak{R}$ provides a representation space for a double irreducible representation of the rotation group. In particular, we show that the 10-dimensional space $\mathfrak{R}^3$ spanned by monomials of degree 3 is the space of a double irreducible representation of weight 2 of the rotation group. We construct a basis for this 10-dimensional space by means of monomials of degree 3. Making use of the eigenvalues $q,\bar q$ of the substitution operator, we split the 10-dimensional space $\mathfrak{R}^3$ into the direct sum of two 5-dimensional subspaces $\mathfrak{R}^3_q,\mathfrak{R}^3_{\bar q}$. In each of these subspaces, we have an irreducible weight 2 representation $\mathrm{SO}(3)\to\mathrm{SU}(5)$. We show that the eigenvectors corresponding to the eigenvalue q can be constructed by means of a ternary q-commutator

$$[\theta^a,\theta^b,\theta^c]_q=\theta^a\theta^b\theta^c+\bar q\,\theta^b\theta^c\theta^a+q\,\theta^c\theta^a\theta^b, \tag{8}$$

and the eigenvectors corresponding to the eigenvalue $\bar q$ by means of a ternary $\bar q$-commutator

$$[\theta^a,\theta^b,\theta^c]_{\bar q}=\theta^a\theta^b\theta^c+q\,\theta^b\theta^c\theta^a+\bar q\,\theta^c\theta^a\theta^b. \tag{9}$$

We propose a cyclic $\mathbb{Z}_3$-extension of a Lie algebra based on the properties of the ternary commutators (8), (9), and give an example of such an extension for the Lie algebra induced by a unital associative algebra. We introduce a Hermitian inner product into the 5-dimensional spaces $\mathfrak{R}^3_q,\mathfrak{R}^3_{\bar q}$ by constructing an orthonormal basis. Then we find the explicit formula for irreducible representation in the 5-dimensional space of an infinitesimal rotation in space $\mathfrak{R}^1$. Since this irreducible representation is unitary, the obtained 5×5-matrix is skew-Hermitian and its structure is similar to the structure of the matrix for quarks in the Georgi–Glashow model [8].

2. ALGEBRA WITH TERNARY CYCLIC RELATIONS

The aim of this section is to introduce and study the algebra described in the Introduction. In this section we study the symmetries of algebraic relations of this algebra and find the dimensions of subspaces spanned by homogeneous monomials.

First of all, we define an algebra motivated by the ternary generalization of the Pauli's exclusion principle as follows:

Definition 1. *An algebra $\mathfrak{R}$ is a unital associative algebra over $\mathbb{C}$ generated by $\theta^1, \theta^2, \ldots, \theta^N$, which obey the quadratic relation*

$$(\theta^1)^2 + (\theta^2)^2 + \ldots + (\theta^N)^2 = 0, \tag{10}$$

and for any triple of integers a, b, c, the ternary relation

$$\theta^a\theta^b\theta^c + \theta^b\theta^c\theta^a + \theta^c\theta^a\theta^b = 0. \tag{11}$$

The identity element of an algebra $\mathfrak{R}$ will be denoted by $\mathbf{1}$. In the case of equal superscripts $a = b = c$, it follows from a ternary relation (11) that the cube of every generator θ^a is zero, i.e.

$$(\theta^a)^3 = 0,\ \forall a = 1, 2, \ldots, N.$$

It is useful to denote the sum of squares of generators by Ω and to introduce a polynomial of 3rd degree Ω^a as follows:

$$\begin{aligned} \Omega &= \delta_{ab}\theta^a\theta^b = (\theta^1)^2 + (\theta^2)^2 + \ldots + (\theta^N)^2, \\ \Omega^a &= \delta_{bc}\theta^b\theta^a\theta^c = \theta^1\theta^a\theta^1 + \theta^2\theta^a\theta^2 + \ldots + \theta^N\theta^a\theta^N. \end{aligned}$$

It is also useful to denote the left-hand sides of the ternary relation (11) as

$$\{\theta^a, \theta^b, \theta^c\} = \theta^a\theta^b\theta^c + \theta^b\theta^c\theta^a + \theta^c\theta^a\theta^b.$$

In analogy with a binary case, we call $\{\theta^a, \theta^b, \theta^c\}$ the ternary cyclic anticommutator of the generators $\theta^a, \theta^b, \theta^c$. Using these notations, the relations of the algebra $\mathfrak{R}$ can be written in a compact form as

$$\Omega = 0,\ \ \{\theta^a, \theta^b, \theta^c\} = 0.$$

Recall that the subspace of the algebra $\mathfrak{R}$, spanned by homogeneous monomials of degree p, is denoted by $\mathfrak{R}^p$. In particular, the subspace $\mathfrak{R}^1$ is spanned by the generators $\theta^1, \theta^2, \ldots, \theta^N$. The group of non-degenerate matrices $\mathrm{GL}(N, \mathbb{C})$ acts on this subspace according to $\tilde{\theta}^a = A^a_b\,\theta^b$, where $A = (A^a_b) \in \mathrm{GL}(N, \mathbb{C})$. This action can be considered as a transition to a set of new generators $\tilde{\theta}^a$ of the algebra $\mathfrak{R}$. It is easy to see that the new generators $\tilde{\theta}^a$ satisfy the same ternary relations (11), i.e. the ternary relations of algebra $\mathfrak{R}$ are invariant under the action of the group $\mathrm{GL}(N, \mathbb{C})$. Indeed, we have

$$\{\tilde{\theta}^a, \tilde{\theta}^b, \tilde{\theta}^c\} = A^a_i A^b_j A^c_k\,\{\theta^i, \theta^j, \theta^k\} = 0.$$

However, the system of all the relations of algebra $\mathfrak{R}$ also includes the quadratic relation (10), and this relation reduces the symmetry group $\mathrm{GL}(N, \mathbb{C})$ to its orthogonal subgroup $\mathrm{O}(N, \mathbb{C})$. Indeed, let $\tilde{\theta}^a = A^a_b\,\theta^b$, where $A = (A^a_b)$ is an orthogonal matrix $\delta_{ab}A^a_i A^b_j = \delta_{ij}$. Then

$$\tilde{\Omega} = \delta_{ab}\tilde{\theta}^a\tilde{\theta}^b = \delta_{ab}A^a_i A^b_j\theta^i\,\theta^j = \delta_{ij}\theta^i\,\theta^j = \Omega,$$

and from $\Omega = 0$ it immediately follows that $\tilde{\Omega} = 0$. If we multiply the quadratic relation (10) either from the left or from the right by a generator θ^a, we get the set of independent ternary relations

$$\theta^a\,\Omega = 0,\ \ \Omega\,\theta^a = 0,\ \ \ a = 1, 2, \ldots, N. \tag{12}$$

These relations are also invariant under the action of orthogonal group. Indeed, we have

$$\tilde{\theta}^a\,\tilde{\Omega} = (A^a_b\theta^b)\,\Omega = A^a_b\,(\theta^b\,\Omega) = 0,\ \tilde{\Omega}\,\tilde{\theta}^a = \Omega\,(A^a_b\theta^b) = A^a_b\,(\Omega\,\theta^b) = 0.$$

Hence, the relations (10), (11) are invariant under the action of the orthogonal group $\mathrm{O}(N,\mathbb{C})$. In particular, they are invariant under the action of the real special orthogonal group $\mathrm{SO}(N,\mathbb{R})$.

