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GROUP THEORY

Classifying cubic symmetric graphs of order $52p^2$

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Abstract. An automorphism group of a graph is said to be s-regular if it acts regularly on the set of s-arcs in the graph. A graph is s-regular if its full automorphism group is s-regular. In this paper, we classify all connected cubic symmetric graphs of order $52p^2$ for each prime p.

Keywords: cubic symmetric graph, simple group, s-regular graph.

1. INTRODUCTION

Throughout this paper, all graphs considered are finite, undirected, with no loops and no multiple edges. For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set and the full automorphism group of X, respectively. An *s-arc* in graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s$. A graph X containing at least one s-arc is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A permutation group G on a set Ω is said to be *semiregular* if the stabilizer G_v of v in G is trivial for each $v \in \Omega$. By the orbit-stabilizer theorem it follows that if G is semiregular, then all of its orbits have length equal to $|G|$. A permutation group G is *regular* if it is semiregular and transitive. A subgroup of the automorphism group of a graph X is said to be *s-regular* if it acts regularly on the set of s-arcs of X. In particular, if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of X, then X is said to be *s-regular*. Thus, if a graph X is s-regular then $\mathrm{Aut}(X)$ is transitive on the set of s-arcs and the only automorphism fixing an s-arc is the identity automorphism of X.

Graphs associated with groups and other algebraic structures have been actively investigated because they have valuable applications and are related to automata theory (cf. [13,20]). In fact, many symmetric graphs are Cayley graphs of groups. Tutte [18,19] showed that every finite cubic symmetric graph is s-regular for some $1 \leq s \leq 5$. Since cubic graphs must have an even number of vertices, they must be cubic symmetric graphs. It follows that every cubic symmetric graph has an order of the form $2mp$ for an integer m and a

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prime p. In order to know all cubic symmetric graphs, we need to classify the cubic s-regular graphs of order $2mp$ for a fixed integer m and every prime p.

Cheng and Oxley [2] classified the cubic s-regular graphs of order $2p$ (in fact, they classified the symmetric graphs of order $2p$ with any valency). Since then, the classifications of cubic symmetric graphs of special orders have received considerable attention. For example, Feng et al. [5–11] classified the cubic s-regular graphs with order $2p^2$, $2p^3$, $4p^i$, $6p^i$, $8p^i$, $10p^i$, $12p^i$, $22p^i$ and $8p^3$ for any prime p and each $i = 1,2$. In 2006, Conder [3] searched all cubic symmetric graphs on up to 2048 vertices by the aid of a computer, and uploaded the data of his search on the website. Zhou and Feng [21] classified the cubic symmetric graphs of order $2pq$ for each prime p and each prime q. Alaeiyan and Hosseinipoor [1] and Talebi and Mehdipoor [17] classified the cubic symmetric graphs of order $12p^i$ and $22p^i$ for each $i = 1,2$, respectively. Oh [15] classified the cubic s-regular graphs of order $12p$, $36p$, $44p$, $52p$, $66p$, $68p$ and $76p$.

In this paper, we completely classify all connected cubic symmetric graphs of order $52p^2$ for each prime p. The main result of the paper is the following theorem, which presents a complete classification of connected s-regular cubic graphs of order $52p^2$ for each $s \geq 1$ and each prime p.

Theorem 1.1. *Let p be a prime. Then a connected cubic symmetric graph of order $52p^2$ is isomorphic to one of the graphs in Table 1 that are from [3]. Moreover, all graphs in Table 1 are pairwise non-isomorphic.*

Table 1. Cubic symmetric graphs of order $52p^2$

Graph	s-regular	Girth	Diameter	Bipartite?
C208.1	1	10	9	Yes
C468.1	5	12	13	Yes
C1300.1	3	10	14	No
C8788.1	2	10	32	No

2. PRELIMINARIES

Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. Denote by X_N the quotient graph corresponding to the orbits of N, which is the graph having the orbits of N as a vertex set with two distinct vertices adjacent in X_N whenever there is at least an edge between those orbits in X. For a vertex $v \in V(X)$, denote by $N(v)$ the set of vertices that are adjacent to v. A graph $\tilde{X}$ is called a covering of graph X with a projection $\rho : \tilde{X} \to X$, if ρ is a surjection from $V(\tilde{X})$ to $V(X)$ such that $\rho\mid_{N(\tilde{v})}: N(\tilde{v}) \to N(v)$ is a bijection, for any vertex $v \in V(X)$ and any $\tilde{v} \in \rho^{-1}(v)$. A covering $\rho : \tilde{X} \to X$ is said to be regular if there is a semiregular subgroup N of the automorphism group $\mathrm{Aut}(\tilde{X})$ such that the graph X is isomorphic to the quotient graph $\tilde{X}_N$, say, by an isomorphism τ, and the quotient map $\tilde{X} \to \tilde{X}_N$ is the composition $\tau \circ \rho$.

