CONTROL
THEORY

Linearization of discrete-time control system by state transformation

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Abstract. This paper presents necessary and sufficient linearizability conditions by state transformation for discrete-time multi-input nonlinear control system under the mild assumption on the surjectivity property and describes how to find the state transformation when it exists. The conditions are formulated in terms of backward shifts of vector fields, defined by the system dynamics. The conditions are compared with those that allow additionally the regular static state feedback. The theory is illustrated by two examples.

Key words: nonlinear control system, discrete-time system, backward shifts of vector fields, state linearizability.

1. INTRODUCTION

Linearization of system dynamics by state transformation (and state feedback), when applicable, is an extremely powerful technique. Once such linearization is obtained, one can rely on the linear control theory to control nonlinear systems. The focus of this paper is on discrete-time nonlinear control systems. In [1] the algebraic framework was developed for such systems, based on the shifts of the vector fields, and being dual to the framework based on the differential forms [2,3]. It is important to note that the new framework does not assume system reversibility like the other approaches to nonlinear discrete-time systems [4,5]; only a less restrictive surjectivity property is necessary. Moreover, in [1] the alterative characterization of the static state feedback linearizability property was given in terms of the vector spaces of the backward shifts of certain vector fields, defined by system dynamics. Let us recall that the static state feedback linearizability property means that by using the state transformation and regular static state feedback, one can transform the state equations into the form such that the forward shifts of the state coordinates are described by the linear functions of the states and inputs. The goal of this paper is to find conditions, again in terms of the backward shifts of certain vector fields, under which the multi-input system can be linearized, using *only* the state transformation. Since feedback is not allowed, the conditions are obviously more restrictive than those regarding the static state feedback linearizability in [1]. It should be pointed out that the problem was addressed earlier in [4] for single-input systems by using different tools. Moreover, it is important to mention that we look for conditions in the form that include those from [1] plus some additional conditions,

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so it will be clear by direct inspection how much freedom is lost compared with the case where feedback is also used.

Linearization by state transformation for a single-input system is also studied in [6], where it has been proven that the state linearizability property can be alternatively characterized by the specific structure of system dynamics, described in terms of the composition of two diffeomorphisms. It should be noted that such characterization holds only under the assumption of system reversibility.

Finally, unlike in [4], where local results were given, our results are generic, i.e. valid on an open and dense subset of the state space, provided that they hold at some point of the domain. Let us mention that the concept of the generic property does not make sense, in general, for systems described by smooth functions, whereas it makes sense in the case of analytic functions [7]. For this reason we assume analyticity, whereas the framework in [4,5] requires only smoothness. Since we look at dimensions (or ranks) over the field of functions and not over $\mathbb{R}$, it is meaningless to speak about constant dimensionality of the distributions. A generic rank is a maximal rank on an open and dense set. The rank may drop on some subset. By reducing the set, one can always achieve a constant rank over $\mathbb{R}$.

2. PRELIMINARIES

Let us recall some preliminary results that originate mostly from [1] and consider the discrete-time nonlinear control system

$$x^{\langle 1 \rangle}(t) = \overline{\Phi}(x(t), u(t)), \tag{1}$$

where $x^{\langle 1 \rangle}(t) := x(t+1)$, $t \in \mathbb{Z}$, the variables $x(t) \in \bar{X} \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, and the state transition map $\overline{\Phi} : \bar{X} \times U \to \bar{X}$ is supposed to be analytic. Both $\bar{X}$ and U are assumed to be open sets. We assume that the map $\overline{\Phi}$ can be extended to the map $\Phi = [\overline{\Phi}^T, \chi^T]^T : \bar{X} \times U \to \bar{X} \times \mathbb{R}^m$ so that Φ has the global analytic inverse $[\Lambda^T, \lambda^T]^T : \Phi(\bar{X} \times U) \to \bar{X} \times U$. We introduce the additional variable $z(t) \in \mathbb{R}^m$,

$$z(t) = \chi(x(t), u(t)). \tag{2}$$

The system (1), (2) defines the inversive difference field $\mathscr{K}$ of the meromorphic functions in a finite number of variables from the set $\mathscr{C} = \{x, u^{\langle k \rangle}, z^{\langle -l \rangle}, k \geq 0, l \geq 1\}$. Here $x^{\langle k \rangle}$ denotes the kth order forward shift of x and $z^{\langle -l \rangle}$ the lth order backward shift of z. The 1st order forward shift of the variable x is defined by the equations (1) and the 1st order backward shifts of x and u by

$$x^{\langle -1 \rangle} = \Lambda(x, z^{\langle -1 \rangle}), \quad u^{\langle -1 \rangle} = \lambda(x, z^{\langle -1 \rangle}). \tag{3}$$

