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WAVELETS AND FRACTIONAL CALCULUS

Haar wavelet fractional derivative

Dedicated to Professor Ülo Lepik on his 100th birthday: he enlighted a century with his outstanding life and brilliant work

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Abstract. In this paper, the fundamental properties of fractional calculus are discussed with the aim of extending the definition of fractional operators by using wavelets. The Haar wavelet fractional operator is defined, in a more general form, independently on the kernel of the fractional integral.

Key words: wavelet theory, fractional calculus, Haar wavelet, operational matrix.

1. INTRODUCTION

In recent years, fractional calculus has been applied in several interesting topics from many different fields. This new approach, based on a suitable definition of fractional operators, has enabled to handle fundamental problems by showing unexpected results and new horizons in research (see, e.g., [15,29,43] and references therein).

Unfortunately the definition of fractional derivative is not unique, and therefore, there is a deep controversial debate on the legitimacy of these operators. Moreover, the physical meaning of the fractional order parameter is still missing. As a consenquence, there is a wide choice of fractional operators with newly born candidates, and a deep quest for a physical interpretation of the solution to fractional differential problems. (see, e.g., [13,28,29,40,43].

An interesting definition of fractional derivative, based on the sinc-function, was recently given in [42]. This function is very popular in signal analysis. Moreover, it is the fundamental basic function for the so-called Shannon wavelets [7–12]. In [42] the authors proposed the sinc-function as the kernel of the fractional operator, but since the sinc-function is the mother wavelet (for the Shannon family), this idea suggests us the possibility of considering a more general class of fractional operators, where the kernels are wavelets or (more generally) wavelet series. In this paper, such a class of wavelet fractional operators will be proposed and the corresponding theory discussed.

Wavelet theory has been a very rich and fast-evolving research topic. It has led to a large number of papers dealing with several issues from distant fields, giving rise to a wide distribution of different wavelet families. Among the many families of wavelets the most popular and smart family is the the so-called Haar wavelet family [27], which has been used by Lepik to solve several nonlinear differential problems by a smart method [22–25]. Many applications and models based on Haar wavelets have shown their flexibility and suitability to understand complex issues and to provide solutions to nonlinear problems as well [3,18–20,30–33,37,39]. In [26], Lepik also solved a fractional integral equation by using Haar wavelet series for the integrating function. Proceeding from Lepik's original idea, a generalization is proposed below, where both the integrating function and the kernel can be represented as a wavelet series (in the most general case) or as a Haar wavelet series in the particular case of these special functions.

Wavelets enjoy many special properties such as their localization in time (or frequency) and the multi-scale decomposition. In particular, the multiscale property enables to decompose the approximation space into separate scales [6,16], thus focusing on the main physical contribution at each scale. We will see that the fractional order parameter might have a suitable physical interpretation by the multi-scale approach, so that the interpolation factor could be explained as a zoom in and zoom out through the multi-scales of the differential problem.

The paper is organized as follows. Preliminary remarks on fractional operators are given in Section 2. Section 3 shows how to combine wavelets and fractional operators, and discusses the many possibilities offered by the wavelet series representation. Section 4 provides the basic properties of the Haar wavelet and the Haar wavelet fractional derivative is explicitly computed. Section 5 presents the conclusions.

2. PRELIMINARY REMARKS ON FRACTIONAL CALCULUS

The fractional derivative was conceived in the 18th century. It can be substantially derived as a generalization of the following.

Theorem 1 (Cauchy). *Let $f(x) \in C_0$ be a continuous function on reals and $n > 0$ a given integer, the Cauchy formula for the repeated integral is*

$$J_a^{-n} f(x) \stackrel{\text{def}}{=} \int_a^x \int_a^{t_1} \dots \int_a^{t_{n-1}} f(t_n) dt_n \dots dt_2 dt_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt. \tag{1}$$

Proof. The proof can be easily obtained by induction, by showing that

$$\frac{d}{dx} J_a^{-(n+1)} f(x) = n J_a^{-n} f(x), \qquad \frac{d}{dx} J_a^{-1} f(x) = f(x).$$

□

In particular, it is

$$\frac{d^n}{dx^n} J_a^{-n} f(x) = f(x) \tag{2}$$

so that the n-th integer order derivative

$$D^n \stackrel{\text{def}}{=} \frac{d^n}{dx^n}$$

is the inverse operator of the repeated integral J_a^{-n}.