Now we assume that $N=3$, that is, we assume that the algebra $\mathfrak{R}$ is generated by $\theta^1,\theta^2,\theta^3$. Then the dimension of the subspace $\mathfrak{R}^2$, spanned by binary products of the generators $\theta^a\theta^b$, is 8, as we have nine possible products $\theta^a\theta^b$ and only one binary relation (10). Hence, $\dim\mathfrak{R}^2=9-1=8$. The following relations form the set of ternary relations of algebra $\mathfrak{R}$:

$$\{\theta^a,\theta^b,\theta^c\}=0,\quad \theta^a\,\Omega=0,\quad \Omega\,\theta^a=0, \tag{13}$$

where $a,b,c=1,2,3$. It is clear that ternary relations $\theta^a\,\Omega=0,\ \ \Omega\,\theta^a=0$, where $a=1,2,3$, are independent, and the number of these relations is six. All these relations can be represented by the following equations:

$$\theta^1\left((\theta^2)^2+(\theta^3)^2\right)=0,\ \left((\theta^2)^2+(\theta^3)^2\right)\theta^1=0, \tag{14}$$
$$\theta^2\left((\theta^3)^2+(\theta^1)^2\right)=0,\ \left((\theta^3)^2+(\theta^1)^2\right)\theta^2=0, \tag{15}$$
$$\theta^3\left((\theta^1)^2+(\theta^2)^2\right)=0,\ \left((\theta^1)^2+(\theta^2)^2\right)\theta^3=0. \tag{16}$$

In order to find independent ternary relations with the ternary cyclic anticommutator on the left-hand side, we have to take into account the following symmetry of the ternary cyclic commutator:

$$\{\theta^a,\theta^b,\theta^c\}=\{\theta^b,\theta^c,\theta^a\}=\{\theta^c,\theta^a,\theta^b\}.$$

Thus, if all three superscripts a,b,c are different, i.e. $a=1,b=2,c=3$, we get two independent relations:

$$\{\theta^1,\theta^2,\theta^3\}=0,\ \{\theta^3,\theta^2,\theta^1\}=0.$$

If two of the three values are equal but different from the third, we have only one independent relation. Thus, in this case, we have six independent relations:

$$\begin{aligned}&\{\theta^1,\theta^2,\theta^2\}=0,\quad \{\theta^2,\theta^1,\theta^1\}=0,\\ &\{\theta^1,\theta^3,\theta^3\}=0,\quad \{\theta^3,\theta^1,\theta^1\}=0,\\ &\{\theta^2,\theta^3,\theta^3\}=0,\quad \{\theta^3,\theta^2,\theta^2\}=0.\end{aligned}$$

Finally, equal superscripts $a=b=c$ give us three more independent relations $(\theta^1)^3=0$, $(\theta^2)^3=0$, and $(\theta^3)^3=0$. Summing up the number of independent ternary relations with the ternary cyclic anticommutator on the left-hand side, we obtain 11 relations, and adding to this number six previously found independent relations, we conclude that the number of all independent ternary relations is 17. Hence, the dimension of the subspace $\mathfrak{R}^3$, spanned by triple monomials $\theta^a\theta^b\theta^c$, is 27 – 17 = 10.

We construct a basis for the space $\mathfrak{R}^3$ as follows. Take $a=1,b=2,c=2$. This combination determines the ternary relation

$$\{\theta^1,\theta^2,\theta^2\}=\theta^1\theta^2\theta^2+\theta^2\theta^2\theta^1+\theta^2\theta^1\theta^2=0.$$

Taking two monomials $\theta^1\theta^2\theta^2$, $\theta^2\theta^2\theta^1$ as elements of a basis, we see that the monomial $\theta^2\theta^1\theta^2$ can be expressed in terms of these two monomials. The relations (14) show that the monomials $\theta^1\theta^3\theta^3$ and $\theta^3\theta^3\theta^1$ can also be expressed in terms of the monomials $\theta^1\theta^2\theta^2$, $\theta^2\theta^2\theta^1$. From the ternary relation

$$\{\theta^1,\theta^3,\theta^3\}=\theta^1\theta^3\theta^3+\theta^3\theta^3\theta^1+\theta^3\theta^1\theta^3=0,$$

it follows that the monomial $\theta^3\theta^1\theta^3$ can be expressed in terms of $\theta^1\theta^3\theta^3$, $\theta^3\theta^3\theta^1$, which means that it can also be expressed in terms of $\theta^1\theta^2\theta^2$ and $\theta^2\theta^2\theta^1$. Similar reasoning leads to the conclusion that each pair of integers $(1,2),(2,3),(3,1)$ gives two linearly independent monomials. We obtain four more monomials in the case when all indices are different, i.e. $a=1,b=2,c=3$ and $a=3,b=2,c=1$.

Hence, the following triple monomials

$$\mathfrak{f}_1 = \theta^1\theta^2\theta^2,\ \mathfrak{f}_2 = \theta^2\theta^3\theta^3,\ \mathfrak{f}_3 = \theta^3\theta^1\theta^1,\ \mathfrak{f}_4 = \theta^1\theta^2\theta^3,\ \mathfrak{f}_5 = \theta^3\theta^2\theta^1, \tag{17}$$

$$\mathfrak{f}_6 = \theta^2\theta^2\theta^1,\ \mathfrak{f}_7 = \theta^3\theta^3\theta^2,\ \mathfrak{f}_8 = \theta^1\theta^1\theta^3,\ \mathfrak{f}_9 = \theta^2\theta^3\theta^1,\ \mathfrak{f}_{10} = \theta^2\theta^1\theta^3 \tag{18}$$

form the basis for the 10-dimensional space $\mathfrak{N}^3$. As an index for elements of the basis (17), (18), we will use capital letters from the beginning of the Latin alphabet, assuming that they take integer values from 1 to 5. Then the first five elements of the basis can be written as $\mathfrak{f}_A$ and the next five elements as $\mathfrak{f}_{A+5}$, where $A = 1,2,\ldots,5$.

The dimensions of spaces of higher degree monomials were found by means of the methods of computer algebra. The dimension of the space $\mathfrak{N}^4$, spanned by monomials of the 4th degree, is 7. In the case of monomials of the 5th degree, the number of all possible monomials 2, 4, 3 coincides with the number of independent relations of the 5th order of the algebra $\mathfrak{N}$, and thus the dimension of the space $\mathfrak{N}^5$ (and the spaces $\mathfrak{N}^p, p > 5$) is 0. Hence, the dimension of the vector space of the whole algebra $\mathfrak{N}$ is 29.

3. IRREDUCIBLE REPRESENTATIONS OF THE ROTATION GROUP

The purpose of this section is to show that the algebra $\mathfrak{N}$ is closely related to irreducible tensor representations of the rotation group in a space of tensors of the 3rd order. More precisely, we will show that the 10-dimensional space $\mathfrak{N}^3$ of the algebra $\mathfrak{N}$, spanned by triple monomials, is the space of a double irreducible unitary representation of the rotation group SO(3). In the next section, in order to split this double irreducible representation into two irreducible representations, we will split by means of ternary commutators the 10-dimensional space $\mathfrak{N}^3$ into a direct sum of two 5-dimensional subspaces, and in each subspace we will have an irreducible representation $\mathrm{SO}(3) \to \mathrm{SU}(5)$.