The following proposition is an important result and will be used frequently in the sequel (at times without an explicit reference to it), which is from [14, Theorem 9].

Proposition 2.1. [14, Theorem 9] *Let X be a connected cubic symmetric graph, and G an s-regular subgroup of* $\mathrm{Aut}(X)$ *for some integer $s \geq 1$. If a normal subgroup N of G has more than two orbits, then N acting on $V(X)$ is semiregular and G/N is an s-regular subgroup of* $\mathrm{Aut}(X_N)$. *Moreover, X is a regular covering of X_N.*

The following two results are extracted from [15, Theorem 2.4] and [9, Theorem 6.2], respectively.

Theorem 2.2. [15, Theorem 2.4] *Let X be a connected cubic symmetric graph of order $52p$, where p is a prime number. Then X can be s-regular for each $1 \leq s \leq 5$. Furthermore,*

(i) *X is* 1*-regular if and only if it is isomorphic to* C104.1;

(ii) *X is* 2*-regular if and only if it is isomorphic to* C364.1*,* C364.2*,* C364.3*,* C364.4*,* C364.5 *or* C364.6;

(iii) *X is* 3*-regular if and only if it is isomorphic to* C364.7.

Theorem 2.3. [9, Theorem 6.2] *Let X be a connected cubic symmetric graph of order $4p$ or $4p^2$ for some prime p. Then X is isomorphic to one of the following*:

(a) *the* 2*-regular hypercube Q_3 of order* 8;

(b) *the* 2*-regular generalized Petersen graph $P(8,3)$ of order* 16;

(c) *the* 2*-regular generalized Petersen graph $P(10,7)$ of order* 20;

(d) *the* 3*-regular Dodecahedron graph of order* 20;

(e) *the* 3*-regular Coxeter graph C_{28} of order* 28.

All groups considered in this paper will be finite. Let G be a group and H a subgroup of G. The center and the derived subgroup of G are denoted by $Z(G)$ and G', respectively. $N_G(H)$ and $C_G(H)$ are, respectively, the normalizer of H in G and the centralizer of H in G. We use $[G:H]$ to denote the number of cosets of H in G. Next, we state the N/C theorem in group theory.

Proposition 2.2. [16, Theorem 7.1] *Let G be a group and H a subgroup of G. Then the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of H.*

Suppose that Γ be a connected cubic symmetric graph. We remark that $\mathrm{Aut}(\Gamma)$ acting on the set of s-arcs of Γ is transitive. Since the cardinality of the set of s-arcs of Γ is $2^{s-1}\cdot 3\cdot |V(\Gamma)|$, it follows that

$$|\mathrm{Aut}(\Gamma)| = 2^{s-1}\cdot 3\cdot |V(\Gamma)|. \tag{1}$$

3. PROOF OF THE MAIN THEOREM

In this section, we will prove Theorem 1.1.

Proof. Suppose that $p\leq 13$. Then, according to Conder [3], there are four cubic symmetric graphs of order $52p^2$, that is, C208.1, C468.1, C1300.1, and C8788.1, and it is clear that they are pairwise non-isomorphic. Thus, it suffices to prove that there does not exist a cubic symmetric graph of order $52p^2$ if $p\geq 17$. Assume, to the contrary, that X is a cubic symmetric graph of order $52p^2$ with $p\geq 17$. It is straightforward that

$$|\mathrm{Aut}(X)| = 2^{s+1}\cdot 3\cdot 13\cdot p^2$$

and $1\leq s\leq 5$ by (1), [19, Main Result] and [18, Theorem 22]. In the following, we write

$$A = \mathrm{Aut}(X)$$

and let P be a Sylow p-subgroup of A. We shall finish the proof by the following steps.

Step 1. P is not normal in A.

If P is a normal subgroup, then, by Proposition 2.1, we have that P acting on $V(X)$ is semiregular. It is clear that $|P| = p^2$. So the quotient graph X_P corresponding to the orbits of P is a connected cubic symmetric graph of order 52, which is a contradiction by [3].

Step 2. $O_{13}(A) = 1$, where $O_{13}(A)$ is the largest normal 13-subgroup of A.

