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MATHEMATICS

Inverse problem to identify a space-dependent diffusivity coefficient in a generalized subdiffusion equation from final data

The paper is dedicated to the 100th birthday of Professor Ülo Lepik

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Abstract. An inverse problem to determine a space-dependent diffusivity coefficient in a one-dimensional generalized time fractional diffusion equation from final data is considered. The global uniqueness and local existence and stability of the solution to this problem is proved. Proof of these statements is based on the fixed-point principle and previously obtained results regarding an inverse source problem for a generalized subdiffusion equation.

Key words: inverse problem, final overdetermination, generalized fractional evolution equation.

1. INTRODUCTION

Equations with time fractional derivatives (containing power-type kernels) are used to model subdiffusion processes in self-similar media [1,3,13]. However, in many cases a medium under consideration is self-similar, which means that the character of the process changes when the time is rescaled. In such cases generalized fractional derivatives (containing more general kernels) are introduced to the equations [2,10, 20]. Generalized fractional derivatives involve more degrees of freedom and enable to better fit the models with real situations.

Often the medium parameters are a priori unknown and determined via solution to inverse problems that involve measurements of states of the processes. The states may be measured at final time moments. Final data are suitable for determination of space-dependent paramaters of the equations for two reasons: 1) unknown quantities and data are functions of the same type; 2) the resulting inverse problems are moderately ill-posed.

Problems to reconstruct space-dependent factors of source terms of subdiffusion equations containing usual or generalized fractional derivatives from final data have been studied in several papers [8,9,11,14,18, 21]. The analysis of such problems uses the Fourier expansion or positivity principles with the Fredholm alternative. A more general approach in a Hilbert space setting is presented in the recent paper [15].

Problems to recover reaction coefficents (potentials) from final data can be handled by means of the fixed-point principles on the basis of results obtained for inverse source problems [9]. Another approach

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to inverse coefficient problems for subdiffusion equations with usual fractional derivatives is based on Carleman estimates [17,23]. In case the interior of the domain is not accessible, boundary data can be used instead of the final measurements in order to recover space-dependent coefficients [7].

In the present paper we consider a problem to identify a space-dependent diffusivity coefficient in a one-dimensional subdiffusion equation containing a generalized time fractional derivative from the final data. We will prove the global uniqueness and local existence and stability for that problem. The analysis is based on previously obtained results regarding an inverse source problem for such an equation [9].

A surprising result is that the inverse diffusivity problem is less ill-posed than the corresponding inverse source problem: the solution to the former one depends continuously on the 1st derivative of the final measurement u_T whereas the solution to the latter one depends continuously on the 2nd derivative of u_T.

2. PHYSICAL BACKGROUND AND FORMULATION OF INVERSE PROBLEM

In the derivation of a differential equation to be considered in this paper we follow the approach presented in [16]. We assume the following constitutive relation with memory:

$$Q(t,x) = -a(x)\frac{\partial}{\partial t}\int_0^t M(t-\tau)u_x(\tau,x)d\tau, \tag{1}$$

where t is the time, $x \in \mathbb{R}$ denotes a space variable, Q represents the flux, u is the state of the diffusion process, M refers to a memory kernel and a is the diffusivity. Plugging this relation into the conservation equation $u_t + Q_x = F$, where F is the source function, we obtain the following generalized fractional diffusion equation:

$$u_t(t,x) = \frac{\partial}{\partial t}M * (a(x)u_x(t,x))_x + F(t,x), \tag{2}$$

where $*$ denotes the time convolution, e.g. $z_1 * z_2(t) = \int_0^t z_1(t-\tau)z_2(\tau)d\tau$.

Another method to derive the equation (2) is based on the continuous time random walk. Details can be found, e.g., in [2].

Suppose that there exists a time-dependent function k such that $k*M(t) \equiv 1$. Then, applying the operator $k*$ to (2), we transform it to the following form that contains the explicit elliptic operator at the right-hand side:

$$k * u_t(t,x) = (a(x)u_x(t,x))_x + f(t,x), \quad f = k*F.$$

The term $k*u_t(t,x)$ can be rewritten in a form that does not contain the first order time derivative of u: $k*u_t(t,x) = \frac{\partial}{\partial t}k*[u(t,x)-u(0,x)]$. Thus, we obtain

$$\frac{\partial}{\partial t}k*[u(t,x)-u(0,x)] = (a(x)u_x(t,x))_x + f(t,x). \tag{3}$$

The operators $k*\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}k*$ are the generalized fractional derivatives of Caputo and Riemann–Liouville type, respectively. We will use the following notation for the latter one:

$$D_t^{\{k\}} = \frac{\partial}{\partial t}k*.$$

In the usual fractional diffusion model, the kernels M and k are $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ and $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, where $0<\beta<1$. More examples of kernels M and k occurring in different physical models are given in Subsection 6.1 of this paper.

Let $T,l>0$. Firstly, we formulate the following initial-boundary value problem (*direct problem*) for the state function u:

$$D_t^{\{k\}}(u-u_0)(t,x)=[a(x)u_x(t,x)]_x+f(t,x),\ x\in(0,l),\ t\in(0,T), \tag{4}$$

$$a(0)u_x(t,0)=h_0(t),\ a(l)u_x(t,l)=h_l(t),\ t\in(0,T), \tag{5}$$

$$u(0,x)=u_0,\ x\in(0,l), \tag{6}$$

where a,f,h_0,h_l are given functions and u_0 is a given number.

The boundary conditions (5) are related to fluxes at $x=0$ and $x=l$. In view of (1), the outward pointing fluxes at $x=0$ and $x=l$ are $H_0=a(0)\frac{\partial}{\partial t}M*u(\cdot,0)$ and $H_l=-a(l)\frac{\partial}{\partial t}M*u(\cdot,l)$, respectively. The functions h_0 and h_l involved in (5) can be expressed via fluxes as $h_0=k*H_0$ and $h_l=-k*H_l$, respectively.

Next, let us suppose that the diffusivity coefficient a is unknown but the state is specified at the final time $t=T$, i.e.

$$u(T,x)=u_T(x),\ x\in(0,l), \tag{7}$$

where u_T is a given function. We pose the *inverse problem* to determine a pair of functions (a,u) that satisfy the conditions (4)–(7).

The inverse problem (4)–(7) will be the main research topic of this paper. We will be able to study it in the case of the constant initial state. Therefore, the quantity u_0 was defined as a number already in the formulation of the corresponding direct problem.