Let us assume that $f(x) \in C_1$ is a differentiable function so that from (2) we can easily get

$$D^{n+1} J_a^{-n} f(x) \stackrel{\text{def}}{=} \frac{d}{dx} \left[\frac{d^n}{dx^n} J_a^{-n} f(x) \right] = f'(x). \tag{3}$$

Let α be a rational number, and $n = 1$. In order to proceed, we need to define the generalized repeated integral with a rational number. But unfortunately, we have a variety of integrals and, as a consequence, a variety of fractional operators.

2.1. Some of the most popular fractional derivatives

In this section, some of the most popular definitions of fractional derivatives (see, e.g., [34,38]) are provided.

Let us start with the Riemann–Liouville integral and the corresponding derivative.

Definition 1. *The Riemann–Liouville integral of fractional order* $\nu \geq 0$ *of a function* $f(x)$ *is defined as*

$$(J^{\nu} f)(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau, & \nu > 0, \\ f(t), & \nu = 0. \end{cases}$$

For the Riemann–Liouville fractional operator J^{α} the following properties hold:

$$\begin{aligned} (a)\ J^{\alpha}\left(J^{\beta} f(t)\right) &= J^{\beta}\left(J^{\alpha} f(t)\right), \\ (b)\ J^{\alpha}\left(J^{\beta} f(t)\right) &= J^{\alpha+\beta} f(t), \\ (c)\ J^{\alpha} t^{\nu} &= \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} t^{\nu+\alpha}, \qquad \alpha, \beta \geq 0, \nu > -1, \\ (d)\ J^{\nu} e^{\lambda t} &= \frac{1}{\nu\Gamma(\nu)} e^{\lambda t} t^{\nu}, \quad \nu > 0, \\ (e)\ J^{\nu} c &= \frac{c}{\nu\Gamma(\nu)} t^{\nu}, \quad \nu > 0. \end{aligned}$$

From this definition the corresponding derivative follows as shown below.

Definition 2. *Riemann–Liouville fractional derivative of order* $\alpha > 0$ *is defined as*

$$D^{\alpha}_{RL} f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \qquad n \in \mathbb{N}, \ n-1 < \alpha \leq n. \tag{4}$$

The main problem with this derivative is the unvanishing value for a constant function. Therefore, the following was proposed by Caputo [38].

Let $f(x) \in C^n$ be an n-differentiable function, α a positive value, then the following holds.

Definition 3. *The* α*-order Caputo fractional derivative is defined as*

$$D^{\alpha}_{C} f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \ t > 0, & 0 \leq n-1 < \alpha < n, \end{cases}$$

where n *is an integer,* $x > 0$*, and* $f \in C^n$.

It can be easily shown that

$$(a)\ J^{\alpha}D_C^{\beta}f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^k}{k!},$$

$$(b)\ D_C^{\alpha}J^{\alpha}f(x) = f(x).$$

$$(c)\ D_C^{\alpha}t^n = \begin{cases} 0, & for\ n \in \mathbb{N}_0\ and\ \alpha < n, \\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}, & otherwise, \end{cases}$$

$$(d)\ D_C^{\alpha}D_C^{\beta}f(x) = D_C^{\beta}D_C^{\alpha}f(x).$$

2.2. The kernel of the fractional integral

Riemann–Liouville (RL) and Caputo (C) derivatives are the most popular derivatives and they have been used in many applications (see, e.g., [2,5,11,12,14,17,21,35,36,40,41]). Nevertheless, they both suffer some drawbacks. In fact, the RL-derivative is unvanishing when $f(x)$ is constant. The Caputo derivative, instead, is defined on a singular kernel, which seems to be a problem. Therefore, many authors have tried to avoid these issues by defining some more flexible non-singular derivatives.