The forward shift of a function $\varphi(x, u, u^{\langle 1 \rangle}, ..., u^{\langle k \rangle}, z^{\langle -1 \rangle}, ..., z^{\langle -l \rangle}) \in \mathscr{K}$ is defined as the composition $\varphi^{\langle 1 \rangle} := \varphi(\overline{\Phi}(x, u), u^{\langle 1 \rangle}, u^{\langle 2 \rangle}, ..., u^{\langle k+1 \rangle}, \chi(x, u), ..., z^{\langle -l+1 \rangle})$. The higher order forward shifts of x are defined recursively as

$$\begin{aligned} x^{\langle 1 \rangle} &= \overline{\Phi}^1(x, u) := \overline{\Phi}(x, u), \\ x^{\langle k \rangle} &= \overline{\Phi}^k(x, u, u^{\langle 1 \rangle}, ..., u^{\langle k-1 \rangle}) := \overline{\Phi}^{\langle k-1 \rangle}(\overline{\Phi}(x, u), u^{\langle 1 \rangle}, ..., u^{\langle k-1 \rangle}), \quad k \geq 2. \end{aligned} \tag{4}$$

The backward shift of φ is the composition $\varphi^{\langle -1 \rangle} := \varphi(\Lambda(x, z^{\langle -1 \rangle}), \lambda(x, z^{\langle -1 \rangle}), u, ..., u^{\langle k-1 \rangle}, z^{\langle -2 \rangle}, ..., z^{\langle -l-1 \rangle})$. The higher order backward shifts of x are defined recursively as

$$\begin{aligned} x^{\langle -1 \rangle} &= \Lambda^1(x, z^{\langle -1 \rangle}) := \Lambda(x, z^{\langle -1 \rangle}), \\ x^{\langle -k \rangle} &= \Lambda^k(x, z^{\langle -1 \rangle}, z^{\langle -2 \rangle}, ..., z^{\langle -k \rangle}) := \Lambda^{k-1}(\Lambda(x, z^{\langle -1 \rangle}), z^{\langle -2 \rangle}, ..., z^{\langle -k \rangle}), \quad k \geq 2. \end{aligned} \tag{5}$$

Due to (2) and (4), the higher order forward shifts of z can be computed as $z^{\langle k \rangle} = \chi\left(\overline{\Phi}^k(x, u, ..., u^{\langle k-1 \rangle}), u^{\langle k \rangle}\right)$, and due to (3) and (5), the higher order backward shifts of u as $u^{\langle -k \rangle} = \lambda\left(\Lambda^{k-1}(x, z^{\langle -1 \rangle}, ..., z^{\langle -k+1 \rangle}), z^{\langle -k \rangle}\right)$.

Consider the infinite set of the symbols $\mathrm{d}\mathscr{C} = \{\mathrm{d}x, \mathrm{d}u^{\langle k \rangle}, \mathrm{d}z^{\langle -l \rangle}, k \geq 0, l \geq 1\}$ and let $\mathscr{E} := \mathrm{span}_{\mathscr{K}}\{\mathrm{d}\mathscr{C}\}$ be the vector space spanned over $\mathscr{K}$ by the elements of $\mathrm{d}\mathscr{C}$, called the 1-forms

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k \geq 0} \sum_{j=1}^{m} B_{jk} \mathrm{d}u_j^{\langle k \rangle} + \sum_{l \geq 1} \sum_{q=1}^{m} C_{ql} \mathrm{d}z_q^{\langle -l \rangle},$$

where only the finite number of B_{jk} and C_{ql} differ from zero [2]. Next define the space $\mathscr{E}^*$, dual to $\mathscr{E}$, whose elements are the vector fields

$$\Xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{k \geq 0} \sum_{j=1}^{m} \xi_{jk} \frac{\partial}{\partial u_j^{\langle k \rangle}} + \sum_{l \geq 1} \sum_{q=1}^{m} \bar{\xi}_{ql} \frac{\partial}{\partial z_q^{\langle -l \rangle}}. \tag{6}$$

By duality between $\mathscr{E}$ and $\mathscr{E}^*$ the following holds for scalar products of 1-forms and the vector fields: $\langle \mathrm{d}x_i, \Xi \rangle = \xi_i$, $\langle \mathrm{d}u_j^{\langle k \rangle}, \Xi \rangle = \xi_{jk}$, $\langle \mathrm{d}z_q^{\langle -l \rangle}, \Xi \rangle = \bar{\xi}_{ql}$.