3. PRELIMINARIES

3.1. Abstract functional spaces and Sonine kernels

Let X be a Banach space. We define some spaces of abstract functions that map the intervals $(0,T)$ and $[0,T]$ into X.

By $L_p((0,T);X)$, $p\in[1,\infty]$, we denote the abstract Lebesgue spaces, i.e.

$$L_p((0,T);X)=\left\{v:(0,T)\to X:\|v\|_{L_p((0,T);X)}:=\left[\int_0^T\|v(t)\|_X^p dt\right]^{\frac{1}{p}}<\infty\right\},\ \ p\in[1,\infty),$$

$$L_\infty((0,T);X)=\left\{v:(0,T)\to X:\|v\|_{L_\infty((0,T);X)}:=\operatorname{ess}\sup_{t\in(0,T)}\|v(t)\|_X<\infty\right\}.$$

The space $C([0,T];X)$ contains functions $v:[0,T]\to X$ that are continuous on $[0,T]$. This is a Banach space with the norm $\|v\|_{C([0,T];X)}=\max_{t\in[0,T]}\|v(t)\|_X$. Moreover, we define

$$C_0([0,T];X)=\{v\in C([0,T];X):v(0)=0\}$$

and

$$C^1([0,T];X)=\{v:v,v'\in C([0,T];X)\}.$$

Let $0<\alpha<1$. The abstract Hölder spaces with their norms are defined by

$$C_0^\alpha([0,T];X)=\Big\{v\in C_0([0,T];X):\|v\|_{C_0^\alpha([0,T];X)}:=\sup_{0<t_1<t_2<T}\frac{\|v(t_2)-v(t_1)\|_X}{(t_2-t_1)^\alpha}<\infty\Big\},$$

$$C^\alpha([0,T];X)=C_0^\alpha([0,T];X)+X=\{v:v(t)=v_1(t)+v_2,\ v_1\in C_0^\alpha([0,T];X),\ v_2\in X\},$$

$$\|v\|_{C^\alpha([0,T];X)}=\|v-v(0)\|_{C_0^\alpha([0,T];X)}+\|v(0)\|_X.$$

We also introduce the following space:

$$C_0^{1+\alpha}([0,T];X)=\{v: v,v'\in C_0^{\alpha}([0,T];X)\},\ \ 0<\alpha<1.$$

A function $M\in L_{1,loc}(0,\infty)$ is called the *Sonine kernel* if the equation

$$M*k(t)=1,\quad t>0, \tag{8}$$

has a solution $k\in L_{1,loc}(0,\infty)$ [19]. The solution k, if it exists, is unique [5] and is referred to as *associate* to M. Since the convolution is commutative, k is also the Sonine kernel and M is its associate.

The Sonine kernel is unbounded at $t=0$, because otherwise $M*k(t)\to 0$ as $t\to 0^+$ and this contradicts $M*k(t)\equiv 1$.

Let M be the Sonine kernel and k its associate. Then

$$D_t^{\{k\}}(M*v)=\frac{d}{dt}k*M*v=\frac{d}{dt}1*v=v,\quad \forall v\in L_1((0,T);X). \tag{9}$$

Therefore, the operator $M*$ is a one-to-one mapping from $L_1((0,T);X)$ to the space

$$M*L_1((0,T);X)=\{M*v: v\in L_1((0,T);X)\}$$

and $D_t^{\{k\}}$ is the inverse of $M*$. The reversed relation to (9) is

$$M*(D_t^{\{k\}}v)=v,\quad \forall v\in M*L_1((0,T);X). \tag{10}$$

Next, we define some abstract C- and Hölder spaces related to the Sonine kernel M and its associate k:

$$C_0^{\{k\}}([0,T];X):=M*C([0,T];X),\ \ \|v\|_{C_0^{\{k\}}([0,T];X)}=\|D_t^{\{k\}}v\|_{C([0,T];X)},$$
$$C^{\{k\}}([0,T];X):=C_0^{\{k\}}([0,T];X)+X,\ \ \|v\|_{C^{\{k\}}([0,T];X)}=\|v-v(0)\|_{C_0^{\{k\}}([0,T];X)}+\|v(0)\|_X,$$
$$C_0^{\{k\},\alpha}([0,T];X)=M*C_0^{\alpha}([0,T];X),\ \ \|v\|_{C_0^{\{k\},\alpha}([0,T];X)}=\|D_t^{\{k\}}v\|_{C_0^{\alpha}([0,T];X)}.$$

The following continuous embeddings are valid [9]:

$$C^1([0,T];X)\hookrightarrow C^{\{k\}}([0,T];X)\hookrightarrow C([0,T];X)$$
$$C_0^{1+\alpha}([0,T];X)\hookrightarrow C_0^{\{k\},\alpha}([0,T];X)\hookrightarrow C_0^{\alpha}([0,T];X).$$

Clearly, the integration improves the regularity of a function. Therefore, one may ask the question: does the subspace $C_0^{\{k\},\alpha}([0,T];X)$ of $C_0^{\alpha}([0,T];X)$ consist of functions that are Hölder continuous of an order greater than α? This is true provided the function M satisfies certain additional restrictions, as can be seen from the following lemma.

Lemma 1. [9] *If* $M(t)\le c_1t^{\beta-1}$, $|M'(t)|\le c_2t^{\beta-2}$, $t\in(0,T)$ *for some* $c_1,c_2>0$, $0<\beta\le\alpha<1$, *then* $M*\in\mathscr{B}(C_0^{\alpha-\beta}([0,T];X),C_0^{\alpha}([0,T];X))$, *which implies* $C_0^{\{k\},\alpha-\beta}([0,T];X)\hookrightarrow C_0^{\alpha}([0,T];X)$.

In the particular case $M(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}$ we have $C_0^{\{k\},\alpha-\beta}([0,T];X)=C_0^{\alpha}([0,T];X)$ [6].

3.2. Inverse source problem

In this section we formulate the results regarding an inverse source problem that consists in finding the pair (ϕ, w) satisfying

$$\left.\begin{array}{l} D_t^{\{k\}} w(t,x) = a(x)w_{xx}(t,x) + \phi(x)r(t,x) + q(t,x),\ x \in (0,l),\ t \in (0,T), \\ w(t,0) = w(t,l) = 0,\ t \in (0,T), \\ w(0,x) = 0,\ x \in (0,l), \\ w(T,x) = w_T(x),\ x \in (0,l), \end{array}\right\} \quad (11)$$

where a, r, q, w_T are given functions. Such a problem is an important tool in the analysis of the inverse diffusivity problem (4)–(7) posed in Section 2.