If not, $O_{13}(A)\cong Z_{13}$. In view of Proposition 2.1, one has that $O_{13}(A)$ on $V(X)$ is semiregular and so the quotient graph $X_{O_{13}(A)}$ corresponding to the orbits of $O_{13}(A)$ is a connected cubic symmetric graph of order $4p^2$. Now, by Theorem 2.3 and $p\geq 17$, we can obtain a contradiction.

Step 3. A has no normal p-subgroups of order p.

Suppose that $|K| = p$ and K is normal in A. Then the quotient graph X_K is a connected cubic A/K-arc-transitive graph of order $52p$. By Theorem 2.2, it must be that $p = 7$, a contradiction as $p \geq 17$.

Step 4. A has no normal 2-subgroups.

Assume, to the contrary, that H is a normal 2-subgroup of A. Then H on $V(X)$ is semiregular by Proposition 2.1. As a result, $|H|$ is a divisor of $|X|$, and hence, $|H| = 4$ or 2. If $|H| = 4$ then X_H has an odd number of vertices and valency 3, and a contradiction. Thus, we may assume that $|H| = 2$. Then X_H is a connected cubic A/H-arc-transitive graph of order $26p^2$. Let J/H be a minimal normal subgroup of $\mathrm{Aut}(X_H)$. It is clear that $\mathrm{Aut}(X_H) = 2^s \cdot 3 \cdot 13 \cdot p^2$ and so $2^5 \cdot 3 \cdot 13 \cdot p^2$ is divisible by $\mathrm{Aut}(X_H)$. If $\mathrm{Aut}(X_H)$ has a normal Sylow p-subgroup, then A has a normal subgroup of order $2p^2$, and this forces that P is normal in A, a contradiction by Step 1. Consequently, $p < 2^5 \cdot 3 \cdot 13$, and hence,

$$|\mathrm{Aut}(X_H)| < 2^5 \cdot 3 \cdot 13 \cdot (2^5 \cdot 3 \cdot 13)^2 = 1943764992.$$

By [12, Table 1, p. 8], one has Table 2, and by the list of non-abelian simple groups of order less than 10^{25} in [4, p. 239], one has Table 3. If J/H is unsolvable, then, by Tables 2 and 3, it follows that

$$J/H \cong A_5, PSL(2,7), PSL(2,13), PSL(2,25), PSU(3,4).$$

Note that $p \geq 7$, then it must be that $J/H \cong PSL(2,7)$ or $PSL(2,13)$. It follows that J/H acting on X_H has more than two orbits. Thus, J/H is semiregular by Proposition 2.1. It means that $|26p^2|$ is divisible by $|J/H|$, which is a contradiction. Consequently, J/H is an elementary abelian r-group, where r is a prime number. By Proposition 2.1, again, $|J/H|$ is a divisor of $26p^2$, and clearly $|J/H| \neq 2$. Thus, $|J/H| = 13$, p or p^2. It follows that $|J| = 26$, $2p$ or $2p^2$, respectively. If $|J| = 26$, then J has a normal Sylow 13-subgroup, which is characteristic in J. Since J is normal in A, one has $O_{13}(A) \neq 1$, contrary to Step 2. For $|J| = 2p^2$, it is easy to see that A has normal Sylow p-subgroups, which is also a contradiction. Thus, we may assume that $|J| = 2p$, and hence, A has a normal p-subgroup of order p, a contradiction by Step 3.

Table 2. Non-abelian simple $\{2, p, q\}$-groups G

G	$\|G\|$
A_5	$2^2 \cdot 3 \cdot 5$
A_6	$2^3 \cdot 3^2 \cdot 5$
$PSL(2, 7)$	$2^3 \cdot 3 \cdot 7$
$PSL(2, 8)$	$2^3 \cdot 3^2 \cdot 7$
$PSL(2, 17)$	$2^4 \cdot 3^3 \cdot 17$
$PSL(3, 3)$	$2^4 \cdot 3^3 \cdot 13$
$PSU(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$PSU(4, 2)$	$2^6 \cdot 3^4 \cdot 5$

Table 3. Non-abelian simple $\{2,3,13, q\}$-groups G with $3\ |G|$ and $3^2 \nmid |G|$

G	$\|G\|$
$PSL(2, 13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$
$PSL(2, 25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$
$PSU(3, 4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$

Step 5. Final contradiction.