Let $g = (g^a_b) \in \mathrm{SO}(3)$ be a 3rd order special orthogonal matrix. The rotation group acts in the space $\mathfrak{N}^p$, spanned by monomials of pth degree, as follows:

$$\Pi_g(\theta^{a_1}\theta^{a_2}\ldots\theta^{a_p}) = g^{a_1}_{b_1}g^{a_2}_{b_2}\cdots g^{a_p}_{b_p}\,\theta^{b_1}\theta^{b_2}\ldots\theta^{b_p}. \tag{19}$$

In this section, we will consider a representation of the rotation group in the space $\mathfrak{N}^3$, spanned by the triple monomials. Our aim is to show that this representation is a double irreducible tensor representation of the rotation group. We will consider an infinitesimal version of this representation, i.e. a representation of the Lie algebra so(3). This representation will be denoted by $L \in \mathrm{so}(3) \to \pi_L \in \mathfrak{gl}(\mathfrak{N}^3)$, where $L = (L^a_b)$ is a skew-symmetric 3rd order matrix. Obviously, $\pi_L : \mathfrak{N}^p \to \mathfrak{N}^p$ is a derivation of the algebra $\mathfrak{N}$, i.e.

$$\pi_L(\theta^{a_1}\theta^{a_2}\ldots\theta^{a_p}) = L^{a_1}_b\,\theta^b\theta^{a_2}\ldots\theta^{a_p} + L^{a_2}_b\,\theta^{a_1}\theta^b\ldots\theta^{a_p} + \ldots + L^{a_p}_b\,\theta^{a_1}\theta^{a_2}\ldots\theta^b.$$

Let $C_\pi = \pi^2_{L_1} + \pi^2_{L_2} + \pi^2_{L_3}$ be the Casimir operator of a representation π. Here L_1, L_2, L_3 are the infinitesimal rotations,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is proved in [12] that the tensors of 3rd order, transforming according to an irreducible representation of the rotation group, satisfy a condition, which can be written in terms of triple monomials $\theta^a\theta^b\theta^c$ as follows:

$$C_\pi\,(\theta^a\theta^b\theta^c) + 6\,\theta^a\theta^b\theta^c = 0. \tag{20}$$

In other words, all triple monomials, which satisfy the condition (20), span the representation space of a double irreducible representation of the rotation group.

First of all, we will show that the triple monomials, which span the space $\mathfrak{R}^3$ of the algebra $\mathfrak{R}$, satisfy the condition (20). In other words, the condition (20) follows from the ternary relations (13) of the algebra $\mathfrak{R}$. For further calculations, it is convenient to introduce the 3rd order matrices $\rho^{ab}=\varepsilon^{abc}L_c$, where ε^{abc} is a totally skew-symmetric tensor. If $(\rho^{ab})^c_d$ are the entries of the matrix ρ^{ab}, and we define $\pi^{ab}(\theta^c)=(\rho^{ab})^c_d\,\theta^d$, then

$$\pi^{ab}(\theta^c)=\delta^{bc}\,\theta^a-\delta^{ac}\,\theta^b. \tag{21}$$

Taking into account that π^{ab} acts on a triple monomial as an algebra derivation, we obtain

$$\begin{aligned}\pi^{dh}\big(\theta^a\theta^b\theta^c\big) &= \delta^{ah}\theta^d\theta^b\theta^c+\delta^{bh}\theta^a\theta^d\theta^c+\delta^{ch}\theta^a\theta^b\theta^d\\ &\qquad -\delta^{da}\theta^h\theta^b\theta^c-\delta^{db}\theta^a\theta^h\theta^c-\delta^{dc}\theta^a\theta^b\theta^h.\end{aligned} \tag{22}$$

Making use of (21), we can calculate the square of the operator π^{dh} and, acting by the square of this operator on a triple monomial $\theta^a\theta^b\theta^c$, we obtain the expression

$$\begin{aligned}
&\delta^{ah}\delta^{dh}\,\theta^d\theta^b\theta^c+\delta^{ah}\delta^{bh}\,\theta^d\theta^d\theta^c+\delta^{ah}\delta^{ch}\,\theta^d\theta^b\theta^d-\delta^{ah}\delta^{dd}\,\theta^h\theta^b\theta^c\\
&-\delta^{ah}\delta^{db}\,\theta^d\theta^h\theta^c-\delta^{ah}\delta^{dc}\,\theta^d\theta^b\theta^h+\delta^{ah}\delta^{bh}\,\theta^d\theta^d\theta^c+\delta^{bh}\delta^{dh}\,\theta^a\theta^d\theta^c\\
&+\delta^{bh}\delta^{ch}\,\theta^a\theta^d\theta^d-\delta^{ad}\delta^{bh}\,\theta^h\theta^d\theta^c-\delta^{dd}\delta^{bh}\,\theta^a\theta^h\theta^c-\delta^{dc}\delta^{bh}\,\theta^a\theta^d\theta^h\\
&+\delta^{ah}\delta^{ch}\,\theta^d\theta^b\theta^d+\delta^{bh}\delta^{ch}\,\theta^a\theta^d\theta^d+\delta^{ch}\delta^{dh}\,\theta^a\theta^b\theta^d-\delta^{ad}\delta^{ch}\,\theta^h\theta^b\theta^d\\
&-\delta^{bd}\delta^{ch}\,\theta^a\theta^h\theta^d-\delta^{dd}\delta^{ch}\,\theta^a\theta^b\theta^h-\delta^{ad}\delta^{hh}\,\theta^d\theta^b\theta^c-\delta^{ad}\delta^{bh}\,\theta^h\theta^d\theta^c\\
&-\delta^{ad}\delta^{ch}\,\theta^h\theta^b\theta^d+\delta^{ad}\delta^{dh}\,\theta^h\theta^b\theta^c+\delta^{ad}\delta^{bd}\,\theta^h\theta^h\theta^c+\delta^{ad}\delta^{cd}\,\theta^h\theta^b\theta^h\\
&-\delta^{bd}\delta^{ah}\,\theta^d\theta^h\theta^c-\delta^{bd}\delta^{hh}\,\theta^a\theta^d\theta^c-\delta^{bd}\delta^{ch}\,\theta^a\theta^h\theta^d+\delta^{ad}\delta^{bd}\,\theta^h\theta^h\theta^c\\
&+\delta^{bd}\delta^{hd}\,\theta^a\theta^h\theta^c+\delta^{bd}\delta^{cd}\,\theta^a\theta^h\theta^h-\delta^{ah}\delta^{cd}\,\theta^d\theta^b\theta^h-\delta^{bh}\delta^{cd}\,\theta^a\theta^d\theta^h\\
&-\delta^{cd}\delta^{hh}\,\theta^a\theta^b\theta^d+\delta^{ad}\delta^{cd}\,\theta^h\theta^b\theta^h+\delta^{bd}\delta^{cd}\,\theta^a\theta^h\theta^h+\delta^{cd}\delta^{dh}\,\theta^a\theta^b\theta^h.
\end{aligned} \tag{23}$$

Now we calculate the Casimir operator $C_\pi=(\pi^{23})^2+(\pi^{31})^2+(\pi^{12})^2$. From the formula (21) it follows that $\pi^{aa}=0$ and $\pi^{ab}=-\pi^{ba}$, hence $(\pi^{ab})^2=(\pi^{ba})^2$. If we take the sum of the squares of operators π^{dh}, where the superscripts d,h run the values $1,2,3$, we obtain two times the Casimir operator, $2C_\pi$. This means that if in the expression (23) we take the sum over the superscripts d,h, the resulting expression will be equal to $2C_\pi$. We obtain

$$C_\pi(\theta^a\theta^b\theta^c)=2\big(\delta^{ab}\,\Omega\,\theta^c+\delta^{bc}\,\theta^a\,\Omega+\delta^{ca}\,\Omega^b-\{\theta^c,\theta^b,\theta^a\}-3\,\theta^a\theta^b\theta^c\big), \tag{24}$$

where

$$\Omega=\delta_{ab}\theta^a\theta^b=\sum_a(\theta^a)^2,\ \Omega^b=\delta_{ac}\theta^a\theta^b\theta^c=\sum_a\theta^a\theta^b\theta^a. \tag{25}$$

Substituting the right-hand side of (24) into the basic equation (20), we obtain

$$\delta^{ab}\,\Omega\,\theta^c+\delta^{bc}\,\theta^a\,\Omega+\delta^{ca}\,\Omega^b-\{\theta^c,\theta^b,\theta^a\}=0. \tag{26}$$

