Proceedings of the Estonian Academy of Sciences,
2023, **72**, 1, 1–5
https://doi.org/10.3176/proc.2023.1.01
Available online at www.eap.ee/proceedings

CONTROL THEORY

Transformation of nonlinear discrete-time state equations into the observer form: revision

Tanel Mullari[a*] and Ülle Kotta[b]

[a] Tallinn University of Technology, Department of Cybernetics, Ehitajate tee 5, 19086 Tallinn, Estonia
[b] Tallinn University of Technology, Department of Software Science, Akadeemia tee 21/1, 12618 Tallinn, Estonia

Received 12 October 2022, accepted 24 November 2022, available online 2 January 2023

Abstract. This paper simplifies the existing necessary and sufficient conditions for transformability of state equations into the observer form by state transformation.

Keywords: nonlinear control system, discrete-time system, observer form, state equivalence, algebraic approach.

For nonlinear control systems that are transformable via state transformation into the observer form, which is a linear system appended with nonlinear input-output injection terms that depend only on measured input and output, observers with linear error dynamics can be easily constructed unlike in the general case. This explains the huge popularity of this research topic for different system classes, which have been addressed via different mathematical tools under different assumptions for more than 30 years; see for instance [1–9].

In [10], necessary and sufficient conditions are given to transform the state equations of discrete-time nonlinear control system

$$x^{\langle 1\rangle} = \overline{\Phi}(x,u), \quad y = h(x) \tag{1}$$

by state transformation $X = \Psi(x)$ into the classical observer form

$$\begin{aligned} X_i^{\langle 1\rangle} &= X_{i+1} + \varphi_i(y,u), \quad i = 1,\dots,n-1, \\ X_n^{\langle 1\rangle} &= \varphi_n(y,u), \quad y = X_1. \end{aligned} \tag{2}$$

The aim of this note is to improve the main result (Theorem 6) from [10] in two aspects: to simplify the set of necessary and sufficient transformability conditions and simplify the sufficiency part of the proof. We refer the reader to [10] for preliminaries we do not repeat here, except for the material that is absolutely necessary for understanding the new proof.

In [10], $x^{\langle 1\rangle}$ means the first-order forward shift of the state variable. The higher order forward and backward shifts (also of other variables as well as vector fields and 1-forms) are denoted by upper index $\langle k\rangle$,

* Corresponding author, tanel.mullari@ttu.ee

where $k \in Z$. The state transition map $\overline{\Phi}$ in (1) is supposed to be analytic and such that it can be extended to the map $\Phi = [\overline{\Phi}^T, \chi^T]^T$, having the global analytic inverse. Introduce the additional variable z by $z = \chi(x,u)$. The system (1) defines the inversive difference field $\mathscr{K}$ of meromorphic functions in a finite number of variables $\{x, u^{\langle k\rangle}, k \geq 0, z^{\langle -l\rangle}, l \geq 1\}$ as well as the vector spaces of the 1-forms and the vector fields over $\mathscr{K}$; see [10] for more details. Recall the vector spaces $\mathscr{Y} := \mathrm{span}_{\mathscr{K}}\{\mathrm{d}y^{\langle l\rangle}, l \geq 0\}$, $\mathscr{U} := \mathrm{span}_{\mathscr{K}}\{\mathrm{d}u^{\langle j\rangle}, j \geq 0\}$, $\mathscr{X} := \mathrm{span}_{\mathscr{K}}\{\mathrm{d}x\}$ [10]. The subspace $\mathscr{O} = \mathscr{X} \cap (\mathscr{Y} + \mathscr{U})$ is called the *observable space of system* (1).

Assumption 1. *The system* (1) satisfies the generic observability condition $\dim_{\mathscr{K}} \mathscr{O} = n$.