The backward shift of the vector field Ξ in (6) is another vector field

$$\Xi^{\langle -1 \rangle} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{k \geq 0} \sum_{j=1}^{m} b_{jk} \frac{\partial}{\partial u_j^{\langle k \rangle}} + \sum_{l \geq 1} \sum_{q=1}^{m} c_{ql} \frac{\partial}{\partial z_q^{\langle -l \rangle}}, \tag{7}$$

where

$$a_i = \left\langle \mathrm{d}x_i^{\langle 1 \rangle}, \Xi \right\rangle^{\langle -1 \rangle} = \left\langle \mathrm{d}\overline{\Phi}_i, \Xi \right\rangle^{\langle -1 \rangle}, \quad b_{jk} = \left\langle \mathrm{d}u_j^{\langle k+1 \rangle}, \Xi \right\rangle^{\langle -1 \rangle}, \quad c_{ql} = \left\langle \mathrm{d}z_q^{\langle -l+1 \rangle}, \Xi \right\rangle^{\langle -1 \rangle}. \tag{8}$$

The projection of Ξ in (6) is another vector field

$$\Xi^{\pi} = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}. \tag{9}$$

Note that the shift and the projection operators do not commute. For simplicity of notation, in what follows, we often write

$$\partial/\partial x = (\partial/\partial x_1, ..., \partial/\partial x_n)^T, \quad \partial/\partial u = (\partial/\partial u_1, ..., \partial/\partial u_m)^T, \quad \partial/\partial z^{\langle -l \rangle} = (\partial/\partial z_1^{\langle -l \rangle}, ..., \partial/\partial z_m^{\langle -l \rangle})^T.$$

Recall that the static state feedback linearizability conditions for system (1) are formulated in terms of a non-decreasing sequence of the distributions of vector fields, defined for $k \geq 1$ by

$$\mathscr{D}_k = \mathrm{span}_{\mathscr{K}}\left\{\frac{\partial}{\partial z^{\langle -1 \rangle}}, \left(\frac{\partial}{\partial u}\right)^{\langle -l \rangle \pi}, \quad l = 1, ..., k\right\}. \tag{10}$$

Like in [1], k^* denotes a uniquely defined integer such that $\dim_{\mathscr{K}} \mathscr{D}_k < \dim_{\mathscr{K}} \mathscr{D}_{k^\star}$ for $k < k^\star$, but $\dim_{\mathscr{K}} \mathscr{D}_k = \dim_{\mathscr{K}} \mathscr{D}_{k^\star}$ for all $k \geq k^\star$. Define also the projections $\mathscr{D}_k^{\pi}$ as

$$\mathscr{D}_k^{\pi} = \mathscr{D}_k \cap \mathrm{span}_{\mathscr{K}}\left\{\frac{\partial}{\partial x}\right\} = \mathrm{span}_{\mathscr{K}}\left\{\left(\frac{\partial}{\partial u}\right)^{\langle -l \rangle \pi}, \quad l = 1, ..., k\right\}. \tag{11}$$

Lemma 1. *[1] System (1) is accessible if and only if* $\dim_{\mathscr{K}} \mathscr{D}_{k^*}^{\pi} = n$.

Remark 2. The fact that $\dim_{\mathcal{K}} \mathcal{D}_{k^\star}^\pi = n$ implies that there exist m indices n_i, $n_i \geq 1$, $\sum_{i=1}^m n_i = n$, such that

$$\dim_{\mathcal{K}} \operatorname{span}_{\mathcal{K}} \left\{ \left(\frac{\partial}{\partial u_i}\right)^{\langle -l_i \rangle \pi}, \quad l_i = 1, ..., n_i \right\} = n. \tag{12}$$

Obviously, $k^\star = \max\{n_i\}$.

A distibution $\mathcal{D}$ is called involutive, if for the arbitrary vector fields $\Xi_1, \Xi_2 \in \mathcal{D}$ also their Lie brackets $[\Xi_1, \Xi_2] \in \mathcal{D}$. The vector fields Ξ_1 and Ξ_2 commute, if their Lie brackets are identically equal to zero. It is easy to check by simple calculations that:

$$[\Xi_1, \Xi_2]^{\langle -1 \rangle} = \left[\Xi_1^{\langle -1 \rangle}, \Xi_2^{\langle -1 \rangle}\right], \quad \langle \omega, \Xi_1 \rangle^{\langle -1 \rangle} = \left\langle \omega^{\langle -1 \rangle}, \Xi_1^{\langle -1 \rangle} \right\rangle, \quad \langle \omega, \Xi_1 \rangle^{\langle 1 \rangle} = \left\langle \omega^{\langle 1 \rangle}, \Xi_1^{\langle 1 \rangle} \right\rangle, \tag{13}$$

where $\langle \omega, \Xi_1 \rangle$ denotes the scalar product of a 1-form ω and a vector field Ξ_1. For a 1-form $\omega = \sum_i A_i d\zeta_i$, $A_i \in \mathcal{K}$, $d\zeta_i \in d\mathcal{C}$, the forward and backward shifts are defined as $\omega^{\langle 1 \rangle} = \sum_i A_i^{\langle 1 \rangle} d\zeta_i^{\langle 1 \rangle}$ and $\omega^{\langle -1 \rangle} = \sum_i A_i^{\langle -1 \rangle} d\zeta_i^{\langle -1 \rangle}$, respectively.