Theorem 1. [9] *Let k be the Sonine kernel and M its associate. Assume that*

$$k \in C(0,\infty),\ \ k \text{ is nonincreasing},\ \ \exists t_k > 0 : k(t) \text{ is strictly decreasing in } (0,t_k), \quad (12)$$

$$M \in C^1(0,\infty),\ \ M > 0,\ \ M' \leq 0,\ \ -M' \text{ is nonincreasing and convex.} \quad (13)$$

Moreover, let $a \in C[0,T]$, $a(x) > 0$, $x \in [0,l]$, and one of the following assumptions be valid:
(A1) $r \in C_0^{1+\alpha_1}([0,T];C[0,l])$ *for some* $0 < \alpha_1 < 1$;
(A2) $r \in C_0^{\{k\},\alpha_1}([0,T];C[0,l])$ *and* $M(t) \geq ct^{\gamma-1}$, $t \in (0,T)$, *for some* $c > 0$, $0 < \gamma < \alpha_1 < 1$;
(A3) $r \in C_0^{\{k\},\alpha_1-\beta}([0,T];C[0,l])$ *and* $c_1 t^{\gamma-1} \leq M(t) \leq c_2 t^{\beta-1}$, $|M'(t)| \leq c_3 t^{\beta-2}$, $t \in (0,T)$, *for some* $c_1, c_2, c_3 > 0$, $0 < \beta \leq \gamma < \alpha_1 < 1$.
Additionally, we assume that

$$r \geq 0, \quad D_t^{\{k\}} r \geq 0, \quad (14)$$

$$\text{a.e. } x \in (0,l)\ \ \exists t_x \in (0,T] : r(t_x,x) > 0, \quad (15)$$

$$\text{for any } b \in \{0;l\} \text{ either } r(T,b) > 0 \text{ or } r(\cdot,b) = 0. \quad (16)$$

Finally, let $(\phi,w) \in C[0,l] \times C_0^{\{k\}}([0,T];C[0,l]) \cap C_0([0,T];W_p^2(0,l))$ *for some* $p > 1$ *solve* (11) *for* $q = 0$, $w_T = 0$. *Then* $(\phi,w) = (0,0)$.

Theorem 2. [9] *Let k be the Sonine kernel, M its associate and* (12), (13) *hold. Let r and M satisfy one of the assumptions* (A1)–(A3), *the inequalities* (14) *and* $r(T,x) > 0$, $x \in [0,l]$. *If* $w_T \in C^2[0,l]$, $w_T(0) = w_T(l) = 0$, $q \in C_0^{\{k\},\alpha_2}([0,T];L_p(0,l)) \cap C_0([0,T];C[0,l])$ *for some* $p > 1$ *and* $0 < \alpha_2 < 1$, *then* (11) *has a unique solution* (ϕ,w) *in the space* $C[0,l] \times C_0^{\{k\},\alpha'}([0,T];{}_0W_p^2(0,l)) \cap C_0([0,T];C^2[0,l])$, *where* $\alpha' = \min\{\hat{\alpha};\alpha_2\}$,

$$\hat{\alpha} = \begin{cases} \alpha_1 & \text{in cases (A1), (A2)} \\ \alpha_1 - \beta & \text{in case (A3)} \end{cases} \quad (17)$$

and ${}_0W_p^2(0,l) = \{z \in W_p^2(0,l) : z(0) = z(l) = 0\}$. *Moreover, the estimate*

$$\begin{aligned} &\|\phi\|_{C[0,l]} + \|w\|_{C_0^{\{k\},\alpha'}([0,T];{}_0W_p^2(0,l)) \cap C_0([0,T];C^2[0,l])} \\ &\leq C_1 \left(\|q\|_{C_0^{\{k\},\alpha_2}([0,T];L_p(0,l)) \cap C_0([0,T];C[0,l])} + \|w_T\|_{C^2[0,l]} \right) \end{aligned} \quad (18)$$

is valid where the constant C_1 is independent of q and w_T.

4. STATEMENTS ABOUT THE DIRECT PROBLEM (4)–(6)

Lemma 2. *Let k be the Sonine kernel and M its associate. Let $a \in C[0,l]$. If a function $u \in C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ satisfies $au_x \in C_0([0,T];W_1^1(0,l))$ and solves* (4)–(6)*, then*

$$\int_0^l u(t,y)dy = M * \left[h_l - h_0 + \int_0^l f(\cdot,y)dy\right](t) + lu_0, \ \ t \in [0,T]. \tag{19}$$

Proof. Integrating (4) from 0 to l and taking into account (5), we obtain $D_t^{\{k\}}\left[\int_0^l u(t,y)dy - lu_0\right] = h_l(t) - h_0(t) + \int_0^l f(t,y)dy$. Applying the operator $M*$ to this relation and taking into account (10), we reach (19).

□

Remark 1. The relation (19) is the integral conservation law. Indeed, in Section 2 we saw that $f = k * F$, $h_0 = k * H_0$ and $h_l = -k * H_l$, where F is the physical source function and H_0 and H_l are the outward pointing fluxes. Therefore, since $M * k = 1$, the relation (19) can be rewritten as

$$\int_0^l u(t,y)dy = -\int_0^t [H_l(\tau) + H_0(\tau)]\,d\tau + \int_0^t \int_0^l F(\tau,y)dyd\tau + \int_0^l u_0 dx.$$

Let us define an operator $\mathscr{J} : L_1(0,l) \to \{z \in W_1^1(0,l) : z(0) = z(l) = 0\}$ by the following formula:

$$\mathscr{J}\rho(x) = \frac{l-x}{l}\int_0^x \rho(y)dy + \frac{x}{l}\int_l^x \rho(y)dy, \ \ x \in [0,l]. \tag{20}$$

Proposition 1. *Let k be the Sonine kernel and M its associate. Let $a \in C[0,l]$. Then the following assertions are valid.*