By virtue of our assumption that a triple monomial $\theta^a\theta^b\theta^c$ belongs to the space $\mathfrak{R}^3$, in this case the generators $\theta^1,\theta^2,\theta^3$ obey the ternary relations

$$\theta^a\,\Omega=0,\quad \Omega\,\theta^a=0,\quad \{\theta^a,\theta^b,\theta^c\}=0. \tag{27}$$

We see that the first, second and fourth terms in (26) vanish and the condition (26) takes the form $\delta^{ca}\,\Omega^b=0$. It is easy to show that in the algebra $\mathfrak{R}$, in addition to the above ternary relations, we also have the ternary relations $\Omega^a=0$. Indeed, we can write

$$\delta_{bc}\{\theta^b,\theta^c,\theta^a\}=\Omega\,\theta^a+\Omega^a+\theta^a\,\Omega. \tag{28}$$

Consequently, $\Omega^a = \delta_{bc}\{\theta^b,\theta^c,\theta^a\} - \Omega\,\theta^a - \theta^a\,\Omega$, and the relations $\Omega^a = 0$, where $a = 1,2,3$, follow immediately from the ternary relations (27). Thus, we have proved that the condition (20) of irreducibility of a representation π follows from the ternary relations of the algebra $\mathfrak{R}$.

Conversely, we can now prove that the ternary relations of the algebra $\mathfrak{R}$ follow from the condition (20). Indeed, let us assume that a triple monomial $\theta^a\theta^b\theta^c$ satisfies the condition (20). But this condition can be put in the form (26). Hence, taking the sum over $a = b$ in the left-hand side of this equation, we get $\Omega\,\theta^a = 0$. Analogously, the sum over $b = c$ leads to $\theta^a\Omega = 0$ and the sum over $a = c$ gives $\Omega^a = 0$. Hence, we also have $\{\theta^a,\theta^b,\theta^c\} = 0$. Thus, we have proved that the ternary relations of the algebra $\mathfrak{R}$ follow from the irreducibility condition (20) for a representation π, and the space $\mathfrak{R}^3$ is a representation space for a double irreducible representation of the rotation group.

4. SUBSTITUTION OPERATOR AND ITS EIGENVECTORS

In this section, our aim is to split the 10-dimensional space $\mathfrak{R}^3$ of the algebra $\mathfrak{R}$ into a direct sum of two 5-dimensional subspaces in such a way that this decomposition will be invariant with respect to the representation of the rotation group. Then in each 5-dimensional subspace there will be an irreducible unitary representation of the rotation group $\mathrm{SO}(3) \to \mathrm{SU}(5)$. This decomposition can be done with the help of eigenvectors of a substitution operator. Let σ be a cyclic substitution such that $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. Then we can define a substitution operator S_σ acting on triple monomials as follows:

$$S_\sigma(\theta^{a_1}\theta^{a_2}\theta^{a_3}) = \theta^{a_{\sigma(1)}}\theta^{a_{\sigma(2)}}\theta^{a_{\sigma(3)}} = \theta^{a_2}\theta^{a_3}\theta^{a_1}.$$

The substitution operator S_σ maps each ternary relation from the set of relations (27) either into a ternary relation from the same set of ternary relations or into their linear combination. In other words, the substitution operator preserves the structure of the space $\mathfrak{R}^3$. It is evident that the ternary relations with the ternary cyclic anticommutator on the left-hand side are invariant with respect to the substitution operator S_σ. Indeed, we have

$$S_\sigma(\{\theta^a,\theta^b,\theta^c\}) = \{\theta^b,\theta^c,\theta^a\} = \{\theta^a,\theta^b,\theta^c\}. \tag{29}$$

The second part of the ternary relations (14)–(16) is also invariant with respect to S_σ. Indeed, if we apply the substitution operator S_σ to the first relation from the left in (14), we obtain the second ternary relation in the same row:

$$\begin{aligned} S_\sigma\Big(\theta^1\big((\theta^2)^2+(\theta^3)^2\big)\Big) &= S_\sigma(\theta^1\theta^2\theta^2)+S_\sigma(\theta^1\theta^3\theta^3) = \theta^2\theta^2\theta^1+\theta^3\theta^3\theta^1 \\ &= \big((\theta^2)^2+(\theta^3)^2\big)\theta^1 = 0. \end{aligned}$$

If we apply the substitution operator S_σ to the left-hand side of the second relation in (14), we obtain

$$\begin{aligned} S_\sigma\Big(\big((\theta^2)^2+(\theta^3)^2\big)\theta^1\Big) &= S_\sigma(\theta^2\theta^2\theta^1)+S_\sigma(\theta^3\theta^3\theta^1) \\ &= \theta^2\theta^1\theta^2+\theta^3\theta^1\theta^3. \end{aligned} \tag{30}$$

However, in the previous section it is proved that for any integer $a = 1,2,3$ it holds $\Omega^a = 0$, where

$$\Omega^a = \theta^1\theta^a\theta^1+\theta^2\theta^a\theta^2+\theta^3\theta^a\theta^3.$$

Taking subsequently $a = 1,2,3$ in $\Omega^a = 0$, we get three relations:

$$\theta^2\theta^1\theta^2+\theta^3\theta^1\theta^3 = 0,\ \theta^1\theta^2\theta^1+\theta^3\theta^2\theta^3 = 0,\ \theta^2\theta^3\theta^2+\theta^1\theta^3\theta^1 = 0,$$

which show that (30) does not lead to a new relation, but gives the relation that follows from the set of ternary relations (27).

From $\sigma^3 = id$, where id is the identical substitution, it follows that $S_\sigma^3 = I$, where I is the identity operator in the vector space $\mathfrak{R}^3$. Thus, the eigenvalues of the substitution operator S_σ are cubic roots of unity $1, q, \bar{q}$, where $q = \exp(2\pi i/3)$. This means that we can decompose the space $\mathfrak{R}^3$, spanned by triple monomials, into the direct sum of the subspaces, spanned by the eigenvectors corresponding to the eigenvalues $1, q, \bar{q}$. Let us denote these subspaces by $\mathfrak{R}^3_1$, $\mathfrak{R}^3_q$, $\mathfrak{R}^3_{\bar{q}}$. Then $\mathfrak{R}^3 = \mathfrak{R}^3_1 \oplus \mathfrak{R}^3_q \oplus \mathfrak{R}^3_{\bar{q}}$.

The ternary cyclic anticommutator can be written with the help of the substitution operator as follows:

$$\{\theta^a, \theta^b, \theta^c\} = \theta^a\theta^b\theta^c + S_\sigma(\theta^a\theta^b\theta^c) + S_\sigma^2(\theta^a\theta^b\theta^c). \tag{31}$$

The formula (29) shows that the ternary cyclic anticommutator is an eigenvector of the substitution operator with eigenvalue 1. Because of ternary relations $\{\theta^a, \theta^b, \theta^c\} = 0$, the subspace of eigenvectors of the substitution operator S_σ with eigenvalue 1 is the trivial subspace $\mathfrak{R}^3_1 = \{0\}$, and we have $\mathfrak{R}^3 = \mathfrak{R}^3_q \oplus \mathfrak{R}^3_{\bar{q}}$.

