Free vibration analysis of tapered Timoshenko beam with higher order Haar wavelet method

This paper is dedicated to the 100th birthday of Professor Ülo Lepik.
This study can be considered as a continuation of the research in the area of Haar wavelet methods started by Professor Ülo Lepik

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Abstract. In the current study, the higher order Haar wavelet method based formulation is developed for the analysis of the free vibrations of the tapered Timoshenko beam. The clamped-clamped and clamped-pinned boundary conditions are explored and the results with the 4th order and the 6th order of convergence are presented. The results are found to be in good agreement with the corresponding results of the Ritz method. The proposed approach can be considered as the principal improvement of the widely used Haar wavelet method providing the same accuracy with the several magnitudes lower mesh. Thus, the higher order Haar wavelet method has reduced the computational cost in comparison with the widely used Haar wavelet method since the computational complexity of both methods is determined by the mesh used. In the case of the fixed equal mesh used for both methods, the higher order Haar wavelet method results in the several magnitudes lower absolute error without a remarkable increase in computational complexity. The cost needed to pay for higher accuracy is hidden in a certain increase in the implementation complexity compared with the widely used Haar wavelet method.

Key words: higher order Haar wavelet method, tapered Timoshenko beam, free vibration.

1. INTRODUCTION

Development and adaptation of computational methods and mathematical modelling techniques are rapidly evolving research areas with the main focus on finding more accurate, less time-consuming, and simpler approximations.

The Haar wavelet method (HWM) was first introduced in [1–2]. According to Chen and Hsiao’s approach, the highest order of derivatives included in a differential equation is expanded into a series of Haar functions [1–2]. This method is applied to solving differential and integro-differential equations covering applications in various research areas such as engineering, natural sciences, etc. [3–9]. Furthermore, this method is used as a numerical solution to linear and nonlinear delay differential equations [10], and space derivatives are obtained through the Haar wavelet collocation method to solve 1D and 2D cubic nonlinear Schrodinger equations [11]. In [12] the accuracy and convergence results of the HWM are presented. Based on the obtained
results, it can be concluded that despite its simple implementation, the HWM needs refinement in order to
come up with such widely used numerical methods as the finite difference method and the differential
quadrature method.

Recently, the higher order Haar wavelet method (HOHWM) was introduced in [13] in order to improve
the accuracy and convergence of the previously proposed Haar wavelet method. The HOHWM has been
applied with success to solving differential equations, vibration, and buckling response of beams [14–18].
Theoretical and numerical analyses of the free and forced vibration of homogeneous and functionally graded
Timoshenko beams have been performed [19–22]. In the case of tapered beams, many approaches have been
used for analysing the Timoshenko beam that has a non-uniform cross-section [23–26].

The HOHWM is applied with success to the analysis of plate and shell structures using Euler–Bernoulli
and zig-zag theories. In this paper the HOHWM approach is adapted to the Timoshenko beam theory.

2. HOHWM APPROACH TO FREE VIBRATION ANALYSIS OF THE TIMOSHENKO BEAM

In this section, the formulation of the free vibration of the tapered Timoshenko beam and boundary conditions
are introduced.

2.1. Free vibration of the Timoshenko beam

A schematic view of the Timoshenko beam with a non-uniform cross-section along the length, \( x \)-direction,
is shown in Fig. 1.

Herein, free vibration of homogeneous tapered Timoshenko beams has been investigated. The material
properties of the beams are assumed to be constant. Firstly, the cross-sectional area \( A(x) \) and the moment of
inertia \( I(x) \) are presented as

\[
A(x) = A_0 \left( 1 - \frac{c x}{L} \right), \quad I(x) = I_0 \left( 1 - \frac{c x}{L} \right)^{\frac{3}{2}}, \quad x \in [0, L],
\]  