Indeed, the more general fractional derivative with a given kernel $K(x,\alpha)$, which generalizes the C-derivative, is

$$D^{\alpha}f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \displaystyle\int_0^x f^{(n)}(\tau)K(x-\tau,\alpha)d\tau,\ t > 0, & 0 \le n-1 < \alpha < n. \end{cases} \tag{5}$$

The kernel should be defined in such a way that at least the two conditions

$$\lim_{\alpha\to 0} K(x-\tau,\alpha) = 1, \qquad \lim_{\alpha\to 1} K(x-\tau,\alpha) = \delta(x-\tau) \tag{6}$$

hold true. Moreover, in order to be a non-singular kernel, it should be also

$$\lim_{x\to\tau} K(x-\tau,\alpha) \neq 0, \qquad \forall\alpha\ . \tag{7}$$

Although there are several definitions of derivatives, they all depend on a kernel. In particular, it can be easily seen that the C-derivative [4], the Caputo–Fabrizio (CF) derivative [5] and the Atangana–Baleanu (AB) derivative [1] are some special cases of (5) corresponding to the kernels, respectively:

$$\begin{aligned} (C) \qquad K(x-\tau,\alpha) &= \frac{1}{\Gamma(n-\alpha)}(x-\tau)^{n-\alpha-1} \\ (CF) \qquad K(x-\tau,\alpha) &= \frac{M(\alpha)}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}(x-\tau)} \\ (AB) \qquad K(x-\tau,\alpha) &= \frac{B(\alpha)}{1-\alpha}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(x-\tau)\right), \end{aligned} \tag{8}$$

where the Mittag-Leffler function is defined as

$$E_{\alpha}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k+1)}. \tag{9}$$

It can be easily shown that the proposed kernels fulfill (6) while only (CF) and (AB) also fulfill the condition (7).

3. COMBINING WAVELETS AND FRACTIONAL OPERATORS

The general structure of Caputo-based fractional derivative

$$D_C^\alpha f(x) = \int_0^x f'(\tau) K(x-\tau,\alpha) d\tau, \qquad 0<\alpha<1 \tag{10}$$

is based on the kernel $K(x-\tau,\alpha)$, which is a positive function with decay to infinity (to ensure convergence), similar to the functions (8), to fulfill at least the conditions (6). The general structure of the Riemann–Liouville first order derivative according to (4) is

$$D_{RL}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x f(\tau)(x-\tau)^{-\alpha} d\tau, \qquad 0<\alpha\le 1. \tag{11}$$

In some recent papers it has been proposed to assume for the kernel a power series expression, such as in the case of the Mittag-Leffler function (9) so that the modern fractional Caputo-like operators can be defined as

$$D^\alpha f(x) = \int_0^x f'(\tau) \sum_{i=0}^\infty a_i(x,\alpha)\tau^i d\tau, \tag{12}$$

analogously for the Riemann–Liouville type by expressing the kernel $(8)_1$ by a power series.

As a suitable generalization of (12), we can assume that either or both the function and the kernel could be expressed in terms of wavelet series. Thus, let $\psi_j(\tau)$ be a family of wavelets and we can have the following result.

Definition 4. *The wavelet scale approximation of the classical fractional derivative is defined as*

$$D^\alpha f(x) = \int_0^x \sum_{j=0}^\infty b_j \psi_j(\tau) K(x-\tau,\alpha) d\tau. \tag{13}$$

However, by this definition, the wavelet representation of the integrand function $f(\tau)$ can be easily taken and, therefore, it is easy to have a numerical approximation of the resulting operator at different scales. Nevertheless, it does not give a better description of the fractional order parameter α. Moreover, we can have the wavelet approximation within the so-called modern fractional approach by taking the kernel as a power series.

Definition 5. *The wavelet scale approximation of the modern fractional derivative is defined as*

$$D^\alpha f(x) = \int_0^x \sum_{j=0}^\infty b_j \psi_j(\tau) \sum_{i=0}^\infty a_i(x,\alpha)\tau^i d\tau. \tag{14}$$

However, this operator can be seen as a hybrid fractional operator, which in any case has a wavelet approximation for the integrating function and a power approximation for the kernel.

A more general approach is to define a scale wavelet fractional operator, where the kernel is taken as a wavelet series. In this way we can single out the contribution of the kernel at each scale, so that the fractional parameter α can be seen as the contribution to the rate of change of a function depending on the scale. Thus, it has a more significant physical meaning than an abstract dependence on a pure fractional parameter α so that we can give the following definition.

Definition 6. *The modern wavelet fractional derivative is defined as*

$$D^\alpha f(x) = \int_0^x f'(\tau) \sum_{i=0}^\infty c_i(x,\alpha)\psi_i(\tau) d\tau,$$

analogously for the Riemann–Liouville operator, by expressing $(8)_1$ as a wavelet series. In this approach the kernel, which is the fundamental function for this operator, is decomposed at different scales. It might be related to the fractional order α (as shown in the next section), thus giving to α and to the corresponding operator a scale dependence.