(i) *If a function $u \in C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ satisfies $au_x \in C_0([0,T];W_1^1(0,l))$ and solves* (4)–(6)*, then the function $v \in C_0^{\{k\}}([0,T];W_1^1(0,l)) \cap C_0([0,T];W_1^2(0,l))$ defined by*

$$v(t,x) = \mathscr{J}u(t,x) \tag{21}$$

satisfies $av_{xx} \in C_0([0,T];W_1^1(0,l))$ and solves the following problem:

$$D_t^{\{k\}}v(t,x) = a(x)v_{xx}(t,x) + g(t,x), \ x \in (0,l), \ t \in (0,T), \tag{22}$$

$$v(t,0) = v(t,l) = 0, \ t \in (0,T), \tag{23}$$

$$v(0,x) = 0, \ x \in (0,l), \tag{24}$$

where

$$g(t,x) = \mathscr{J}f(t,x) - \frac{l-x}{l}h_0(t) - \frac{x}{l}h_l(t). \tag{25}$$

(ii) *If $f \in C([0,T];L_1(0,l))$, $h_0, h_l \in C_0[0,T]$ and a function $v \in C_0^{\{k\}}([0,T];W_1^1(0,l)) \cap C_0([0,T];W_1^2(0,l))$ satisfies $av_{xx} \in C_0([0,T];W_1^1(0,l))$ and solves* (22)–(24) *with g of the form* (25)*, then the function $u \in C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ defined by*

$$u(t,x) = v_x(t,x) + \frac{1}{l}M * \left[h_l - h_0 + \int_0^l f(\cdot,y)dy\right](t) + u_0 \tag{26}$$

satisfies $au_x \in C_0([0,T];W_1^1(0,l))$ and solves (4)–(6).

Proof. (i) The assertions regarding the regularity of v and the conditions (23), (24) immediately follow from (21), (20), the assumed regularity of u and (6). Applying the operator $\mathscr{J}$ to (4), observing that $\mathscr{J}[(au_x)_x] = au_x - \frac{l-x}{l}a(0)u_x(t,0) - \frac{x}{l}a(l)u_x(t,l)$ and taking the boundary conditions (5) and the relation $u_x = v_{xx}$ into account, we obtain (22) with g of the form (25).

(ii) The asserted regularity of u and the condition (6) follow from the formula (26), the assumed regularity of f, h and v and the relation (24). Passing to the limit $x \to 0^+$ in (22) and observing (23), (25), we have $0 = a(0)v_{xx}(t,0) - h_0(t)$. Since $v_{xx} = u_x$, we obtain the boundary condition $a(0)u_x(t,0) = h_0(t)$. In a similar manner we prove another boundary condition $a(l)u_x(t,l) = h_l(t)$, too. Next, we apply the operator $\frac{\partial}{\partial x}$ to the equation (22). Replacing v_x by $u - u_0 - \frac{1}{l}M * \left[h_l - h_0 + \int_0^l f(\cdot,y)dy\right]$ in the left-hand side of the obtained equation and using (9), we have

$$D_t^{\{k\}}(u-u_0)(t,x) - Q(t) = (a(x)v_{xx}(t,x))_x + g_x(t,x),$$

where $Q(t) = \frac{1}{l}\left[h_l(t) - h_0(t) + \int_0^l f(t,y)dy\right]$. On the other hand, (25) implies $g_x(t,x) = f(t,x) - Q(t)$. Therefore, since $v_{xx} = u_x$, we obtain the equation (4). □

5. RESULTS ABOUT THE INVERSE PROBLEM (4)–(7)

Firstly, we prove the global uniqueness of the solution to (4)–(7).

Theorem 3. *Let k be the Sonine kernel, M its associate and* (12), (13) *hold. Let the inverse problem* (4)–(7) *have two solutions* $(a,u),(a_1,u_1) \in C[0,l] \times C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ *such that* $au_x, a_1u_{1,x} \in C_0([0,T];W_1^1(0,l))$ *and* $u_1 - u \in C([0,T];W_p^1(0,l))$ *for some* $p > 1$. *Assume that* $a_1(x) > 0$, $x \in [0,l]$, *and one of the following conditions is valid:*

(A4) $u_x \in C_0^{1+\alpha_1}([0,T];C[0,l])$ *for some* $0 < \alpha_1 < 1$;

(A5) $u_x \in C_0^{\{k\},\alpha_1}([0,T];C[0,l])$ *and* $M(t) \geq ct^{\gamma-1}$, $t \in (0,T)$, *for some* $c > 0$, $0 < \gamma < \alpha_1 < 1$;

(A6) $u_x \in C_0^{\{k\},\alpha_1-\beta}([0,T];C[0,l])$ *and* $c_1t^{\gamma-1} \leq M(t) \leq c_2t^{\beta-1}$, $|M'(t)| \leq c_3t^{\beta-2}$, $t \in (0,T)$, *for some* $c_1,c_2,c_3 > 0$, $0 < \beta \leq \gamma < \alpha_1 < 1$.

Additionally, let

$$u_x \geq 0, \quad D_t^{\{k\}}u_x \geq 0, \tag{27}$$

$$a.e.\ x \in (0,l)\ \exists t_x \in (0,T] : u_x(t_x,x) > 0, \tag{28}$$

$$\text{for any } b \in \{0;l\} \text{ either } u_x(T,b) > 0 \text{ or } u_x(\cdot,b) = 0. \tag{29}$$

Then $(a,u) = (a_1,u_1)$.

Proof. Let us denote $v(t,x) = \mathscr{J}u(t,x)$ and $v_1(t,x) = \mathscr{J}u_1(t,x)$. Due to Proposition 1 (i), v is a solution to the problem (22)–(24) and v_1 is a solution to the following problem:

$$D_t^{\{k\}}v_1(t,x) = a_1(x)v_{1,xx}(t,x) + g(t,x),\ x \in (0,l),\ t \in (0,T), \tag{30}$$

$$v_1(t,0) = v_1(t,l) = 0,\ t \in (0,T), \tag{31}$$

$$v_1(0,x) = 0,\ x \in (0,l). \tag{32}$$

Moreover, $v(T,x) = v_1(T,x)$ and $v_{xx} = u_x$, $v_{1,xx} = u_{1,x}$. The pair of differences $(\phi,w) = (a_1 - a, v_1 - v)$ belongs to $C[0,l] \times C_0^{\{k\}}([0,T];W_1^1(0,l)) \cap C_0([0,T];W_p^2(0,l))$ and is a solution to the following problem:

$$\begin{aligned} &D_t^{\{k\}}w(t,x) = a_1(x)w_{xx}(t,x) + \phi(x)u_x(t,x),\ x \in (0,l), t \in (0,T),\\ &w(t,0) = w(t,l) = 0,\ t \in (0,T),\\ &w(0,x) = 0,\ x \in (0,l),\\ &w(T,x) = 0,\ x \in (0,l). \end{aligned} \tag{33}$$

This is the inverse source problem (11) with the data $r = u_x$, $q = 0$ and $w_T = 0$. The assumptions of Theorem 1 are satisfied for this problem. Theorem 1 implies $\phi = 0$ and $w = 0$. Thus, $a = a_1$ and $v = v_1$.