Consider a vector field $\Xi = \sum_{i=1}^n \xi_i(x) \partial/\partial x_i$, whose coefficients depend only on x. Under the state transformation

$$X = \Psi(x) \tag{14}$$

this vector field takes, in the new coordinates, the form [8]

$$\Psi_* \Xi = \sum_{i,j=1}^n \frac{\partial \Psi_j(x)}{\partial x_i} \xi_j(x) \bigg|_{x=\Psi^{-1}(X)} \frac{\partial}{\partial X_j}. \tag{15}$$

Here the operator Ψ_* is called the *tangent map* of Ψ, and the vector field $\Psi_* \Xi$ is called the *image* of the vector field Ξ under the map Ψ_*.

A set of vector fields $\Xi_1, ..., \Xi_n$ is called linearly independent if there do not exist non-zero coefficients $\beta_i \in \mathcal{K}$, $i = 1, ..., n$, such that $\sum_{i=1}^n \beta_i \Xi_i \equiv 0$.

Lemma 3. *Let $\Xi_j = \sum_{i=1}^n \xi_{ij}(x) \partial/\partial x_i \in \operatorname{span}_{\mathcal{K}} \{\partial/\partial x\}$, $j = 1, ..., n$, be linearly independent. If the vector fields Ξ_j commute, then generically one can define the state transformation $X_i = \Psi_i(x)$, $\Psi_i \in \mathcal{K}$, $i = 1, ..., n$, such that*

$$\langle d\Psi_i, \Xi_j \rangle = \delta_{ij}, \quad i, j = 1, ..., n. \tag{16}$$

Moreover, in the new coordinates

$$\Psi_* \Xi_j = \frac{\partial}{\partial X_j}. \tag{17}$$

Proof. See Appendix A.

This lemma is an analogue of Theorem 2.36 from [8], which proves the local validity of the result around a fixed point on a manifold. However, the proof is not easy to extend for the generic case, since it relies on shifting the point under the flows of the vector fields Ξ_j, $j = 1, ..., n$. The coordinate transformation can, under commutativity assumption, be constructed as the composition of these flows. Moreover, the approach in this paper is algebraic, not differential geometric. For the above reasons, the alternative proof is given.

The problem statement. System (1) is said to be (generically) linearizable by state transformation (briefly, state linearizable), if there exists an analytic state transformation $X = \Psi(x)$, $\Psi_i \in \mathcal{K}$, $i = 1, ..., n$, such that the system in the new coordinates takes the form

$$X^{\langle 1 \rangle} = AX + Bu, \tag{18}$$

where the pair (A, B) satisfies the Kalman controllability condition. The main goal of the paper is to derive the necessary and sufficient conditions for the existence of the state transformation (14) that allows to transform the equations (1) into the form (18).

3. BACKWARD SHIFTS OF THE VECTOR FIELDS

In this section we prove several properties of the backward shifts of the vector fields $\partial/\partial u_i$, $i = 1, ..., m$, necessary for proving the main result. Note that, in general, the vector fields Ξ defined by (6) belong to the infinite dimensional space $\mathscr{E}^*$. However, in this paper we operate with the vector fields $\partial/\partial u_i$, $i = 1, ..., m$, and their backward shifts up to order $k^\star + 1$. Below we will show that such vector fields belong to the finite space $\overline{\mathscr{E}}^* = \operatorname{span}_{\mathscr{K}}\{\partial/\partial x, \partial/\partial u, \partial/\partial z^{\langle -l \rangle}, l = 1, ..., k^\star + 1\}$. Although the backward shifts of $\partial/\partial u_i$ for $l \leq k^\star + 1$, used in the following theorems, belong to the lower dimensional subspaces of $\overline{\mathscr{E}}^*$, we consider them as the elements of $\overline{\mathscr{E}}^*$.

Lemma 4. *The backward shifts of the vector field $\partial/\partial u_i$ have, for $l = 1, ..., k^\star + 1$, the following properties:*
(i)

$$\left(\frac{\partial}{\partial u_i}\right)^{\langle -l \rangle} \in \operatorname{span}_{\mathscr{K}}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z^{\langle -1 \rangle}}, ... \frac{\partial}{\partial z^{\langle -l \rangle}}\right\},$$

(ii) their coefficients depend only on the variables $x, z^{\langle -1 \rangle}, ..., z^{\langle -l \rangle}$.

Proof. See Appendix B.