Also, in this case we can have a wavelet approximation of the integrating function and, as a result, the following holds.

Definition 7. *The wavelet scale approximation of the modern wavelet fractional derivative is defined as*

$$D^{\alpha} f(x) = \int_0^x \sum_{j=0}^{\infty} b_j \psi_j(\tau) \sum_{i=0}^{\infty} c_i(x,\alpha) \psi_i(\tau) d\tau. \tag{15}$$

It should be noted that these definitions are given in the most general approach without fixing a particular integrating function and/or a kernel. It is not a definition of a new fractional operator, but merely a new perspective from where these operators can be seen and a means to give the fractional parameter a physical meaning (represented by the dependence of the kernel on the scale of the wavelet approximation). Thus, at the coarse scale we re-obtain the ordinary differential operator.

Although the product of two series might look a cumbersome task, we can give a simpler form to the above equation (15) by using the Cauchy product approximation

$$D^{\alpha} f(x) \cong \int_0^x \sum_{j=0}^{N} b_j^{(f)} c_{N-j}^{(K)}(x,\alpha) \psi_j(\tau) \psi_{N-j}(\tau) d\tau.$$

As a consequence, the following definition holds.

Definition 8. *The N-scale approximation of the wavelet fractional derivative is*

$$D^{\alpha} f(x) = \sum_{j=0}^{N} b_j^{(f)} c_{N-j}^{(K)}(x,\alpha) \int_0^x \psi_j(\tau) \psi_{N-j}(\tau) d\tau \tag{16}$$

and we can get the following result.

Definition 9. *The wavelet fractional derivative is defined as the limit*

$$D^{\alpha} f(x) = \lim_{N\to\infty} \sum_{j=0}^{N} b_j^{(f)} c_{N-j}^{(K)}(x,\alpha) \int_0^x \psi_j(\tau) \psi_{N-j}(\tau) d\tau, \tag{17}$$

where a fundamental role is played by the symmetric operational matrix

$$P_{j,N-j}(x) \stackrel{\text{def}}{=} \int_0^x \psi_j(\tau) \psi_{N-j}(\tau) d\tau. \tag{18}$$

In the next section we will explicitly compute this operational matrix for the Haar wavelet basis and the Riemann–Liouville kernel.

4. HAAR WAVELETS

Let us consider the Haar function

$$h(\tau) = \begin{cases} 1, & \tau \in [0,1/2) \\ -1, & \tau \in [1/2,1) \\ 0, & \tau \notin [0,1). \end{cases}$$

The compressed and translated instances of this function give rise to the Haar wavelets in the interval $[0,1]$:

$$h_i(\tau) = \begin{cases} 1, & \tau \in [a,b) \\ -1, & \tau \in [b,c), \quad i = m+k+1, \\ 0, & \tau \notin [a,c) \end{cases} \tag{19}$$

where $a = \dfrac{k}{m}$, $b = \dfrac{k+1/2}{m}$, $c = \dfrac{k+1}{m}$, $m = 2^j,\ j = 0,1,\dots,N,\ k = 0,1,\dots,m-1$.

The corresponding integrals are

$$\int_0^x h_i(\tau)d\tau = \begin{cases} x-a, & x \in [a,b) \\ (b-a)+(b-x), & x \in [b,c), \quad i = m+k+1, \\ 0, & x > c \end{cases}$$

where $a = \dfrac{k}{m}$, $b = \dfrac{k+1/2}{m}$, $c = \dfrac{k+1}{m}$, $m = 2^j,\ j = 0,1,\dots,N,\ k = 0,1,\dots,m-1$.

Moreover, in order to compute the operational matrix (18), we also need the following values

$$\int_0^x h_i(\tau)h_j(\tau)d\tau = \begin{cases} 2^{-(i+j)/2}(x-a)\delta^{ij}, & x \in [a,b) \\ 2^{-(i+j)/2}[(b-a)+(b-x)]\delta^{ij}, & x \in [b,c), \quad i = m+k+1, \\ 0, & x > c \end{cases} \tag{20}$$

where $a = \dfrac{k}{m}$, $b = \dfrac{k+1/2}{m}$, $c = \dfrac{k+1}{m}$, $m = 2^j,\ j = 0,1,\dots,N,\ k = 0,1,\dots,m-1$ and δ^{ij} is the Kronecher symbol. They can be easily obtained by using the definition (19) and a direct computation.