Due to the assumed regularity of a and u the functions f, h_0 and h_l satisfy $f \in C([0,T];L_1(0,l))$ and $h_0, h_l \in C_0[0,T]$. Therefore, we can apply Proposition 1 (ii) for the problems (22)–(24) and (30)–(32). We have $u = v_x + \frac{1}{l} M * \left[h_l - h_0 + \int_0^l f(\cdot,y)dy\right] + u_0$ and $u_1 = v_{1,x} + \frac{1}{l} M * \left[h_l - h_0 + \int_0^l f(\cdot,y)dy\right] + u_0$. Since $v = v_1$, we obtain $u = u_1$. This completes the proof. □

Next, we are going to establish the local existence and stability for the inverse coefficient problem. Let us formulate a problem that contains approximate data:

$$D_t^{\{k\}}(\widetilde{u} - \widetilde{u}_0)(t,x) = [\widetilde{a}(x)\widetilde{u}_x(t,x)]_x + \widetilde{f}(t,x),\ x \in (0,l),\ t \in (0,T), \tag{34}$$

$$\widetilde{a}(0)\widetilde{u}_x(t,0) = \widetilde{h}_0(t),\ \widetilde{a}(l)\widetilde{u}_x(t,l) = \widetilde{h}_l(t),\ t \in (0,T), \tag{35}$$

$$\widetilde{u}(0,x) = \widetilde{u}_0,\ x \in (0,l), \tag{36}$$

$$\widetilde{u}(T,x) = \widetilde{u}_T(x),\ x \in (0,l). \tag{37}$$

Let us denote the data vectors of the exact and approximate problems as follows:

$$D = (f, h_0, h_l, u_0, u_T), \quad \widetilde{D} = (\widetilde{f}, \widetilde{h}_0, \widetilde{h}_l, \widetilde{u}_0, \widetilde{u}_T).$$

The aim is to show that (34)–(37) has a solution that is close to a solution to (4)–(7) provided the difference of $\widetilde{D}$ and D is sufficiently small.

Theorem 4. *Let k be the Sonine kernel, M its associate and* (12), (13) *hold. Let* (4)–(7) *have a solution* $(a,u) \in C[0,l] \times C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ *such that* $au_x \in C_0([0,T];W_1^1(0,l))$, $a(x) > 0$, $x \in [0,l]$, *u_x and M satisfy one of the assumptions* (A4)–(A6)*, the inequalities* (27) *and* $u_x(T,x) > 0$, $x \in [0,l]$. *Assume that* $\widetilde{f} \in C([0,T];L_1(0,l))$, $\widetilde{h}_0, \widetilde{h}_l \in C_0[0,T]$, $\widetilde{u}_T \in C[0,l]$ *and*

$$\int_0^l \widetilde{u}_T(y)dy = M * \left[\widetilde{h}_l - \widetilde{h}_0 + \int_0^l \widetilde{f}(\cdot,y)dy\right](T) + l\widetilde{u}_0. \tag{38}$$

Let $0 < \alpha_2 < 1$ *and* $p > 1$. *Then there exists a constant* $\delta > 0$*, depending on* k, a, u, α_2, p *such that if*

$$\widetilde{D} - D \in \mathscr{D} \quad \text{and} \quad \|\widetilde{D} - D\|_{\mathscr{D}} \le \delta \quad \text{where} \quad \mathscr{D} = C_0^{\{k\},\alpha_2}([0,T];L_1(0,l)) \times (C_0^{\{k\},\alpha_2}[0,T])^2 \times \mathbb{R} \times C^1[0,l],$$

then there exist functions $\widetilde{a} \in C[0,l]$ *and* $\widetilde{u} \in C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ *such that* $\widetilde{a}\widetilde{u}_x \in C_0([0,T];W_1^1(0,l))$ *and the pair* $(\widetilde{a},\widetilde{u})$ *is a solution to the problem* (34)–(37). *Moreover,* $\widetilde{u} - u \in (C_0^{\{k\},\alpha'}([0,T];W_p^1(0,l)) + \mathbb{R}) \cap C([0,T];C^1[0,l])$ *and the following estimate is valid:*

$$\begin{aligned}&\|\widetilde{a} - a\|_{C[0,l]} + \|\widetilde{u} - u\|_{\left(C_0^{\{k\},\alpha'}([0,T];W_p^1(0,l)) + \mathbb{R}\right) \cap C([0,T];C^1[0,l])}\\ &+ \|\widetilde{a}\widetilde{u}_x - au_x\|_{C_0([0,T];W_1^1(0,l))} \le K\|\widetilde{D} - D\|_{\mathscr{D}},\end{aligned} \tag{39}$$

where $\alpha' = \min\{\hat{\alpha};\alpha_2\}$, $\hat{\alpha} = \begin{cases} \alpha_1 & \text{in cases (A4), (A5)} \\ \alpha_1 - \beta & \text{in case (A6)} \end{cases}$ *and the constant K is independent of* $\widetilde{D} - D$.

Proof. Let us consider the problem to find a pair (ϕ, w) that satisfies the following relations:

$$D_t^{\{k\}} w(t,x) = a(x)w_{xx}(t,x) + \phi(x)u_x(t,x) + \phi(x)w_{xx}(t,x) + q(t,x),\ x \in (0,l),\ t \in (0,T), \tag{40}$$

$$w(t,0) = w(t,l) = 0,\ t \in (0,T), \tag{41}$$

$$w(0,x) = 0,\ x \in (0,l), \tag{42}$$

$$w(T,x) = w_T(x),\ x \in (0,l), \tag{43}$$

where $q(t,x) = \mathscr{J}(\widetilde{f}-f)(t,x) - \frac{l-x}{l}(\widetilde{h}_0-h_0)(t) - \frac{x}{l}(\widetilde{h}_l-h_l)(t)$ and $w_T(x) = \mathscr{J}(\widetilde{u}_T-u_T)(x)$. In case $\widetilde{D}-D \in \mathscr{D}$, we have $q \in C_0^{\{k\},\alpha_2}([0,T];W_1^1(0,T))$, $w_T \in C^2[0,T]$, $w_T(0) = w_T(l) = 0$ and

$$\|q\|_{C_0^{\{k\},\alpha_2}([0,T];W_1^1(0,T))} + \|w_T\|_{C^2[0,l]} \le C_2\|\widetilde{D}-D\|_{\mathscr{D}} \tag{44}$$

with some constant $C_2 > 0$. Further, let $\mathscr{F}_{a,r}$ stand for the a- and r-dependent operator that maps the pair of functions (q,w_T) to the solution of the inverse source problem (11). Let us fix some $p > 1$. Due to Theorem 2, the problem (40)–(43) is in the space

$$\mathscr{W} = C[0,l] \times C_0^{\{k\},\alpha'}([0,T];{}_0W_p^2(0,l)) \cap C_0([0,T];C^2[0,l])$$

equivalent to the following operator equation:

$$S = F(S), \quad \text{where} \quad S = (\phi,w),\ F(S) = \mathscr{F}_{a,u_x}(\phi w_{xx}+q,w_T). \tag{45}$$