(1)

where \( A_0 \) and \( I_0 \) are the area and the moment of inertia at the base of the beam, respectively. \( L \) is the length
of the beam, \( E \) denotes Young’s modulus, \( G \) refers to shear modulus, \( \rho \) represents mass density, and \( k \) is the
shear correction factor which is chosen to be 5/6. For the described Timoshenko beam, the basic governing
differential equations for transverse vibration of the tapered beam can be presented as

\[
\begin{align*}
\frac{\partial}{\partial x} \left( E I(x) \frac{\partial \varphi}{\partial x} \right) + \kappa G A(x) \left( \frac{\partial w}{\partial x} - \varphi \right) - \rho I \frac{\partial^2 \varphi}{\partial t^2} = 0 \\
\frac{\partial}{\partial x} \left( \kappa G A(x) \left( \frac{\partial w}{\partial x} - \varphi \right) \right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0
\end{align*}
\]  

(2)

Fig. 1. Schematic view of a tapered beam.
where \( w \) and \( \varphi \) are the transverse deflection and rotation of the cross-section, respectively. The bending moment \( M \) and the shear force \( Q \) at any cross-section can be read as

\[
M = EI(x) \frac{\partial \varphi}{\partial x}, \quad Q = kGA(x) \left( \frac{\partial w}{\partial x} + \varphi \right).
\]  

(3)

The boundary conditions for the beam can be expressed

for the clamped edge as: \( w = 0, \varphi = 0 \),
for the pinned edge as: \( w = 0, M = 0 \).

(4)

\[ \textbf{2.2. Higher order Haar wavelet method} \]

The higher order Haar wavelet method (HOHWM) is developed as an improvement of the widely used Haar wavelet method (HWM) [13].

The \( n \)-th order ordinary differential equation, in general, can be presented as

\[
G(x,u,u',u'',...u^{(n-1)},u^{(n)}) = 0,
\]  

(5)

where \( n \) represents the order of the highest derivative involved in the differential equation. In the HOHWM, in comparison to the Haar wavelet method, the order of expansion is increased by \( 2s \), Eq. (6). Based on the Haar wavelet, the expansion is presented as

\[
f(x) = \frac{d^{n+2s}u(x)}{dx^{n+2s}} = \sum_{i=1}^{\infty} a_i h_i(x), \quad s = 1,2, ..., \text{ Eq. (6)}
\]

in which \( h_i(x) \) is the Haar function [18]

\[
h_i(x) = \begin{cases} 
1 & \text{for } x \in [\xi_1(i),\xi_2(i)) \\
-1 & \text{for } x \in [\xi_2(i),\xi_3(i)) \\
0 & \text{elsewhere}
\end{cases}
\]  

(7)

where \( i = m + k + 1, m = 2^j \) is a maximum number of square waves arranged in the interval \([A,B]\) and the parameter \( k \) indicates the location of the particular square wave [18]

\[
\xi_1(i) = A + 2k\mu \Delta x, \quad \xi_2(i) = A + (2k + 1)\mu \Delta x, \quad \xi_3(i) = A + 2(k + 1)\mu \Delta x,
\]

\[
\mu = M/m, \quad \Delta x = (B - A)/(2M).
\]  

(8)

The integrals of the Haar functions (7) of order \( n \) can be expressed as [13]

\[
p_{n,i}(x) = \begin{cases} 
0 & \text{for } x \in [A,\xi_1(i)) \\
\frac{(x-\xi_1(i))^n}{n!} & \text{for } x \in [\xi_1(i),\xi_2(i)) \\
\frac{(x-\xi_1(i))^{n-2}(x-\xi_2(i))^n}{n!} & \text{for } x \in [\xi_2(i),\xi_3(i)) \\
\frac{(x-\xi_1(i))^{n-2}(x-\xi_2(i))^{n-2}(x-\xi_3(i))^n}{n!} & \text{for } x \in [\xi_3(i),B) \\
0 & \text{elsewhere}
\end{cases}
\]  

(9)

The differential equation can be satisfied in selected uniform grid points

\[
x_{il} = \frac{i}{2M}, \quad x_{ir} = 1 - \frac{i}{2M}, \quad i = 0,...,s - 1,
\]  

(10)
where $L$ and $R$ are the added collocation points on the left and right boundary, respectively. Then the numerical order of the convergence of the method can be estimated by

$$\text{Convergence rate} = \frac{\log \left( \frac{F_{i-1} - F_{\text{Ref}}}{F_i - F_{\text{Ref}}} \right)}{\log(2)}, \quad (11)$$

where $F_{\text{Ref}}$ is the existing solution, which in the current solution is obtained from the Ritz method [21].