Define the set of 1-forms:

$$\omega_k := \sum_{i=1}^{n} \frac{\partial y^{\langle k\rangle}}{\partial x_i} \mathrm{d}x_i, \quad k = 0, ..., n-1. \tag{3}$$

Assumption 1 is equivalent to the condition that the 1-forms ω_k, $k = 0, ..., n-1$, are linearly independent:

$$\dim_{\mathscr{K}} (\mathrm{span}_{\mathscr{K}} \{\omega_k, k = 0, ..., n-1\}) = n.$$

Define the vector field $\Xi \in \mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$ such that

$$\langle \omega_k, \Xi \rangle \equiv \delta_{k,n-1}, \quad k = 0, ..., n-1, \tag{4}$$

where by $\delta_{k,n-1}$ is denoted the Kronecker delta. As shown in [10], under Assumption 1 the vector field Ξ is uniquely determined. By the definition of Ξ and Lemma 1 in [10], the vector fields $\Xi^{\langle -l\rangle}$, $l = 0, ..., n$, belong to $\mathrm{span}_{\mathscr{K}}\{\partial/\partial x, \partial/\partial z^{\langle -1\rangle}, ..., \partial/\partial z^{\langle -l\rangle}\}$. However, for $l < n$ one can consider all such vector fields as the elements of a larger dimensional space $\mathrm{span}_{\mathscr{K}}\{\partial/\partial x, \partial/\partial z^{\langle -1\rangle}, ... \partial/\partial z^{\langle -n\rangle}\}$. In a similar manner one can consider all the 1-forms ω_k and $\mathrm{d}y^{\langle k\rangle}$, $k = 0, ..., n-1$, as the elements of the space $\mathrm{span}_{\mathscr{K}}\{\mathrm{d}x, \mathrm{d}u, ..., \mathrm{d}u^{\langle n-2\rangle}\}$ and write

$$\mathrm{d}y = \omega_0, \quad \mathrm{d}y^{\langle k\rangle} = \omega_k + \sum_{j=0}^{k-1} \frac{\partial y^{\langle k\rangle}}{\partial u^{\langle j\rangle}} \mathrm{d}u^{\langle j\rangle}, \; k = 1, ..., n-1. \tag{5}$$

Let

$$\Xi^{\langle -l\rangle \pi} = \sum_{i=1}^{n} \left\langle \mathrm{d}x_i, \Xi^{\langle -l\rangle} \right\rangle \frac{\partial}{\partial x_i} \in \mathrm{span}_{\mathscr{K}} \left\{ \frac{\partial}{\partial x} \right\}$$

be the projection of $\Xi^{\langle -l\rangle}$. Note that for $k = 0, ..., n-1$, $l = 0, ..., n$, the following holds from [10]:

$$\left\langle \mathrm{d}y^{\langle k\rangle}, \Xi^{\langle -l\rangle} \right\rangle = \left\langle \omega_k + \sum_{j=0}^{k-1} \frac{\partial y^{\langle k\rangle}}{\partial u^{\langle j\rangle}} \mathrm{d}u^{\langle j\rangle}, \Xi^{\langle -l\rangle \pi} + \sum_{q=1}^{l} \left\langle \mathrm{d}z^{\langle -q\rangle}, \Xi^{\langle -l\rangle} \right\rangle \frac{\partial}{\partial z^{\langle -q\rangle}} \right\rangle = \left\langle \omega_k, \Xi^{\langle -l\rangle \pi} \right\rangle. \tag{6}$$

Under Assumption 1 the vector fields $\Xi^{\langle -l\rangle \pi}$, $l = 0, ..., n-1$, are linearly independent over $\mathscr{K}$ [10]. Therefore, from Lemma 3 in [11] one can conclude that the lemma below holds.

Lemma 2. *If the vector fields $\Xi^{\langle -l\rangle \pi}$, $l = 0, ..., n-1$, commute and their coefficients depend only on x, then, generically, one can define the state transformation $X_i = \Psi_i(x)$, $\Psi_i \in \mathscr{K}$, $i = 1, ..., n$, such that $\langle \mathrm{d}\Psi_i, \Xi^{\langle -l\rangle \pi} \rangle = \delta_{i,n-l}$.*

Note that because the vector fields $\Xi^{\langle -l\rangle \pi}$, $l = 0, ..., n-1$, span the vector space $\mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$, the total differentials of $\Psi_i(x)$ are uniquely defined.

Theorem 6 in [10] has *three* solvability conditions. Theorem 3 below demonstrates that actually the claim of the theorem holds under the first two conditions, meaning that the third condition is not independent. Note also that the new proof is more transparent and much shorter.