Using (18) and (44), we deduce the estimate

$$\|F(S)\|_{\mathscr{W}} \le C_1\left(\|\phi w_{xx}+q\|_{C_0^{\{k\},\alpha_2}([0,T];L_p(0,l))\cap C_0([0,T];C[0,l])} + \|w_T\|_{C^2[0,l]}\right) \le C_3\|S\|_{\mathscr{W}}^2 + C_4\|\widetilde{D}-D\|_{\mathscr{D}}, \tag{46}$$

where $C_3 = C_1\max\{\omega_1;1\}$, $C_4 = C_1C_2\max\{\omega_2;1\}$ and ω_1 and ω_2 are the norms of the embedding operators $C_0^{\{k\},\alpha'}([0,T];L_p(0,l)) \hookrightarrow C_0^{\{k\},\alpha_2}([0,T];L_p(0,l))$ and $C_0^{\{k\},\alpha_2}([0,T];W_1^1(0,l)) \hookrightarrow C_0^{\{k\},\alpha_2}([0,T];L_p(0,l)) \cap C_0([0,T];C[0,l])$, respectively. Similarly, for $S_j = (\phi_j,w_j)$, $j=1,2$, we have

$$\begin{aligned}&\|F(S_1)-F(S_2)\|_{\mathscr{W}} = \|\mathscr{F}_{a,u_x}\big((\phi_1-\phi_2)w_{1,xx} + \phi_2(w_{1,xx}-w_{2,xx}),0\big)\|_{\mathscr{W}}\\ &\le C_3\big(\|S_1\|_{\mathscr{W}}+\|S_2\|_{\mathscr{W}}\big)\|S_1-S_2\|_{\mathscr{W}}.\end{aligned} \tag{47}$$

Let $\|\widetilde{D}-D\|_{\mathscr{D}} \le \delta = \frac{1}{8C_3C_4}$. Then, for any S such that $\|S\|_{\mathscr{W}} \le \rho = K_0\|\widetilde{D}-D\|_{\mathscr{D}}$ where $K_0 = 2C_4$ from (46), we have

$$\begin{aligned}&\|F(S)\|_{\mathscr{W}} \le C_3\rho^2 + C_4\|\widetilde{D}-D\|_{\mathscr{D}} = C_3K_0^2\|\widetilde{D}-D\|_{\mathscr{D}}^2 + C_4\|\widetilde{D}-D\|_{\mathscr{D}} = C_3K_0^2\|\widetilde{D}-D\|_{\mathscr{D}}\|\widetilde{D}-D\|_{\mathscr{D}}\\ &+C_4\|\widetilde{D}-D\|_{\mathscr{D}} \le C_3K_0^2\delta\|\widetilde{D}-D\|_{\mathscr{D}} + C_4\|\widetilde{D}-D\|_{\mathscr{D}} = \frac{3K_0}{4}\|\widetilde{D}-D\|_{\mathscr{D}} < \rho.\end{aligned}$$

This implies that the operator F leaves the ball $\|S\|_{\mathscr{W}} \le \rho$ invariant. Further, for any S_j, $j=1,2$ such that $\|S_j\|_{\mathscr{W}} \le \rho$ from (47), we obtain

$$\begin{aligned}&\|F(S_1)-F(S_2)\|_{\mathscr{W}} \le 2C_3\rho\|S_1-S_2\|_{\mathscr{W}} = 2C_3K_0\|\widetilde{D}-D\|_{\mathscr{D}}\|S_1-S_2\|_{\mathscr{W}}\\ &\le 2C_3K_0\delta\|S_1-S_2\|_{\mathscr{W}}^2 = \frac{1}{2}\|S_1-S_2\|_{\mathscr{W}}.\end{aligned}$$

This shows that the operator F is a contraction in the ball $\|S\|_{\mathscr{W}} \le \rho$. Consequently, the equation (45) and the equivalent problem (40)–(43) have a unique solution $S = (\phi,w)$ in such a ball. Due to the definition of ρ we have the estimate

$$\|\phi\|_{C[0,l]} + \|w\|_{C_0^{\{k\},\alpha'}([0,T];{}_0W_p^2(0,l))\cap C_0([0,T];C^2[0,l])} \le K_0\|\widetilde{D}-D\|_{\mathscr{D}}. \tag{48}$$

Let $v = \mathscr{J}u$. By Proposition 1 (i), v belongs to $C_0^{\{k\}}([0,T];W_1^1(0,l)) \cap C_0([0,T];W_1^2(0,l))$, satisfies $av_{xx} \in C_0([0,T];W_1^1(0,l))$ and is a solution to (22)–(24), where g is given by (25). Further, let $\widetilde{v} = v+w$. Due to the properties of v and w we have $\widetilde{v} \in C_0^{\{k\}}([0,T];W_1^1(0,l)) \cap C_0([0,T];W_1^2(0,l))$. Adding the equations

(40), (41) and (42) to the equations (22), (23) and (24), respectively, and replacing u_x by v_{xx}, we see that $\widetilde{v}$ is a solution to the following problem:

$$D_t^{\{k\}}\widetilde{v}(t,x) = \widetilde{a}(x)\widetilde{v}_{xx}(t,x) + \widetilde{g}(t,x),\ x \in (0,l),\ t \in (0,T), \tag{49}$$

$$\widetilde{v}(t,0) = \widetilde{v}(t,l) = 0,\ t \in (0,T), \tag{50}$$

$$\widetilde{v}(0,x) = 0,\ x \in (0,l), \tag{51}$$

where $\widetilde{a} = a + \phi$ and

$$\widetilde{g}(t,x) = g(t,x) + q(t,x) = \mathscr{J}\widetilde{f}(t,x) - \frac{l-x}{l}\widetilde{h}_0(t) - \frac{x}{l}\widetilde{h}_l(t).$$