Let us consider the following ternary expressions of generators:

$$\begin{aligned} [\theta^a, \theta^b, \theta^c]_q &= \theta^a\theta^b\theta^c + \bar{q}S_\sigma(\theta^a\theta^b\theta^c) + qS_\sigma^2(\theta^a\theta^b\theta^c) \\ &= \theta^a\theta^b\theta^c + \bar{q}\,\theta^b\theta^c\theta^a + q\,\theta^c\theta^a\theta^b \end{aligned} \tag{32}$$

and

$$\begin{aligned} [\theta^a, \theta^b, \theta^c]_{\bar{q}} &= \theta^a\theta^b\theta^c + qS_\sigma(\theta^a\theta^b\theta^c) + \bar{q}S_\sigma^2(\theta^a\theta^b\theta^c) \\ &= \theta^a\theta^b\theta^c + q\,\theta^b\theta^c\theta^a + \bar{q}\,\theta^c\theta^a\theta^b. \end{aligned} \tag{33}$$

It is easy to verify that (32) and (33) are the eigenvectors of S_σ. Indeed, we have

$$\begin{aligned} S_\sigma\big([\theta^a, \theta^b, \theta^c]_q\big) &= S_\sigma(\theta^a\theta^b\theta^c) + \bar{q}\,S_\sigma^2(\theta^a\theta^b\theta^c) + qS_\sigma^3(\theta^a\theta^b\theta^c) \\ &= q\,(\theta^a\theta^b\theta^c + \bar{q}\,S_\sigma(\theta^a\theta^b\theta^c) + qS_\sigma^2(\theta^a\theta^b\theta^c)) = q\,[\theta^a, \theta^b, \theta^c]_q, \end{aligned} \tag{34}$$

and we see that the expression $[\theta^a, \theta^b, \theta^c]_q$ is an eigenvector of the substitution operator with the eigenvalue q. Analogously, one can verify that $[\theta^a, \theta^b, \theta^c]_{\bar{q}}$ is an eigenvector of the substitution operator with the eigenvalue $\bar{q}$.

5. CYCLIC $\mathbb{Z}_3$-EXTENSION OF A LIE ALGEBRA

In this section, we will study the two expressions (32) and (33), introduced at the end of the previous section, in order to construct the eigenvectors of the substitution operator S_σ. Our aim in this section is to show that these two expressions have properties similar to the properties of the binary commutator $[A, B] = A\cdot B - B\cdot A$, and they can be used to construct a ternary extension of a Lie algebra different from the one proposed by Filippov and Nambu.

We start with general considerations to show that both the binary commutator and its n-ary generalization, proposed in the approach of Filippov and Nambu, are special cases of the same general construction. Let G be a subgroup of a symmetric group of order n. Suppose this subgroup has a representation $\rho : G \to \mathbb{C}$ in the field of complex numbers such that

$$\sum_{\sigma\in G} \rho(\sigma) = 0. \tag{35}$$

Let L be a vector space over complex numbers. Then an n-ary commutator based on the group G and its representation ρ is a multilinear mapping

$$(u_1, u_2, \ldots, u_n) \in L \times L \times \ldots \times L \to [u_1, u_2, \ldots, u_n] \in L$$

that has the property

$$[u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)}]_\rho = \rho(\sigma)[u_1, u_2, \ldots, u_n]. \tag{36}$$

Note that from (35) and (36) it follows that

$$\sum_{\sigma \in G} [u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)}]_\rho = 0. \tag{37}$$

The binary commutator and its n-ary generalization, proposed by Filippov and Nambu, fit this definition as a particular case when G is a symmetric group of order n and ρ maps an even permutation to 1 and an odd permutation to –1. Then the property (36) means a skew-symmetry of an n-ary commutator. Note that in this construction, 1 and –1 can be considered as the square roots of unity. Moreover, the representation ρ is faithful only in the case of the symmetric group of order 2, i.e. in the case of the binary commutator.

Now, let G be the group $\mathbb{Z}_3$ of cyclic substitutions of the set $\{1,2,3\}$. Then there are two faithful representations of $\mathbb{Z}_3$ by cubic roots of unity:

$$\rho : \{\mathrm{id}, \sigma, \sigma^2\} \to \{1, q, \bar{q}\}, \ \bar{\rho} : \{\mathrm{id}, \sigma, \sigma^2\} \to \{1, \bar{q}, q\},$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ q = \exp(2\pi i/3).$$

Thus, applying the above given general definition (36), we obtain two ternary commutators $[u_1, u_2, u_3]_q$ and $[u_1, u_2, u_3]_{\bar{q}}$ based on the cyclic group $\mathbb{Z}_3$ and its two faithful representations by cubic roots of unity. These ternary commutators have the properties

$$[u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}]_q = \rho(\sigma)\,[u_1, u_2, u_3]_q, \tag{38}$$

$$[u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}]_{\bar{q}} = \bar{\rho}(\sigma)\,[u_1, u_2, u_3]_{\bar{q}}, \tag{39}$$

and, subsequently, we will refer to them as the ternary cyclic q-commutator and the ternary cyclic $\bar{q}$-commutator, respectively. It is easy to verify that the two expressions (32), (33), introduced in the previous section, in order to construct the eigenvectors of the cyclic substitution operator S_σ, have exactly the same properties, i.e. they are ternary cyclic commutators.

Naturally, those properties of a commutator that we use to extend the notion of a binary Lie bracket to algebras with an n-ary multiplication law are only a part of the structure that could be called a generalization of a Lie algebra. The second part of this structure, and the most important one, is an analogue of the Jacobi identity. We propose such an identity in the following definition.

Definition 2. *Let $\mathfrak{g}$ be a Lie algebra and let $\rho : a \in \mathfrak{g} \to \rho_a \in \mathfrak{gl}(V)$ be its representation in a vector space V. A trilinear mapping*

$$(x, y, z) \in V \times V \times V \to [\![x, y, z]\!] \in V \tag{40}$$

is called a ternary $\mathbb{Z}_3$-bracket compatible with a representation ρ of a Lie algebra $\mathfrak{g}$ if it has the property

$$[\![x, y, z]\!] + [\![y, z, x]\!] + [\![z, x, y]\!] = 0, \ x, y, z \in V \tag{41}$$

and it satisfies the identity

$$\rho_a([\![x, y, z]\!]) = [\![\rho_a(x), y, z]\!] + [\![x, \rho_a(y), z]\!] + [\![x, y, \rho_a(z)]\!]. \tag{42}$$

A vector space V equipped with a ternary $\mathbb{Z}_3$-bracket, compatible with a representation ρ of a Lie algebra $\mathfrak{g}$, will be called a ternary $\mathbb{Z}_3$-extension of a Lie algebra $\mathfrak{g}$.

As a comment on this definition, we note that the ternary cyclic q-commutator and the ternary cyclic $\bar{q}$-commutator satisfy the condition (41). This follows from the general definition and the formula (37).

In order to give an example of a ternary $\mathbb{Z}_3$-extension of a Lie algebra, we consider a unital associative algebra $\mathscr{A}$ over the complex numbers. The identity element of this algebra will be denoted by e. We can consider the algebra $\mathscr{A}$ as a Lie algebra by equipping it with the commutator $[u,v] = uv - vu$, where $u, v \in \mathscr{A}$. To emphasize the fact that we are considering $\mathscr{A}$ as a Lie algebra, we will denote it by $\mathscr{A}_L$. As a representation of this Lie algebra, we use the adjoint representation $ad : u \in \mathscr{A}_L \to \mathrm{ad}_u \in \mathfrak{gl}(\mathscr{A})$, where $\mathrm{ad}_u(x) = [u,x]$.