Lemma 5. *For $i, j = 1, ..., m$ and $l, q = 1, ..., k^\star + 1$,*

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -l \rangle}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -q \rangle}\right] \equiv 0. \tag{19}$$

Proof. The proof is by induction. Since the partial derivative operators always commute, (19) holds for $l, q = 0$. Suppose that (19) is valid for $l, q = 0, ..., k$, and let us prove its validity for $k + 1$. First, shift (19) backwards by one step, using (13)

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -l-1 \rangle}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -q-1 \rangle}\right] \equiv 0, \quad l, q = 0, ..., k.$$

Using notations $\bar{l} := l + 1$, $\bar{q} := q + 1$ gives

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -\bar{l} \rangle}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -\bar{q} \rangle}\right] \equiv 0, \quad \bar{l}, \bar{q} = 1, ..., k+1. \tag{20}$$

From Lemma 4, the coefficients of $(\partial/\partial u_i)^{\langle -l \rangle}$, $l = 1, ..., k^\star + 1$, do not depend on u, and therefore

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -\bar{l} \rangle}, \frac{\partial}{\partial u_j}\right] \equiv 0, \quad \bar{l} = 1, ..., k+1. \tag{21}$$

From (20) and (21) taking into account also the commutativity of all $\partial/\partial u_i$, $i = 1, ..., m$, we get

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -\bar{l} \rangle}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -\bar{q} \rangle}\right] \equiv 0, \quad \bar{l}, \bar{q} = 0, ..., k+1.$$

$\square$

Observe that due to Lemma 4,

$$\left(\frac{\partial}{\partial u_i}\right)^{\langle -l\rangle} = a_{il}(x, \mathbf{z}_{-l})\frac{\partial}{\partial x} + c_{il}(x, \mathbf{z}_{-l})\frac{\partial}{\partial \mathbf{z}_{-l}}, \quad l = 1, ..., k^\star + 1, \tag{22}$$

where $\mathbf{z}_{-l} := \{z^{\langle -1\rangle}, ..., z^{\langle -l\rangle}\}$. The projections of $(\partial/\partial u_i)^{\langle -l\rangle}$ read, according to (9) as

$$\left(\frac{\partial}{\partial u_i}\right)^{\langle -l\rangle \pi} = a_{il}(x, \mathbf{z}_{-l})\frac{\partial}{\partial x}. \tag{23}$$

Lemma 6. *If the coefficients of $(\partial/\partial u_i)^{\langle -l\rangle \pi}$ and the coefficients of $(\partial/\partial u_j)^{\langle -q\rangle \pi}$, $i, j = 1, ..., m$ and $l, q = 1, ..., k^\star + 1$, depend only on x, then*

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -l\rangle \pi}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -q\rangle \pi}\right] \equiv 0. \tag{24}$$

Proof. According to (22) and the definition of the Lie brackets

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -l\rangle}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -q\rangle}\right]$$

$$= \left(\frac{\partial a_{jq}}{\partial x}a_{il} - \frac{\partial a_{il}}{\partial x}a_{jq} + \frac{\partial a_{jq}}{\partial \mathbf{z}_{-p}}c_{il} - \frac{\partial a_{il}}{\partial \mathbf{z}_{-p}}c_{jq}\right)\frac{\partial}{\partial x} + \left(\frac{\partial c_{jq}}{\partial x}a_{il} - \frac{\partial c_{il}}{\partial x}a_{jq} + \frac{\partial c_{jq}}{\partial \mathbf{z}_{-p}}c_{il} - \frac{\partial c_{il}}{\partial \mathbf{z}_{-p}}c_{jq}\right)\frac{\partial}{\partial \mathbf{z}_{-p}},$$

where $p = \max\{l, q\}$. Due to Lemma 5, the left hand side of the above equality is identically equal to zero. Therefore, both expressions in the parentheses on the right hand side of the equality must be also identically equal to zero and thus:

$$\frac{\partial a_{jq}}{\partial x}a_{il} - \frac{\partial a_{il}}{\partial x}a_{jq} + \frac{\partial a_{jq}}{\partial \mathbf{z}_{-p}}c_{il} - \frac{\partial a_{il}}{\partial \mathbf{z}_{-p}}c_{jq} \equiv 0. \tag{25}$$

By the assumption of the lemma, the quantities a_{il} and a_{jq} as the coefficients of the vector fields $(\partial/\partial u_i)^{\langle -l\rangle}$ and $(\partial/\partial u_j)^{\langle -q\rangle}$, respectively, depend only on x, and so the equality (25) takes the form

$$\frac{\partial a_{jq}}{\partial x}a_{il} - \frac{\partial a_{il}}{\partial x}a_{jq} \equiv 0. \tag{26}$$

According to (23) and the definition of the Lie bracket, the quantities on the left hand side of (26) are the coefficients of the Lie brackets in (24). □

Lemma 7. *If the coefficients of $(\partial/\partial u_i)^{\langle -l_i\rangle \pi}$, $i = 1, ..., m$, $l_i = 1, ..., n_i + 1$, do not depend on $z^{\langle -1\rangle}$, then they are the functions of x only.*

Proof. According to Lemma 4 the coefficients of $(\partial/\partial u_i)^{\langle -l_i\rangle \pi}$ depend on the variables $x, z^{\langle -1\rangle}, ..., z^{\langle -l_i\rangle}$ only, see (23). Consequently, we have to show that if these coefficients do not depend on $z^{\langle -1\rangle}$, then they do not depend on the higher order backward shifts of z either. The proof is by induction. Due to Lemma 4 the coefficients of $(\partial/\partial u_i)^{\langle -1\rangle \pi}$ are the functions of x and $z^{\langle -1\rangle}$ only, i.e. in case $l_i = 1$, the lemma holds.