If we define the box function $h_1(\tau)$ as

$$h_1(\tau) = \begin{cases} 1, & \tau \in [0,1) \\ 0, & \tau \notin [0,1) \end{cases},$$

then we can approximate a continuous function $u(x)$ in the interval $[0,1]$ by the Haar series

$$u(\tau) = \sum_{i=1}^{\infty} c_i h_i(\tau), \quad (i = m+k+1),$$

$$c_i = 2^j \int_0^1 u(\tau)h_i(\tau)d\tau,$$

where $m = 2^j,\ j = 0,1,\dots,N,\ k = 0,1,\dots,m-1$.

The orthogonality property holds as

$$\int_0^1 h_i(\tau)h_s(\tau)d\tau = \begin{cases} 2^{-j}, & i = s \\ 0, & i \neq s \end{cases},$$

where $m = 2^j,\ j = 0,1,\dots,N,\ k = 0,1,\dots,m-1$.

Shifted Haar wavelets are a family of Haar functions on the interval $[0,x]$ defined as

$$h_i(\tau,\, x) = \begin{cases} 1, & \tau \in [\dfrac{k}{m}x, \dfrac{2k+1}{2m}x) \\ -1, & \tau \in [\dfrac{2k+1}{2m}x, \dfrac{k+1}{m}x)\,, \quad i = m+k+1, \\ 0, & \tau \notin [kx, \dfrac{k+1}{m}x) \end{cases}$$

where $m = 2^j,\ j = 0,1,\dots,J,\ k = 0,1,\dots,m-1,\ N = 2^J$.

4.1. Fractional Haar derivative at the N-scale approximation

In this section the Haar wavelet fractional derivative (17) can be explicitly defined. By taking into account (20) from (17),(18), we have

$$D^\alpha f(x) = \lim_{N\to\infty} \sum_{j=0}^{N} b_j^{(f)} c_{N-j}^{(K)}(x,\alpha) \int_0^x h_j(\tau,x) h_{N-j}(\tau,x) d\tau.$$

It means

$$D^\alpha f(x) = \lim_{N\to\infty} \sum_{j=0}^{N} 2^{-N} b_j^{(f)} c_{N-j}^{(K)}(x,\alpha)(x-a)\delta^{j,N-j} \tag{21}$$

and since $\delta^{j,N-j} \neq 0$ only when $j = N/2$, we have

$$D^\alpha f(x) = \lim_{N\to\infty} 2^{-N} b_{N/2}^{(f)} c_{N/2}^{(K)}(x,\alpha)(x-a),$$

that is

$$D^\alpha f(x) = 2^{-N} b_{N/2}^{(f)} \lim_{N\to\infty} c_{N/2}^{(K)}(x,\alpha)(x-a), \tag{22}$$

so that this limit will depend only on the wavelet coefficients of the kernel. It should be noted that (22) gives a very simple expression of the Haar fractional derivative thanks to the use of the Haar wavelet. In this case, we have a simple limit depending only on the Haar wavelet coefficients of the kernel. Therefore, some new perspectives in fractional calculus can be opened by grouping kernels according to their wavelet coefficients. The equation (22) also confirms that Haar wavelets are the most suitable tool to handle differential-integral problems as shown by Lepik in his seminal and fundamental papers [22–27], which remain a milestone in modern mathematics.

5. CONCLUSIONS

In this paper, the Haar wavelet fractional operator has been defined in order to have a more general form of a fractional operator which is independent on the chosen kernel. This definition, based on wavelets, combines the many advantages of both wavelets and fractional operators, thus opening new perspectives in the modern fractional calculus.

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Murrulist järku tuletis Haari lainikute abil

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Käesolevas artiklis käsitletakse murrulise diferentsiaal- ja integraalarvutuse põhiomadusi eesmärgiga laiendada murruliste operaatorite definitsiooni. Töös defineeritakse murrulist järku tuletis, kasutades Haari lainikuid piirväärtusena, mis sõltub tuuma Haari lainikute kordajatest.