Since $\widetilde{v} \in C_0^{\{k\}}([0,T];W_1^1(0,l))$ and $\widetilde{g} \in C_0([0,T];W_1^1(0,l))$, from the equation (49) we obtain that $\widetilde{a}\widetilde{v}_{xx} \in C_0([0,T];W_1^1(0,l))$. Applying Proposition 1 (ii) to (49)–(51), we obtain that the function $\widetilde{u} \in C^{\{k\}}([0,T];L_1(0,l)) \cap C([0,T];W_1^1(0,l))$ defined by

$$\widetilde{u}(t,x) = \widetilde{v}_x(t,x) + \frac{1}{l}M * \left[\widetilde{h}_l - \widetilde{h}_0 + \int_0^l \widetilde{f}(\cdot,y)dy\right](t) + \widetilde{u}_0 \tag{52}$$

satisfies $\widetilde{a}\widetilde{u}_x \in C_0([0,T];W_1^1(0,l))$ and solves (34)–(36). Since $v(T,x) = \mathscr{J}u_T(x)$ and $w(T,x) = w_T(x) = \mathscr{J}(\widetilde{u}_T - u_T)(x)$, we have $\widetilde{v}(T,x) = \mathscr{J}\widetilde{u}_T(x)$. Hence, from (52) we obtain

$$\widetilde{u}(T,x) = \widetilde{u}_T(x) - \frac{1}{l}\int_0^l \widetilde{u}_T(y)dy + \frac{1}{l}M * \left[\widetilde{h}_l - \widetilde{h}_0 + \int_0^l \widetilde{f}(\cdot,y)dy\right](T) + \widetilde{u}_0.$$

Using (38), we obtain (37). Therefore, $(\widetilde{a},\widetilde{u})$ is a solution to (34)–(37).

It remains to prove $\widetilde{u} - u \in (C_0^{\{k\},\alpha'}([0,T];W_p^1(0,l)) + \mathbb{R}) \cap C([0,T];C^1[0,l])$ and the estimate (39). From

$$\widetilde{u}(t,x) - u(t,x) = w_x(t,x) + \frac{1}{l}M * \left[\widetilde{h}_l - h_l - \widetilde{h}_0 + h_0 + \int_0^l (\widetilde{f} - f)(\cdot,y)dy\right](t) + \widetilde{u}_0 - u_0, \tag{53}$$

in view of $w \in C_0^{\{k\},\alpha'}([0,T];{}_0W_p^2(0,l)) \cap C_0([0,T];C^2[0,l])$ and the assumptions on $\widetilde{D} - D$, we have $\widetilde{u} - u \in (C_0^{\{k\},\alpha'}([0,T];W_p^1(0,l)) + \mathbb{R}) \cap C([0,T];C^1[0,l])$. Next, we note that $\widetilde{a}\widetilde{v}_{xx} - av_{xx} = D_t^{\{k\}}w + g - \widetilde{g}$. Thus, due to (48) and the definitions of g and $\widetilde{g}$ as well as the norm $\|\cdot\|_{\mathscr{D}}$, we obtain

$$\|\widetilde{a}\widetilde{v}_{xx} - av_{xx}\|_{C_0([0,T];W_1^1(0,l))} \leq K_1\|\widetilde{D} - D\|_{\mathscr{D}}, \tag{54}$$

where K_1 is a constant. By means of (48), (53), (54) as well as the relations $\phi = \widetilde{a} - a$ and $\widetilde{a}\widetilde{v}_{xx} - av_{xx} = \widetilde{a}\widetilde{u}_x - au_x$, we obtain the estimate (39). This completes the proof. □

Remark 2. The estimate (39) shows that the solution to (34)–(37) continuously depends on the first order derivative of the measured function u_T. The degree of ill-posedness of the inverse diffusivity problem (4)–(7) is lower than that of the inverse problem to reconstruct an x-dependent source factor from final measurements [9]. The solution to the latter problem continuously depends on the second order derivatives of the final data.

6. ADDITIONAL REMARKS

6.1. Examples of kernels

Let us define the following subset of the set of completely monotonic functions:

$$\mathscr{CM} = \{z \in L_{1,loc}(0,\infty) \cap C^\infty(0,\infty) : \lim_{t\to 0^+} z(t) = \infty,\ (-1)^i z^{(i)}(t) > 0, t > 0,\ i = 0,1,2,\ldots\}.$$

It holds the following statement:

Lemma 3. [4] *Let $k \in \mathscr{CM}$. Then k is the Sonine kernel and its associate M also belongs to $\mathscr{CM}$.*

Clearly, for $k, M \in \mathscr{CM}$, the conditions (12) and (13) are satisfied.

There are many examples of generalized fractional derivatives with kernels of the class $\mathscr{CM}$ in models of subdiffusion. Let us list some of them. A more detailed description can be found, e.g., in [10].

• Distributed fractional derivatives. Then

$$\text{either} \quad k(t) = \int_0^1 \tfrac{t^{-\beta}}{\Gamma(1-\beta)} dp(\beta) \quad \text{or} \quad M(t) = \int_0^1 \tfrac{t^{\beta-1}}{\Gamma(\beta)} dp(\beta),$$

where p is a Borel measure. Such derivatives occur in the modelling of accelerating and retarding subdiffusion, also of ultraslow diffusion [12]. A particular case is the multiterm derivatives when

$$\text{either} \quad k(t) = \sum_{j=1}^{s} \varkappa_j \tfrac{t^{-\beta_j}}{\Gamma(1-\beta_j)} \quad \text{or} \quad M(t) = \sum_{j=1}^{s} \varkappa_j \tfrac{t^{\beta_j - 1}}{\Gamma(\beta_J)}, \quad \text{where} \quad 0 < \beta_1 < \ldots < \beta_s < 1, \quad \varkappa_j > 0.$$

• Tempered fractional derivatives. They are used in the modelling of slow transition from anomalous diffusion to the normal one. We can point out three different cases that occur in the literature:

$$M(t) = \frac{1}{\Gamma(\beta)} e^{-\lambda t} t^{\beta-1}, 0 < \beta < 1, \lambda > 0$$