Theorem 3. (Revision of Theorem 6 in [10]) *Under Assumption 1 the equations* (1) *can be transformed via state transformation into the observer form* (2) *if and only if the following two conditions are satisfied*:
(i) the vector fields $\Xi^{\langle -l\rangle\pi}$ *commute*:

$$\left[\Xi^{\langle -l\rangle\pi},\Xi^{\langle -j\rangle\pi}\right]\equiv 0,\quad l,j=0,\dots,n-1,$$

(ii) the coefficients of $\Xi^{\langle -l\rangle\pi}$, $l=0,\dots,n-1$, *depend only on the variable* x:

$$\left[\frac{\partial}{\partial u^{\langle j\rangle}},\Xi^{\langle -l\rangle\pi}\right]\equiv 0,\quad \left[\frac{\partial}{\partial z^{\langle -k\rangle}},\Xi^{\langle -l\rangle\pi}\right]\equiv 0,\quad j=0,\dots,n-2,\quad k=1,\dots,n-1.$$

Proof. Sufficiency. If (i) and (ii) hold, then by Lemma 2 one can define the state transformation

$$X_i=\Psi_i(x):\quad \left\langle \mathrm{d}\Psi_i,\Xi^{\langle -l\rangle\pi}\right\rangle\equiv\delta_{i,n-l},\quad i=1,\dots,n,\quad l=0,\dots,n-1. \tag{7}$$

Note that computing the total differentials of the both sides of (2), we get, replacing $\mathrm{d}X_i$ by $\mathrm{d}\Psi_i$,

$$\begin{aligned}\mathrm{d}\Psi_i^{\langle 1\rangle}&=\left.\frac{\partial\varphi_i(X_1,u)}{\partial X_1}\right|_{X_1=\Psi_1(x)}\mathrm{d}\Psi_1+\mathrm{d}\Psi_{i+1}+\left.\frac{\partial\varphi_i(X_1,u)}{\partial u}\right|_{X_1=\Psi_1(x)}\mathrm{d}u,\quad i=1,\dots,n-1,\\ \mathrm{d}\Psi_n^{\langle 1\rangle}&=\left.\frac{\partial\varphi_n(X_1,u)}{\partial X_1}\right|_{X_1=\Psi_1(x)}\mathrm{d}\Psi_1+\left.\frac{\partial\varphi_n(X_1,u)}{\partial u}\right|_{X_1=\Psi_1(x)}\mathrm{d}u.\end{aligned} \tag{8}$$

Consequently, we need to show that if (i) and (ii) hold, then the total differentials of the forward shift of $\Psi_i(x)$, defined by (7), have the form (8).

Since the functions $\Psi_i(x)$ defined by (7) are linearly independent and their number is n, then

$$\mathrm{span}_{\mathscr{K}}\{\mathrm{d}\Psi_i,\, i=1,\dots,n\}=\mathrm{span}_{\mathscr{K}}\{\mathrm{d}x\}. \tag{9}$$

Obviously $\mathrm{d}\Psi_i^{\langle 1\rangle}\in\mathrm{span}_{\mathscr{K}}\{\mathrm{d}x,\mathrm{d}u\}$, therefore due to (9) one has

$$\mathrm{d}\Psi_i^{\langle 1\rangle}=\sum_{j=1}^{n}\alpha_{ij}(x,u)\mathrm{d}\Psi_j+\beta_i(x,u)\mathrm{d}u. \tag{10}$$

Note that because the left-hand side of (10) is a total differential, then also its right-hand side must be a total differential. This is possible if and only if there exist some functions $\phi_i(X,u)$, $i=1,\dots,n$, such that

$$\alpha_{ij}=\left.\frac{\partial\phi_i(X,u)}{\partial X_j}\right|_{X=\Psi(x)},\quad \beta_i=\left.\frac{\partial\phi_i(X,u)}{\partial u}\right|_{X=\Psi(x)}. \tag{11}$$