Theorem 1. *Define*

$$[\![x,y,z]\!] = xyz + \bar{q}\, yzx + q\, zxy, \;\; x,y,z \in \mathscr{A}. \tag{43}$$

Then (43) *is a ternary cyclic q-bracket compatible with the adjoint representation of the Lie algebra $\mathscr{A}_L$, and $\mathscr{A}$ equipped with* (43) *is a ternary $\mathbb{Z}_3$-extension of the Lie algebra $\mathscr{A}_L$.*

Proof. The statement that (43) is a ternary cyclic q-bracket follows from the formula (34) proved at the end of the previous section. Direct calculations, which will be not presented in this paper, show that in this case for any elements x, y, z, v of an algebra $\mathscr{A}$ we have the identity

$$[x, [\![y,z,v]\!]] = [\![[x,y], z, v]\!] + [\![y, [x,z], v]\!] + [\![y, z, [x,v]]\!]. \tag{44}$$

This identity shows that the adjoint representation of the Lie algebra $\mathscr{A}_L$ is compatible with the ternary cyclic q-commutator (43). Hence, the vector space $\mathscr{A}$ equipped with the ternary cyclic q-commutator (43) is a ternary $\mathbb{Z}_3$-extension of the Lie algebra $\mathscr{A}_L$. □

It is worth noting that the ternary bracket (43) is not associative. Recall that in the case of a ternary operation there are two types of associativity. These are associativity of the first kind,

$$(xyz)uv = x(yzu)v = xy(zuv)$$

and associativity of the second kind,

$$(xyz)uv = x(uzy)v = xy(zuv),$$

where $x, y, z, u, v \in \mathscr{A}$. In case of the ternary bracket (43), neither the associativity of the first kind nor the associativity of the second kind is applicable. This is natural, since in the binary case the commutator on an associative algebra defines a Lie algebra, which is a non-associative algebra.

The final remark is that the identity element e of an algebra $\mathscr{A}$ makes it possible to reduce the ternary bracket (43) to the usual commutator of two elements. Indeed, we have

$$\begin{aligned} [\![x,e,y]\!] &= xey + \bar{q}\, eyx + q\, yxe = xy + (\bar{q} + q)\, yx \\ &= xy - yx = [x,y]. \end{aligned} \tag{45}$$

It is also important to note here that if we take $z = e$ in the identity (44), the identity (44) takes the form of the Jacobi identity. Hence, the identity (44) can be considered as a $\mathbb{Z}_3$-extension of the Jacobi identity. We think that this justifies the use of our proposed term '$\mathbb{Z}_3$-extension of a Lie algebra'.

6. EIGENVECTOR BASES

Our next aim is to prove that the polynomials

$$\begin{gathered} \mathfrak{f}_1' = [\theta^1, \theta^2, \theta^2]_q, \;\; \mathfrak{f}_2' = [\theta^2, \theta^3, \theta^3]_q, \;\; \mathfrak{f}_3' = [\theta^3, \theta^1, \theta^1]_q, \\ \mathfrak{f}_4' = [\theta^1, \theta^2, \theta^3]_q, \;\; \mathfrak{f}_5' = [\theta^3, \theta^2, \theta^1]_q \end{gathered} \tag{46}$$

and

$$\bar{\mathfrak{f}}'_1 = [\theta^1,\theta^2,\theta^2]_{\bar{q}},\ \bar{\mathfrak{f}}'_2 = [\theta^2,\theta^3,\theta^3]_{\bar{q}},\ \bar{\mathfrak{f}}'_3 = [\theta^3,\theta^1,\theta^1]_{\bar{q}},$$
$$\bar{\mathfrak{f}}'_4 = [\theta^1,\theta^2,\theta^3]_{\bar{q}},\ \bar{\mathfrak{f}}'_5 = [\theta^3,\theta^2,\theta^1]_{\bar{q}} \tag{47}$$

form the bases for the subspaces $\mathfrak{R}^3_q$ and $\mathfrak{R}^3_{\bar{q}}$, respectively. To prove this, it is necessary to prove that these polynomials form a complete system of linearly independent polynomials. We start by showing that this system of polynomials is complete, i.e. that any q-eigenvector is a linear combination of these polynomials. We will use the basis $\{\mathfrak{f}_A,\mathfrak{f}_{A+5}\}$, where $A=1,2,\ldots,5$, for the subspace $\mathfrak{R}^3$, spanned by triple monomials. Note that $\mathfrak{f}_{A+5}=S_\sigma(\mathfrak{f}_A)$, and we can write the basis in the form $\{\mathfrak{f}_A,S_\sigma(\mathfrak{f}_A)\}$. Each vector X of the space $\mathfrak{R}^3$ can be written as a linear combination of vectors of the basis

$$X=\sum_A \big(x_A\,\mathfrak{f}_A+x_{A+5}\,S_\sigma(\mathfrak{f}_A)\big). \tag{48}$$

A vector X is a λ-eigenvector of S_σ if $S_\sigma(X)=\lambda X$, where $\lambda\in\{1,q,\bar{q}\}$. Hence, for a λ-eigenvector, we have the equation

$$\sum_A \big(x_A\,S_\sigma(\mathfrak{f}_A)+x_{A+5}\,S^2_\sigma(\mathfrak{f}_A)\big)=\sum_A(\lambda\,x_A\,\mathfrak{f}_A+\lambda\,x_{A+5}\,S_\sigma(\mathfrak{f}_A)). \tag{49}$$

By virtue of the relations (11) of the algebra $\mathfrak{R}$, for any $1\le A\le 5$ we have

$$\mathfrak{f}_A+S_\sigma(\mathfrak{f}_A)+S^2_\sigma(\mathfrak{f}_A)=0.$$

Therefore, $S^2_\sigma(\mathfrak{f}_A)=-\mathfrak{f}_A-S_\sigma(\mathfrak{f}_A)$. Substituting this expression into the left-hand side of the equation (49), we obtain

$$\sum_A \big(-x_{A+5}\,\mathfrak{f}_A+(x_A-x_{A+5})\,S_\sigma(\mathfrak{f}_A)\big)=\sum_A \big(\lambda\,x_A\,\mathfrak{f}_A+\lambda\,x_{A+5}\,S_\sigma(\mathfrak{f}_A)\big). \tag{50}$$

This leads to a system of linear equations

$$x_{A+5}=-\lambda x_A,\ \ x_A-x_{A+5}=\lambda\,x_{A+5}. \tag{51}$$

Substituting $-\lambda x_A$ instead of x_{A+5} in the second equation, we obtain

$$(1+\lambda+\lambda^2)\,x_A=0.$$

Taking the first eigenvalue $\lambda=1$, we see that this equation, and, hence, the system of equations (51), has only the trivial solution $x_A=x_{A+5}=0$. This proves that the subspace of eigenvectors with eigenvalue 1 is trivial, as stated above.

Since $q,\bar{q}$ are the roots of the equation $1+\lambda+\lambda^2=0$, in the case of $\lambda=q$ or $\lambda=\bar{q}$, the system (51) has non-trivial solutions, which can be written as

$$x_{A+5}=-q\,x_A\ \ (q\text{-eigenvectors}),\quad x_{A+5}=-\bar{q}\,x_A\ \ (\bar{q}\text{-eigenvectors}).$$

Therefore, if X is a q-eigenvector of the substitution operator S_σ, it can be written in the form

$$X=\sum_A x_A\,(\mathfrak{f}_A-q\,S_\sigma(\mathfrak{f}_A)). \tag{52}$$

If X is a $\bar{q}$-eigenvector of the substitution operator S_σ, it can be written in the form

$$X=\sum_A x_A\,(\mathfrak{f}_A-\bar{q}\,S_\sigma(\mathfrak{f}_A)). \tag{53}$$

It is easily verified that the vectors $\mathfrak{f}_A - q\,S_\sigma(\mathfrak{f}_A)$ and $\mathfrak{f}_A - \bar{q}\,S_\sigma(\mathfrak{f}_A)$ are linearly independent q-eigenvectors and $\bar{q}$-eigenvectors of the substitution operator S_σ, and they form the bases for the subspaces $\mathfrak{R}^3_q$ and $\mathfrak{R}^3_{\bar{q}}$, respectively. The formulae (52) and (53) show that linearly independent q-eigenvectors and $\bar{q}$-eigenvectors form complete systems of vectors for the subspaces $\mathfrak{R}^3_q$ and $\mathfrak{R}^3_{\bar{q}}$, respectively. Thus, they are the bases for these subspaces.