[20] $\left(\text{then } k(t) = \frac{1}{\Gamma(1-\beta)} e^{-\lambda t} t^{-\beta} + \frac{\lambda}{\Gamma(1-\beta)} \int_0^t e^{-\lambda\tau} \tau^{-\beta} d\tau \right)$;

$$M(t) = \frac{1}{\Gamma(\beta)} e^{-\lambda t} t^{\beta-1} + \frac{\lambda}{\Gamma(\beta)} \int_0^t e^{-\lambda\tau} \tau^{\beta-1} d\tau,\ 0 < \beta < 1, \lambda > 0$$

[3] $\left(\text{then } k(t) = \frac{1}{\Gamma(1-\beta)} e^{-\lambda t} t^{-\beta} \right)$;

$$M(t) = e^{-\lambda t} t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta),\ 0 < \beta < 1,\ \lambda > 0,$$

where $E_{\beta,\beta}$ is the two-parametric Mittag-Leffler function [22] $\left(\text{then } k(t) = \frac{1}{\Gamma(1-\beta)} e^{-\lambda t} t^{-\beta} + \frac{\lambda}{\Gamma(1-\beta)} \int_0^t e^{-\lambda\tau} \tau^{-\beta} d\tau - \lambda^\beta \right)$.

6.2. Positivity assumptions on u_x

In this subsection we consider the direct problem (4)–(6), assume that its solution u is sufficiently smooth and ask the question: which sufficient conditions on the data f, h_0 and h_l guarantee the validity of the conditions $u_x \geq 0$, $D_t^{\{k\}} u_x \geq 0$ and $u_x(T,x) > 0$, $x \in [0,l]$ in Theorems 3 and 4?[1]

[1] Evidently, the other two inequality type conditions (28) and (29) in Theorem 3 follow from $u_x(T,x) > 0$, $x \in [0,l]$.

From (4)–(6) we deduce the following problem for $U = au_x$:

$$D_t^{\{k\}}U(t,x) = a(x)U_{xx}(t,x) + a(x)f_x(t,x),\ x \in (0,l),\ t \in (0,T),$$
$$U(t,0) = h_0(t),\ U(t,l) = h_l(t),\ t \in (0,T),$$
$$U(0,x) = 0,\ x \in (0,l).$$

Assume that $a(x) > 0$, $x \in [0,l]$. Then the conditions $u_x \geq 0$, $D_t^{\{k\}}u_x \geq 0$ and $u_x(T,x) > 0$, $x \in [0,l]$ are equivalent to $U \geq 0$, $D_t^{\{k\}}U \geq 0$ and $U(T,x) > 0$, $x \in [0,l]$, respectively. Moreover, let k satisfy (12). Then we can apply a positivity principle for the generalized subdiffusion equation (Lemma 4 in [9]). The assumptions $f_x \geq 0$, $h_0 \geq 0$ and $h_l \geq 0$ imply the inequality $U \geq 0$.

The problem for $V = D_t^{\{k\}}U$ reads

$$D_t^{\{k\}}[V(t,x) - V(0,x)] = a(x)V_{xx}(t,x) + a(x)D_t^{\{k\}}[f_x(t,x) - f_x(0,x)],\ x \in (0,l),\ t \in (0,T),$$
$$V(t,0) = D_t^{\{k\}}h_0(t),\ V(t,l) = D_t^{\{k\}}h_l(t),\ t \in (0,T),$$
$$V(0,x) = a(x)f_x(0,x),\ x \in (0,l).$$

Due to the above mentioned positivity principle, the assumptions $D_t^{\{k\}}[f_x(t,x) - f_x(0,x)] \geq 0$, $D_t^{\{k\}}h_0 \geq 0$ and $D_t^{\{k\}}h_l \geq 0$ imply the inequality $D_t^{\{k\}}U \geq 0$.

It remains to deal with the strict inequality $U(T,x) > 0$, $x \in [0,l]$. We are going to use a method presented in pp. 257–258 of [9]. Let us assume that

$$\exists \mu \in C[0,T],\ \mu \geq 0,\ \mu \neq 0,\ \mu - \text{nondecreasing}:$$
$$a(x)f_x(t,x) \geq \mu(t),\ x \in [0,l],\ t \in [0,T],\quad h_0(t) \geq \mu(t),\ h_l(t) \geq \mu(t),\ t \in [0,T].$$

Let us denote $W = U - \delta 1 * \mu$ where $\delta > 0$. The function W is a solution to the problem

$$D_t^{\{k\}}W(t,x) = a(x)W_{xx}(t,x) + a(x)f_x(t,x) - \delta k * \mu(t),\ x \in (0,l),\ t \in (0,T), \tag{55}$$
$$W(t,0) = h_0(t) - \delta 1 * \mu(t),\ W(t,l) = h_l(t) - \delta 1 * \mu(t),\ t \in (0,T), \tag{56}$$
$$W(0,x) = 0,\ x \in (0,l). \tag{57}$$

Let $\delta \leq \min\left\{\frac{1}{\int_0^T k(\tau)d\tau}; \frac{1}{T}\right\}$. Then we have

$$a(x)f_x(t,x) - \delta k * \mu(t) \geq a(x)f_x(t,x) - \delta \int_0^t k(\tau)d\tau\,\mu(t) \geq a(x)f_x(t,x) - \mu(t) \geq 0$$

and $h_0(t) - \delta 1 * \mu(t) \geq h_0(t) - \delta t\mu(t) \geq h_0(t) - \mu(t) \geq 0$. Similarly, we obtain $h_l(t) - \delta 1 * \mu(t) \geq 0$. The positivity principle implies $W \geq 0$. Thus, $U(T,x) = W(T,x) + \delta 1 * \mu(T) > 0$, $x \in [0,l]$.

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Pöördülesanne difusioonikordaja määramiseks üldistatud subdifusioonivõrrandis lõpphetkel tehtud mõõtmiste alusel

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Vaadeldakse pöördülesannet difusioonikordaja määramiseks ühemõõtmelises subdifusioonivõrrandis, mis sisaldab üldistatud murrulist tuletist ajamuutuja suhtes. Ülesandes on lisatingimusena antud olekufunktsiooni jälg lõpphetkel $t = T$. On tõestatud pöördülesande lahendi ühesus. Seejärel on formuleeritud ligikaudsete algandmetega ülesanne ja tõestatud, et juhul, kui täpne ülesanne omab lahendit ja algandmete viga on piisavalt väike, siis omab ka ligikaudne ülesanne lahendit. Lisaks on tuletatud mainitud lahendite vahe hinnang algandmete vea kaudu.