Assume now that $l_i = k$. Suppose that the coefficients of $(\partial/\partial u_i)^{\langle -k\rangle \pi}$ depend only on x, i.e.

$$\left(\frac{\partial}{\partial u_i}\right)^{\langle -k\rangle \pi} = \sum_{j=1}^{n} a_{jik}(x)\frac{\partial}{\partial x_j}.$$

Show that in this case the coefficients of $(\partial/\partial u_i)^{\langle -k-1\rangle \pi}$ depend only on x and $z^{\langle -1\rangle}$. According to (7), (8) and (9),

$$\left(\frac{\partial}{\partial u_i}\right)^{\langle -k-1\rangle \pi} = \sum_{p=1}^{n} \left\langle \mathrm{d}\overline{\Phi}_p, \left(\frac{\partial}{\partial u_i}\right)^{\langle -k\rangle \pi} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_p} = \sum_{j,p=1}^{n} \left(\frac{\partial \overline{\Phi}}{\partial x_j} a_{jik}\right)^{\langle -1\rangle} \frac{\partial}{\partial x_p}.$$

Note that the product in the parentheses above depends only on x and u. Therefore, due to (3), its backward shift depends only on x and $z^{\langle -1\rangle}$. It means that if the coefficients of $(\partial/\partial u_i)^{\langle -k-1\rangle \pi}$ do not depend on $z^{\langle -1\rangle}$, they cannot depend on the higher order backward shifts of z either, i.e. they are the functions of x only. $\square$

Corollary 8. *If the coefficients of $(\partial/\partial u_i)^{\langle -l\rangle \pi}$ and $(\partial/\partial u_j)^{\langle -q\rangle \pi}$, $i,j = 1,...,m$, $l = 1,...,n_i$, $q = 1,...,n_j$, do not depend on $z^{\langle -1\rangle}$, then (24) holds.*

Proof. Follows directly from Lemma 6 and Lemma 7. $\square$

4. STATE LINEARIZABILITY

We formulate now the main result of the paper.

Theorem 9. *System (1) is generically state linearizable if and only if*
(a) there exist n vector fields $(\partial/\partial u_i)^{\langle -l_i\rangle}$, $i = 1,...,m$, $l_i = 1,...,n_i$, $\sum_{i=1}^{m} n_i = n$, such that (12) holds,
(b) the coefficients of the vector fields $(\partial/\partial u_i)^{\langle -l_i\rangle \pi}$, $l_i = 1,...,n_i+1$, depend only on the variable x.

Proof. Necessity. Suppose that (1) is state linearizable and compute from (18), taking into account that $A^0 = I$,

$$\left(\frac{\partial}{\partial u}\right)^{\langle -l\rangle \pi} = A^{l-1} B \frac{\partial}{\partial X}, \quad l = 1,...,k^\star+1, \tag{27}$$

where $k^\star = \max\{n_i\}$. For such vector fields the condition (b) obviously holds. According to Lemma 1 and Remark 2, the condition (a) is the generic accessibility condition and since the linearized system (18) satisfies the Kalman controllability condition, i.e is accessible, the condition (a) has to be satisfied for system (1) since the state transformation does not change the accessibility property of the system.

Sufficiency. In the sufficiency part of the proof we assume that $i,j = 1,...,m$. We first construct, under the conditions (a) and (b), a state transformation and then show that in the new coordinates the system equations are linear. Then, from condition (a) follows that the linear system satisfies the Kalman controllability condition.

Due to Lemma 4 and Corollary 8, the condition (b) is equivalent to

$$\left[\frac{\partial}{\partial z^{\langle -1\rangle}}, \left(\frac{\partial}{\partial u_i}\right)^{\langle -l_i\rangle \pi}\right] \equiv 0, \quad l_i = 1,...,n_i+1. \tag{28}$$

Now, by Lemma 6, for $l_i = 1,...,n_i+1$, $l_j = 1,...,n_j+1$,

$$\left[\left(\frac{\partial}{\partial u_i}\right)^{\langle -l_i\rangle \pi}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle \pi}\right] \equiv 0. \tag{29}$$

By (12) and Lemma 4, the projections of the backward shifts of $\partial/\partial u_i$ up to order n_i build the (commutative) basis for $\mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$, i.e.