The transition from the bases $\mathfrak{f}_A - q\,S_\sigma(\mathfrak{f}_A)$ and $\mathfrak{f}_A - \bar{q}\,S_\sigma(\mathfrak{f}_A)$ to the systems of polynomials (46) and (47), respectively, is carried out by multiplying each polynomial $\mathfrak{f}_A - q\,S_\sigma(\mathfrak{f}_A)$ ($\mathfrak{f}_A - \bar{q}\,S_\sigma(\mathfrak{f}_A)$) by the factor $1-q$ ($1-\bar{q}$). Indeed, for the first polynomial of the basis $\mathfrak{f}_A - q\,S_\sigma(\mathfrak{f}_A)$, i.e. $A=1$, we have

$$\begin{aligned}(1-q)\,(\theta^1\theta^2\theta^2 - q\,\theta^2\theta^2\theta^1) &= \theta^1\theta^2\theta^2 - q\,\theta^1\theta^2\theta^2 - q\,\theta^2\theta^2\theta^1 + q^2\,\theta^2\theta^2\theta^1\\ &= \theta^1\theta^2\theta^2 + q^2\,\theta^2\theta^2\theta^1 + q\,(-\theta^1\theta^2\theta^2 - \theta^2\theta^2\theta^1)\\ &= \theta^1\theta^2\theta^2 + q^2\,\theta^2\theta^2\theta^1 + q\,\theta^2\theta^1\theta^2 = [\theta^1,\theta^2,\theta^2]_q = \mathfrak{f}'_1.\end{aligned}$$

In deriving this relation, we used the relation $\theta^1\theta^2\theta^2 + \theta^2\theta^2\theta^1 + \theta^2\theta^1\theta^2 = 0$ of the algebra $\mathfrak{R}$, written in the form $-\theta^1\theta^2\theta^2 - \theta^2\theta^2\theta^1 = \theta^2\theta^1\theta^2$. Relations for the remaining polynomials can be derived similarly. Therefore, we have

$$\mathfrak{f}'_A = (1-q)\,(\mathfrak{f}_A - q\,S_\sigma(\mathfrak{f}_A)),\;\; \bar{\mathfrak{f}}'_A = (1-\bar{q})\,(\mathfrak{f}_A - \bar{q}\,S_\sigma(\mathfrak{f}_A)).$$

Obviously, after the multiplication by $1-q$ and $1-\bar{q}$, the resulting systems of polynomials $\mathfrak{f}'_A$ and $\bar{\mathfrak{f}}'_A$ are still the bases for $\mathfrak{R}^3_q$ and $\mathfrak{R}^3_{\bar{q}}$, respectively.

7. MATRIX OF REPRESENTATION SO(3) → SU(5)

The algebra we have introduced is closely related to irreducible representations of the rotation group SO(3). A rotation in the space $\mathfrak{R}^1$, spanned by the generators of the algebra θ^a, generates linear transformations in the 5-dimensional subspaces $\mathfrak{R}^3_q, \mathfrak{R}^3_{\bar{q}}$, spanned by the ternary polynomials $\mathfrak{f}'_A$ and $\bar{\mathfrak{f}}'_A$, respectively. These linear transformations form the 5-dimensional irreducible representations of the rotation group. To describe the infinitesimal form of these irreducible representations, we introduce a derivation of the algebra $\mathfrak{R}$. Let L be an element of the Lie algebra so(3), i.e. L is a skew-symmetric matrix and

$$L = \begin{pmatrix} 0 & u^3 & -u^2\\ -u^3 & 0 & u^1\\ u^2 & -u^1 & 0\end{pmatrix} = -u^1\,L_1 - u^2\,L_2 - u^3\,L_3 = -\sum_a u^a\,L_a.$$

To each such matrix we associate a derivation $\mathscr{D}_L$ of the algebra $\mathfrak{R}$, defining it on the identity element and the generators of the algebra as

$$\mathscr{D}_L(\mathbf{1}) = 0,\; \mathscr{D}_L(\theta^1) = u^3\theta^2 - u^2\theta^3,\; \mathscr{D}_L(\theta^2) = u^1\theta^3 - u^3\theta^1,\; \mathscr{D}_L(\theta^3) = u^2\theta^1 - u^1\theta^2.$$

The derivation $\mathscr{D}_L$ extends to the whole algebra $\mathfrak{R}$ by linearity and the Leibniz rule.

Our next goal is to introduce an orthonormal basis in the space $\mathfrak{R}^3$ in such a way that the matrix of the derivation $\mathscr{D}_L$ would be a skew-Hermitian matrix. Of course, this would correspond to a unitary (irreducible) representation of the rotation group. First of all, we calculate the matrix of $\mathscr{D}_L$ in the basis $\mathfrak{f}'_A$ to find

$$\begin{aligned}\mathscr{D}_L(\mathfrak{f}'_1) &= -u^3\,\mathfrak{f}'_2 + u^2\,\mathfrak{f}'_3 + u^1\,(\mathfrak{f}'_4 + q^2\,\mathfrak{f}'_5),\\ \mathscr{D}_L(\mathfrak{f}'_2) &= u^3\,\mathfrak{f}'_1 - u^1\,\mathfrak{f}'_3 + u^2\,(q\,\mathfrak{f}'_4 + q\,\mathfrak{f}'_5),\\ \mathscr{D}_L(\mathfrak{f}'_3) &= -u^2\,\mathfrak{f}'_1 + u^1\,\mathfrak{f}'_2 + u^3\,(q^2\,\mathfrak{f}'_4 + \mathfrak{f}'_5),\\ \mathscr{D}_L(\mathfrak{f}'_4) &= -2u^1\,\mathfrak{f}'_1 - 2q^2u^2\,\mathfrak{f}'_2 - 2qu^3\,\mathfrak{f}'_3,\\ \mathscr{D}_L(\mathfrak{f}'_5) &= -2qu^1\,\mathfrak{f}'_1 - 2q^2u^2\,\mathfrak{f}'_2 - 2u^3\,\mathfrak{f}'_3.\end{aligned}$$

Now, we introduce a Hermitian scalar product in the 5-dimensional complex space $\mathfrak{R}^3_q$ by means of an orthonormal basis $\mathfrak{e}_A$ in such a way that the derivation operator $\mathscr{D}_L$ is skew-Hermitian, i.e. for any subscripts A, B the Hermitian scalar product satisfies

$$\langle \mathscr{D}_L(\mathfrak{e}_A), \mathfrak{e}_B \rangle + \langle \mathfrak{e}_A, \mathscr{D}_L(\mathfrak{e}_B) \rangle = 0. \tag{54}$$

Simple calculations show that, assuming the vectors of the initial basis $\mathfrak{f}'_A$ to be orthogonal, i.e. $\langle \mathfrak{f}'_A, \mathfrak{f}'_B \rangle = 0$ and $|\mathfrak{f}'_1| = |\mathfrak{f}'_2| = |\mathfrak{f}'_3|$, $|\mathfrak{f}'_4| = |\mathfrak{f}'_5|$, where $|\mathfrak{f}'_A|^2 = \langle \mathfrak{f}'_A, \mathfrak{f}'_A \rangle$, we obtain the solution of the equation (54), except for those cases when one vector $\mathfrak{e}_A$ belongs to the set $\{\mathfrak{f}'_1, \mathfrak{f}'_2, \mathfrak{f}'_3\}$, while the other vector $\mathfrak{e}_B$ belongs to $\{\mathfrak{f}'_4, \mathfrak{f}'_5\}$. In these cases, the condition (54) is satisfied if

$$|\mathfrak{f}'_4| = |\mathfrak{f}'_5| = 2|\mathfrak{f}'_1| = 2|\mathfrak{f}'_2| = 2|\mathfrak{f}'_3|.$$