Compare (8) and (10). We need to prove that if (i) and (ii) are satisfied, then the coefficients (11) on the right side of (10) have the following form:

$$\alpha_{i1}=\left.\frac{\partial\varphi_i(X_1,u)}{\partial X_1}\right|_{X_1=\Psi_1(x)},\quad \beta_i=\left.\frac{\partial\varphi_i(X_1,u)}{\partial u}\right|_{X_1=\Psi_1(x)}, \tag{12}$$

$$\alpha_{ij}=\delta_{i,j-1},\quad i=1,\dots,n,\quad j=2,\dots,n. \tag{13}$$

Show first the validity of (13), multiplying both sides of (10) by $\Xi^{\langle -l\rangle\pi}$, $l=0,\dots,n-2$:

$$\left\langle \mathrm{d}\Psi_i^{\langle 1\rangle},\Xi^{\langle -l\rangle\pi}\right\rangle=\sum_{j=1}^{n}\alpha_{ij}\left\langle \mathrm{d}\Psi_j,\Xi^{\langle -l\rangle\pi}\right\rangle+\beta_i\left\langle \mathrm{d}u,\Xi^{\langle -l\rangle\pi}\right\rangle,$$

which, taking into account (7) and the definition of Kronecker delta, obtains the following form:

$$\alpha_{i,n-l} = \left\langle \mathrm{d}\Psi_i^{\langle 1\rangle}, \Xi^{\langle -l\rangle\pi} \right\rangle, \quad i = 1,\dots,n, \quad l = 0,\dots,n-2. \tag{14}$$

Note that due to the definition of the projection of the vector field

$$\Xi^{\langle -l\rangle} = \Xi^{\langle -l\rangle\pi} + \sum_{k=1}^{l} \left\langle \mathrm{d}z^{\langle -k\rangle}, \Xi^{\langle -l\rangle} \right\rangle \frac{\partial}{\partial z^{\langle -k\rangle}}. \tag{15}$$

From $\mathrm{d}\Psi_i^{\langle 1\rangle} \in \mathrm{span}_{\mathscr{K}}\{\mathrm{d}x, \mathrm{d}u\}$ and (15) one can rewrite (14) as $\alpha_{i,n-l} = \langle \mathrm{d}\Psi_i^{\langle 1\rangle}, \Xi^{\langle -l\rangle}\rangle = \langle \mathrm{d}\Psi_i, \Xi^{\langle -l-1\rangle}\rangle^{\langle 1\rangle}$. Since $\mathrm{d}\Psi_i$ has no components in the directions of $\mathrm{d}z^{\langle -k\rangle}$, $k \geq 1$, and $\mathrm{d}u^{\langle j\rangle}$, $j \geq 0$, one can, in the expressions of $\alpha_{i,n-l}$, replace the vector fields by their projections:

$$\alpha_{i,n-l} = \left\langle \mathrm{d}\Psi_i, \Xi^{\langle -l-1\rangle\pi} \right\rangle^{\langle 1\rangle}, \; i = 1,\dots,n, \quad l = 0,\dots,n-2. \tag{16}$$

From (7), $\langle \mathrm{d}\Psi_i, \Xi^{\langle -l-1\rangle\pi}\rangle \equiv \delta_{i,n-l-1}$, $i = 1,\dots,n$, $l = 0,\dots,n-2$, and since the value of a constant is invariant with respect to shifting, one may write $\alpha_{i,n-l} = \langle \mathrm{d}\Psi_i, \Xi^{\langle -l-1\rangle\pi}\rangle^{\langle 1\rangle} \equiv \delta_{i,n-l-1}$, $i = 1,\dots,n$, $l = 0,\dots,n-2$. Using notation $j := n-l$, one gets $\alpha_{ij} = \delta_{i,j-1}$, $i = 1,\dots,n$, $j = 2,\dots,n$, meaning that (13) holds.

To show the validity of (12), note that the 1-form (10) can be rewritten by (11) and (13) as

$$\begin{aligned}
\mathrm{d}\Psi_i^{\langle 1\rangle} &= \left.\frac{\partial \phi_i(X,u)}{\partial X_1}\right|_{X=\Psi(x)} \mathrm{d}\Psi_1 + \mathrm{d}\Psi_{i+1} + \left.\frac{\partial \phi_i(X,u)}{\partial u}\right|_{X=\Psi(x)} \mathrm{d}u, \quad i = 1,\dots,n-1, \\
\mathrm{d}\Psi_n^{\langle 1\rangle} &= \left.\frac{\partial \phi_n(X,u)}{\partial X_1}\right|_{X=\Psi(x)} \mathrm{d}\Psi_1 + \left.\frac{\partial \phi_n(X,u)}{\partial u}\right|_{X=\Psi(x)} \mathrm{d}u.
\end{aligned} \tag{17}$$