Let N be a minimal normal subgroup of A. Then, by Tables 2 and 3, N is solvable, and hence, N is an elementary abelian r-group for some prime r. By Proposition 2.1, one can see that N acting on $V(X)$ is semiregular, and so $|N|$ is a divisor of $|X|$. Since A is not simple, we have that N is non-trivial. It follows that N is a normal 2-subgroup, 13-subgroup or p-subgroup of A. Now we get the final contradiction from Steps 2–4. □

4. CONCLUSION

In this paper, for every prime p, all connected cubic symmetric graphs of order $52p^2$ are classified.

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REFERENCES

1. Alaeiyan, M. and Hosseinipoor M. K. A classification of the cubic s-regular graphs of orders $12p$ and $12p^2$. *Acta Univ. Apulensis Math. Inform.*, 2011, **25**, 153–158.
2. Cheng, Y. and Oxley, J. On weakly symmetric graphs of order twice a prime. *J. Combin. Theory Ser. B*, 1987, **42**(2), 196–211.
3. Conder, M. *Trivalent (Cubic) Symmetric Graphs on up to 10000 Vertices*. 2011. https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt
4. Conway, J. H., Curties, R. T., Norton, S. P., Parker, R. A. and Wilson, R. A. *Atlas of Finite Groups*. Clarendon Press, Oxford, 1985.
5. Feng, Y. Q., Ghasemi, M. and Yang, D. W. Cubic symmetric graphs of order $8p^3$. *Discrete Math.*, 2014, **318**, 62–70.
6. Feng, Y. Q. and Kwak, J. H. One-regular cubic graphs of order a small number times a prime or a prime square. *J. Aust. Math. Soc.*, 2004, **76**(3), 345–356.
7. Feng, Y. Q. and Kwak, J. H. Classifying cubic symmetric graphs of order $10p$ or $10p^2$. *Sci. China Ser. A*, 2006, **49**(3), 300–319.
8. Feng, Y. Q. and Kwak, J. H. Cubic symmetric graphs of order twice an odd prime-power. *J. Aust. Math. Soc.*, 2006, **81**(2), 153–164.
9. Feng, Y. Q. and Kwak, J. H. Cubic symmetric graphs of order a small number times a prime or a prime square. *J. Combin. Theory Ser. B*, 2007, **97**(4), 627–646.
10. Feng, Y. Q., Kwak, J. H. and Wang, K. Classifying cubic symmetric graphs of order $8p$ or $8p^2$, *European J. Combin.*, 2005, **26**(7), 1033–1052.
11. Feng, Y. Q., Kwak, J. H. and Xu, M. Y. Cubic s-regular graphs of order $2p^3$. *J. Graph Theory*, 2006, **52**(4), 341–352.
12. Gorenstein, D., Lyons, R. and Solomon, R. *The Classification of the Finite Simple Groups*. Mathematical Surveys and Monographs, Vol. 40, American Mathematical Society, Providence, RI, 1994.
13. Kelarev, A. V. *Graph Algebras and Automata*. Marcel Dekker, New York, 2003.
14. Lorimer, P. Vertex-transitive graphs: Symmetric graphs of prime valency. *J. Graph Theory*, 1984, **8**(1), 55–68.
15. Oh, J. M. Cubic s-regular graphs of order $12p$, $36p$, $44p$, $52p$, $66p$, $68p$ and $76p$. *J. Appl. Math. Inform.*, 2013, **31**(5–6), 651–659.
16. Rotman, J. J. *An Introduction to the Theory of Groups*. Springer, New York, 1995.
17. Talebi, A. A. and Mehdipoor, N. Classifying cubic s-regular graphs of orders $22p$ and $22p^2$. *Algebra Discrete Math.*, 2013, **16**(2), 293–298.
18. Tutte, W. T. A family of cubical graphs. *Proc. Cambridge Philos. Soc.*, 1947, **43**, 459–474.
19. Tutte, W. T. On the symmetry of cubic graphs. *Canadian J. Math.*, 1959, **11**, 621–624.
20. Zhang, M. Software defined network energy efficient algorithm based on degree sequence of nodes. *Microelectr. Comp.*, 2021, **38**(10), 65–72.
21. Zhou, J. X. and Feng, Y. Q. Cubic vertex-transitive graphs of order $2pq$. *J. Graph Theory*, 2010, **65**(4), 285–302.

$52p^2$ järgu sümmeetriliste kuubiliste graafide klassifikatsioon

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Graafi automorfismide rühma nimetatakse s-regulaarseks, kui see toimib regulaarselt graafi s-kaartel, st tema automorfismide rühm on s-regulaarne. Artiklis antakse sidusate $52p^2$ järku sümmeetriliste kuubiliste graafide klassifikatsioon kõigi algarvude p jaoks.