This indicates that we have to renormalize the vectors $\mathfrak{f}'_1, \mathfrak{f}'_2, \mathfrak{f}'_3$. Hence, the Hermitian scalar product in the 5-dimensional complex space $\mathfrak{R}^3_q$ is determined by the orthonormal basis

$$\begin{aligned}
\mathfrak{e}_1 &= \sqrt{2}\,[\theta^1, \theta^2, \theta^2]_q = \sqrt{2}(1-q)\,(\mathfrak{f}_1 - qS_\sigma(\mathfrak{f}_1)),\\
\mathfrak{e}_2 &= \sqrt{2}\,[\theta^2, \theta^3, \theta^3]_q = \sqrt{2}(1-q)\,(\mathfrak{f}_2 - qS_\sigma(\mathfrak{f}_2)),\\
\mathfrak{e}_3 &= \sqrt{2}\,[\theta^3, \theta^1, \theta^1]_q = \sqrt{2}(1-q)\,(\mathfrak{f}_3 - qS_\sigma(\mathfrak{f}_3)),\\
\mathfrak{e}_4 &= [\theta^1, \theta^2, \theta^3]_q = (1-q)\,(\mathfrak{f}_4 - qS_\sigma(\mathfrak{f}_4)),\\
\mathfrak{e}_5 &= [\theta^3, \theta^2, \theta^1]_q = (1-q)\,(\mathfrak{f}_5 - qS_\sigma(\mathfrak{f}_5)),
\end{aligned}$$

and the matrix of the derivation $\mathscr{D}_L$ in this basis is skew-Hermitian. Similar constructions in the 5-dimensional complex space $\mathfrak{R}^3_{\bar q}$ lead to the orthonormal basis

$$\begin{aligned}
\bar{\mathfrak{e}}_1 &= \sqrt{2}\,[\theta^1, \theta^2, \theta^2]_{\bar q} = \sqrt{2}(1-\bar q)\,(\mathfrak{f}_1 - \bar qS_\sigma(\mathfrak{f}_1)),\\
\bar{\mathfrak{e}}_2 &= \sqrt{2}\,[\theta^2, \theta^3, \theta^3]_{\bar q} = \sqrt{2}(1-\bar q)\,(\mathfrak{f}_2 - \bar qS_\sigma(\mathfrak{f}_2)),\\
\bar{\mathfrak{e}}_3 &= \sqrt{2}\,[\theta^3, \theta^1, \theta^1]_{\bar q} = \sqrt{2}(1-\bar q)\,(\mathfrak{f}_3 - \bar qS_\sigma(\mathfrak{f}_3)),\\
\bar{\mathfrak{e}}_4 &= [\theta^1, \theta^2, \theta^3]_{\bar q} = (1-\bar q)\,(\mathfrak{f}_4 - \bar qS_\sigma(\mathfrak{f}_4)),\\
\bar{\mathfrak{e}}_5 &= [\theta^3, \theta^2, \theta^1]_{\bar q} = (1-\bar q)\,(\mathfrak{f}_5 - \bar qS_\sigma(\mathfrak{f}_5)).
\end{aligned}$$

Calculating the matrix of the derivation $\mathscr{D}_L$ in the basis $\mathfrak{e}_A$, we obtain the matrix

$$\begin{pmatrix}
0 & u^3 & -u^2 & -\sqrt{2}\;u^1 & -\sqrt{2}q\,u^1\\
-u^3 & 0 & u^1 & -\sqrt{2}\bar q u^2 & -\sqrt{2}\bar q u^2\\
u^2 & -u^1 & 0 & -\sqrt{2}q u^3 & -\sqrt{2}\;u^3\\
\sqrt{2}\;u^1 & \sqrt{2}q u^2 & \sqrt{2}\bar q u^3 & 0 & 0\\
\sqrt{2}\bar q u^1 & \sqrt{2}q u^2 & \sqrt{2}\;u^3 & 0 & 0
\end{pmatrix}, \tag{55}$$

and a similar matrix (with the replacement of q by $\bar q$ and vice versa) in the basis $\bar{\mathfrak{e}}_A$.

8. CONCLUSION

We have introduced and studied the structure of an algebra generated by θ^a, subject to a single binary and a set of ternary relations. This algebra is motivated by a ternary generalization of the Pauli's exclusion principle in the framework of the quark model. We have shown that the 10-dimensional space spanned by triple monomials of our algebra is a representation space of a double irreducible representation of the rotation group. Based on primitive 3rd order roots of unity, we have proposed a ternary $\mathbb{Z}_3$-generalization of the notion of a commutator. Using these ternary commutators, we where able to split the 10-dimensional representation space into two 5-dimensional subspaces, and to prove that each subspace is a representation space for the irreducible representation of the rotation group. We have constructed an orthonormal basis in each of these subspaces and computed the matrix of the derivation operator.

The standard model for quarks and leptons fits nicely into the representations of $\mathrm{SU}(5)$. The two matrices used in the Georgi–Glashow model can be found in [8] and they have the form

$$\bar{\mathbf{5}} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ --- \\ e^- \\ -\nu_e \end{pmatrix}, \mathbf{10} = \begin{pmatrix} 0 & u_3^c & -u_2^c & | & -u^1 & -d^1 \\ -u_3^c & 0 & u_1^c & | & -u^2 & -d^2 \\ u_2^c & -u_1^c & 0 & | & -u^3 & -d^3 \\ -- & -- & -- & -- & -- & -- \\ u^1 & u^2 & u^3 & | & 0 & -e^c \\ d^1 & d^2 & d^3 & | & e^c & 0 \end{pmatrix}.$$

The comparison of the matrix **10** with the matrix of the derivation operator $\mathscr{D}_L$ of the algebra $\mathfrak{R}$ shows the similarity of the structures of these matrices. If we limit ourselves to the quark part of the matrix **10** (without the electron e^c), the similarity in structure becomes even larger. We think that this is another evidence in favor of the fact that a ternary generalization of the Pauli's exclusion principle and the algebra constructed on this basis are an adequate description of the quark model.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge that Stefan Groote was supported by the European Regional Development Fund under Grant No. TK133. The publication costs of this article were partially covered by the Estonian Academy of Sciences.

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Algebra ternaarsete tsükliliste seoste, esituste ja kvarkmudeliga

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Lähtudes Pauli tõrjutusprintsiibi üldistusest kvarkmudelile, konstrueerisime ühikuga assotsiatiivse algebra, mille generaatorid rahuldavad kaht seost. Esiteks: kõigi generaatorite ruutude summa võrdub nulliga (binaarne seos) ja teiseks: generaatorite kolmikute korrutiste tsükliliste permutatsioonide summa võrdub nulliga (ternaarne seos). Uurisime sellise algebra ehitust ning leidsime selle homogeensete monoomide lineaarkatte mõõtme. Näitame, kuidas see algebra on seotud pöörete rühma taandumatute esitustega. Täpsemalt osutub, et 10-mõõtmeline ruum, mille saame monoomide kolmikute kattest, on pöörete rühma taandumatu unitaarne esitus. Saadud 10-mõõtmelise monoomide kolmikute katte jagame kahe 5-mõõtmelise alamruumi otsesummaks, milleks kasutame ternaarseid q- ja $\bar{q}$-kommutaatoreid, kus $q, \bar{q}$ on ühikelemendi kolmandat järku algjuured. Mõlemas otselahutuse alamruumis leidub taandumatu esitus $\mathrm{so}(3) \to \mathrm{su}(5)$, mille maatriks on välja arvutatud. Saadud maatriksi struktuur viitab väljapakutud algebra ja Georgi–Glashow mudeli võimalikele sarnasustele.