$$\mathrm{span}_{\mathscr{K}}\left\{\left(\frac{\partial}{\partial u_i}\right)^{\langle -l_i\rangle\pi}, i=1,...,m, l_i=1,...,n_i\right\} = \mathrm{span}_{\mathscr{K}}\left\{\frac{\partial}{\partial x}\right\}, \tag{30}$$

and so the projections of the higher order backward shifts of $\partial/\partial u_i$ are their linear combinations

$$\left(\frac{\partial}{\partial u_i}\right)^{\langle -n_i-1\rangle\pi} = \sum_{j=1}^{m}\sum_{l_j=1}^{n_j} \tilde{a}_{ijl_j}(x)\left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle\pi}. \tag{31}$$

We next prove that all coefficients $\tilde{a}_{ijl_j}(x)$ are constants. Note that, due to (28), the coefficients $\tilde{a}_{ijl_j}$ do not depend on $\mathbf{z}$. Let us show that they do not depend on x either. From (29) it is immediate that the vector field (31) must commute with all $(\partial/\partial u_k)^{\langle -l_k\rangle\pi}$, $k=1,...,m$, $l_k=1,...,n_k$, i.e.

$$\sum_{j=1}^{m}\sum_{l_j=1}^{n_j}\left(\left\langle \mathrm{d}\tilde{a}_{ijl_j},\left(\frac{\partial}{\partial u_k}\right)^{\langle -l_k\rangle\pi}\right\rangle\left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle\pi} +\tilde{a}_{ijl_j}\left[\left(\frac{\partial}{\partial u_k}\right)^{\langle -l_k\rangle\pi},\left(\frac{\partial}{\partial u_k}\right)^{\langle -n_k\rangle\pi}\right]\right) \equiv 0.$$

By (29), the above equality holds only if for $k=1,...,m$, $l_k=1,...,n_k$,

$$\left\langle \mathrm{d}\tilde{a}_{ijl_j},\left(\frac{\partial}{\partial u_k}\right)^{\langle -l_k\rangle\pi}\right\rangle \equiv 0. \tag{32}$$

Given (30), the vector fields $\partial/\partial x_q$, $q=1,...,n$, can be written as linear combinations of $(\partial/\partial u_i)^{\langle -l_i\rangle\pi}$, $i=1,...,m$, $l_i=1,...,n_i$. Then from (32) really follows that $\partial\tilde{a}_{ijl_j}/\partial x_q \equiv 0$. Consequently, all $\tilde{a}_{ijl_j}$ are constants.

We will construct now the state transformation $X=\Psi(x)$. Since (29) and (30) hold, then according to Lemma 3, one can define a coordinate transformation $X_{i,n_i-l_i+1}=\Psi_{i,n_i-l_i+1}(x)$, $l_i=1,...,n_i$, such that

$$\left\langle \mathrm{d}\Psi_{i,n_i-l_i+1},\left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle\pi}\right\rangle \equiv \delta_{ij}\delta_{l_i l_j}, \tag{33}$$

where the vector fields $(\partial/\partial u_j)^{\langle -l_j\rangle\pi}$, $l_j=1,...,n_j$, will be transformed into the partial derivative operators

$$\Psi_*\left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle\pi} = \frac{\partial}{\partial X_{j,n_j-l_j+1}}. \tag{34}$$

Note that here we define the components of the coordinate transformation $\Psi(x)$ (and the new coordinates X) by using two indices, unlike in Lemma 3. The reason is that Ψ_{i,n_i-l_i+1}, $i=1,...,m$, $l_i=1,...,n_i$, in (33) are defined as the canonical parameters of the vector fields $(\partial/\partial u_i)^{\langle -l_i\rangle\pi}$. The first index i points to the component of the input u, i.e. u_i, and the second index n_i-l_i+1 refers to the canonical parameter of the projection of the l_i-th backward shift of $\partial/\partial u_i$.

It remains to prove that in the new coordinates the state equations have the form (18), meaning that the forward shift of the vector variable X is the *linear* vector function of the variables X and u. The task is, therefore, to show that for $l_i=1,...,n_i$, $l_j=1,...,n_j$,

$$\left\langle \mathrm{d}X_{i,n_i-l_i+1}^{\langle 1\rangle},\frac{\partial}{\partial X_{j,n_j-l_j+1}}\right\rangle \equiv \mathrm{const}, \quad \left\langle \mathrm{d}X_{i,n_i-l_i+1}^{\langle 1\rangle},\frac{\partial}{\partial u_j}\right\rangle \equiv \mathrm{const}. \tag{35}$$

Since the constants are invariant under the coordinate transformations, the identities (35) are likewise valid in the original coordinates and, taking into account also (34), we get, for $l_i = 1,...,n_i$, $l_j = 1,...,n_j$, the system of equations

$$\left\langle d\Psi^{\langle 1\rangle}_{i,n_i-l_i+1}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle \pi}\right\rangle \equiv \text{const}, \quad \left\langle d\Psi^{\langle 1\rangle}_{i,n_i-l_i+1}, \frac{\partial}{\partial u_j}\right\rangle \equiv \text{const}. \tag{36}$$