The right-hand sides of (17) are total differentials if and only if there exist the functions $\varphi_i(X_1,u)$ such that

$$\begin{aligned}
\phi_i(X,u) &= \varphi_i(X_1,u) + X_{i+1}, \quad i = 1,\dots,n-1, \\
\phi_n(X,u) &= \varphi_n(X_1,u).
\end{aligned} \tag{18}$$

Then also (12) is valid, and (17) takes the form (8), meaning that in the new coordinates X, defined by (7), the forward shifts of the new coordinates X have the form (2). It remains to be proved that $\mathrm{d}X_1 = \mathrm{d}y$, or alternatively, taking into account (7), that

$$\left\langle \mathrm{d}y, \Xi^{\langle -l\rangle\pi} \right\rangle \equiv \delta_{1,n-l}, \quad l = 0,\dots,n-1. \tag{19}$$

Due to (5) and the fact that $\Xi \in \mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$, the definition formula (4) can be rewritten as

$$\left\langle \mathrm{d}y^{\langle k\rangle}, \Xi \right\rangle \equiv \delta_{k,n-1}, \quad k = 0,\dots,n-1. \tag{20}$$

Shifting (20) backward k steps results in $\langle \mathrm{d}y, \Xi^{\langle -k\rangle}\rangle \equiv \delta_{k,n-1}$, which is equivalent to $\langle \mathrm{d}y, \Xi^{\langle -k\rangle\pi}\rangle \equiv \delta_{k,n-1}$ due to (6). Since the addition of the same number $1-k$ to both indices of $\delta_{k,n-1}$ does not change its value, the last equality is equivalent to $\langle \mathrm{d}y, \Xi^{\langle -k\rangle\pi}\rangle = \delta_{1,n-k}$, $k = 0,\dots,n-1$, which is (19) for $k = l$.

Necessity. The necessity part of this theorem is proved in [10]. □

ACKNOWLEDGEMENT

The publication costs of this article were covered by the Estonian Academy of Sciences.

REFERENCES

1. Conte, G., Moog, C. H. and Perdon, A. M. *Algebraic Methods for Nonlinear Control Systems*. Springer, London, 2007.
2. Krener, A. J. and Isidori, A. Linearization by output injection and nonlinear observers. *Syst. Control Lett.*, 1983, **3**(1), 47–52.
3. Krener, A. J. and Respondek, W. Nonlinear observers with linearizable error dynamics. *SIAM J. Control Optim.*, 1985, **23**(2), 197–216.
4. Lee, H. G. Verifiable conditions for discrete-time multioutput observer error linearizability. *IEEE Trans. Autom. Control*, 2019, **64(**4), 1632–1639.
5. Zhang, J., Feng, G. and Xu, H. Observer design for nonlinear discrete-time systems: Immersion and dynamic observer error linearization techniques. *Int. J. Robust Nonlinear Control*, 2010, **20**(5), 504–514.
6. Califano, C. and Moog, C. H. The observer error linearization problem via dynamic compensation. *IEEE Trans. Autom. Control*, 2014, **14**(9), 2502–2508.
7. Besancon, G. On output transformations for state linearization up to output injection. *IEEE Trans. Autom. Control*, 1999, **44**(10), 1975–1981.
8. Kaparin, V. and Kotta, Ü. Transformation of nonlinear discrete-time system into the extended observer form. *Int. J. Control*, 2018, **91**(4), 848–858.
9. Simha, A., Respondek, W., Kaldmäe, A., Kaparin, V. and Kotta, Ü. Intrinsic conditions for extended observer forms for nonlinear systems. In *IEEE Conference on Decision and Control, Miami Beach, FL, USA, 17–19 December 2018*. IEEE, 1373–1378.
10. Mullari, T. and Kotta, Ü. Transformation of nonliear discrete-time state equations into the observer form: extension to nonreversible case. *Proc. Estonian Acad. Sci.*, 2021, **70**(3), 235–247.
11. Mullari, T. and Kotta, Ü. Linearization of discrete-time control systems by state transformation. *Proc. Estonian Acad. Sci.*, 2021, **70**(1), 62–79.

Mittelineaarsete diskreetsete olekuvõrrandite teisendamine vaatlejakujule: täiustamine

Tanel Mullari ja Ülle Kotta

Artiklis lihtsustatakse varem leitud tarvilikke ja piisavaid tingimusi olekuvõrrandite vaatlejakujule viidavuseks olekuteisenduste abil.