3. NUMERICAL RESULTS

In order to showcase the accuracy of the formulation proposed above, the values of natural frequencies of the Timoshenko beam under two arbitrary boundary conditions are presented. Table 1 presents the effect of the taper ratio on the non-dimensional natural frequencies of the C-C Timoshenko beam.

<table>
<thead>
<tr>
<th>N</th>
<th>HWM Frequency</th>
<th>A. error</th>
<th>Conv. rate</th>
<th>HOHWM 4th Frequency</th>
<th>A. error</th>
<th>Conv. rate</th>
<th>HOHWM 6th Frequency</th>
<th>A. error</th>
<th>Conv. rate</th>
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<tbody>
<tr>
<td>$c = 0$</td>
<td>4</td>
<td>13.96275845</td>
<td>1.28e-01</td>
<td>13.84197845</td>
<td>7.22e-03</td>
<td>13.83477975</td>
<td>2.13e-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>13.86655612</td>
<td>3.18e-02</td>
<td>13.83519214</td>
<td>4.34e-04</td>
<td>13.83460011</td>
<td>1.56e-06</td>
<td>5.4358</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>13.84269132</td>
<td>7.93e-03</td>
<td>13.83478525</td>
<td>2.68e-05</td>
<td>13.83475854</td>
<td>8.43e-08</td>
<td>5.7522</td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>13.83674066</td>
<td>1.98e-03</td>
<td>13.83476012</td>
<td>1.67e-06</td>
<td>13.83475846</td>
<td>1.51e-09</td>
<td>5.9467</td>
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<td></td>
<td>64</td>
<td>13.83525392</td>
<td>4.95e-04</td>
<td>13.83475856</td>
<td>1.04e-07</td>
<td>13.83475846</td>
<td>7.71e-10</td>
<td>5.9955</td>
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<tr>
<td></td>
<td>256</td>
<td>13.83478942</td>
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<td>13.83475845</td>
<td>4.08e-10</td>
<td>13.83475845</td>
<td>1.40e-13</td>
<td>6.0000</td>
<td></td>
</tr>
</tbody>
</table>

Existing result = 13.834758

| $c = 0.4$ | 4 | 13.38213007 | 9.60E-01 | 12.42216142 | 3.56E-02 | 12.42216313 | 4.40E-03 |
| 128 | 12.4221807 | 1.76E-05 | 12.42216317 | 4.02E-08 | 12.42216357 | 7.28E-10 | 6.0017 |

Existing result = 12.421263

| $c = 0.8$ | 4 | 10.7701461 | 1.04E+00 | 9.738846102 | 1.17E-02 | 9.727997702 | 8.52E-04 |
| 8 | 10.2871461 | 5.60E-01 | 9.72786102 | 7.40E-04 | 9.72718202 | 3.51E-05 | 6.0698 |
| 32 | 9.728230102 | 1.08E-03 | 9.72715702 | 1.10E-05 | 9.72714615 | 8.28E-08 | 6.0606 |
| 64 | 9.727253202 | 1.07E-04 | 9.72715332 | 5.23E-06 | 9.72714610 | 6.74E-10 | 6.0326 |

Existing result = 9.727146

A. error – Absolute error
Table 2. Effect of boundary conditions on non-dimensional natural frequencies of the tapered Timoshenko beam ($c = 0.2$)

<table>
<thead>
<tr>
<th>N</th>
<th>C-P</th>
<th>HWM</th>
<th></th>
<th>HOHWM 4th</th>
<th></th>
<th>HOHWM 6th</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Frequency</td>
<td>A. error</td>
<td>Conv. rate</td>
<td>Frequency</td>
<td>A. error</td>
<td>Conv. rate</td>
<td>Frequency</td>
</tr>
<tr>
<td>4</td>
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<td></td>
<td>11.05355589</td>
<td>3.67E-01</td>
<td></td>
<td>10.77030682</td>
</tr>
<tr>
<td>8</td>
<td>11.05808739</td>
<td>3.70E-01</td>
<td>2.3120</td>
<td>10.73676607</td>
<td>6.99E-02</td>
<td>4.8521</td>
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<tr>
<td>16</td>
<td>10.75601079</td>
<td>6.91E-02</td>
<td>2.0624</td>
<td>10.6877406</td>
<td>8.53E-03</td>
<td>4.0629</td>
<td>10.68692751</td>
</tr>
<tr>
<td>32</td>
<td>10.69501206</td>
<td>8.12E-03</td>
<td>2.0039</td>
<td>10.6869284</td>
<td>4.10E-04</td>
<td>4.0132</td>
<td>10.68688961</td>
</tr>
<tr>
<td>256</td>
<td>10.68695081</td>
<td>6.30E-05</td>
<td>2.0002</td>
<td>10.6868874</td>
<td>7.31E-07</td>
<td>4.0005</td>
<td>10.6868739</td>
</tr>
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</table>

Existing result = 10.68689

<table>
<thead>
<tr>
<th>N</th>
<th>C-C</th>
<th>HWM</th>
<th></th>
<th>HOHWM 4th</th>
<th></th>
<th>HOHWM 6th</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency</td>
<td>A. error</td>
<td>Conv. rate</td>
<td>Frequency</td>
<td>A. error</td>
<td>Conv. rate</td>
<td>Frequency</td>
</tr>
<tr>
<td>4</td>
<td>14.32226684</td>
<td>1.10E+00</td>
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<td>13.26456684</td>
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<td></td>
<td>13.22543684</td>
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<tr>
<td>256</td>
<td>13.22272227</td>
<td>5.43E-06</td>
<td>2.0021</td>
<td>13.22266844</td>
<td>4.85E-09</td>
<td>4.0013</td>
<td>13.22266844</td>
</tr>
</tbody>
</table>

Existing result = 13.222267

taper ratio ($c$) for the beam under clamped-clamped (C-C) boundary conditions. The results are compared with the existing results obtained from the Ritz method and alternative methods employed in [22, 25].

As expected, for the beam with the taper ratio other than $c = 0$, the non-dimensional natural frequency decreases for the higher value of $c$. Moreover, as it can be observed, the results of the higher order Haar wavelet method prove that in the case of the 4th and the 6th order of convergence, the absolute error reduces much faster by increasing the number of terms in the Haar wavelet method. This matter could be essential in the case of more complex problems, thus the accurate result can be obtained faster and with a smaller number of terms.

The effect of boundary conditions is shown in Table 2. For the tapered Timoshenko beam ($c = 0.2$), the results of two boundary conditions – clamped-clamped (C-C) and clamped-pinned (C-P) – are produced, which prove the above-mentioned point for the higher order Haar wavelet method. In the future study, the HOHWM is planned to be applied to design optimization of plate and shell structures [27–31].

4. CONCLUSIONS

During the last two years, the HOHWM has been applied with success to the analysis of plate and shell structures by using Euler–Bernoulli and zig-zag theories. In the current study, the HOHWM is extended to the vibration analysis of Timoshenko beams. The solution has been used to analyse the beam under two boundary conditions, clamped-clamped and clamped-pinned. The results for beams with different taper ratios prove that the higher order Haar wavelet method is accurate, and for the versions with the higher order of convergence (4th and 6th order) the absolute error drops extremely fast. These results can be translated to a faster, simpler, and more accurate solution for other structural analyses where the analytical solution is difficult to obtain.
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Muutuva ristlöikega Timoshenko tala vabavõnkumist analüüs kõrgemat järku Haari lainikute meetodi abil

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