In the first equality of (36) one can replace the projections of $(\partial/\partial u_j)^{\langle -l_j\rangle}$ by the vector fields themselves, since the functions $\Psi^{\langle 1\rangle}_{i,n_i-l_i+1}$ depend only on x. This allows (36) to be rewritten, for $l_i = 1,...,n_i$, $l_j = 0,...,n_j$, as a single equality

$$\left\langle d\Psi^{\langle 1\rangle}_{i,n_i-l_i+1}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j\rangle}\right\rangle \equiv \text{const}. \tag{37}$$

In what follows we will show the validity of (37). Let us prove it first for $l_i = 1,...,n_i$ and $l_j = 0,...,n_j - 1$. Since all Ψ_{i,n_i-l_i+1} depend only on x, one can replace in (33) $(\partial/\partial u_j)^{\langle -l_j\rangle \pi}$ by $(\partial/\partial u_j)^{\langle -l_j\rangle}$. Shifting (33) forward and taking the above observation into account gives, for $l_i = 1,...,n_i$, $l_j = 1,...,n_j$,

$$\left\langle d\Psi^{\langle 1\rangle}_{i,n_i-l_i+1}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -l_j+1\rangle}\right\rangle \equiv \delta_{ij}\delta_{l_i l_j}. \tag{38}$$

If we use the notation $\bar{l}_j := l_j - 1$ in (38), it is obvious that (37) holds for the values of $l_j = 0,...,n_j - 1$. Next prove the validity of (37) for $l_j = n_j$. From (31), (33) and the fact that Ψ_{i,n_i-l_i+1} depends only on the variable x, one gets

$$\left\langle d\Psi_{i,n_i-l_i+1}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -n_j-1\rangle}\right\rangle = \sum_{q=1}^{m}\sum_{l_q=1}^{n_q} \tilde{a}_{jql_q}\left\langle d\Psi_{i,n_i-l_i+1}, \left(\frac{\partial}{\partial u_q}\right)^{\langle -l_q\rangle \pi}\right\rangle \equiv \tilde{a}_{jil_i}. \tag{39}$$

Recall that $\tilde{a}_{jil_i}$ is constant. Shifting next (39) forward by one step and taking into account that the forward shift operator does not change a constant, we get

$$\left\langle d\Psi^{\langle 1\rangle}_{i,n_i-l_i+1}, \left(\frac{\partial}{\partial u_j}\right)^{\langle -n_j\rangle}\right\rangle \equiv \tilde{a}_{jil_i}.$$

It means that the condition (37) holds for $l_i = 1,...,n_i$, $l_j = 0,...,n_j$. $\qquad\qquad\square$

Below we will show how to find the functions $\Psi_{i,n_i-l_i+1}(x)$ in (33). By construction, the scalar products (33) are elements of the unit matrix I_n, being a product of two $(n \times n)$-matrices: $M_1 M_2 = I_n$, where

$$M_1 = \begin{bmatrix} dX_{1,n_1} \\ \vdots \\ dX_{1,1} \\ \vdots \\ dX_{m,n_m} \\ \vdots \\ dX_{m,1} \end{bmatrix}$$

and

$$M_2 = \left[\left(\frac{\partial}{\partial u_1}\right)^{\langle -1\rangle \pi} \cdots \left(\frac{\partial}{\partial u_1}\right)^{\langle -n_1\rangle \pi} \cdots \left(\frac{\partial}{\partial u_m}\right)^{\langle -1\rangle \pi} \cdots \left(\frac{\partial}{\partial u_m}\right)^{\langle -n_m\rangle \pi} \right].$$

Since $M_1 = M_2^{-1}$, we obtain the following rule to compute the new coordinates $X = \Psi(x)$:

1. Compute M_2^{-1}, to get from its rows the differentials of the components of dX.
2. Integrate them to get the coordinate transformation.

Example 10. Consider the non-reversible state equations

$$x_1^{\langle 1\rangle} = x_2 u_1 u_2, \quad x_2^{\langle 1\rangle} = x_1/x_5, \quad x_3^{\langle 1\rangle} = x_2, \quad x_4^{\langle 1\rangle} = x_5 u_1^2/x_3, \quad x_5^{\langle 1\rangle} = x_2 u_2. \tag{40}$$

Check first the linearizability conditions from Theorem 9. Observe that the choice $z_1 = x_1$, $z_2 = x_4$ results in the backward shift equations

$$x_1^{\langle -1\rangle} = z_1^{\langle -1\rangle}, \quad x_2^{\langle -1\rangle} = x_3, \quad x_3^{\langle -1\rangle} = \frac{x_1^2 z_1^{\langle -1\rangle}}{x_2 x_4 x_5^2}, \quad x_4^{\langle -1\rangle} = z_2^{\langle -1\rangle}, \quad x_5^{\langle -1\rangle} = \frac{z_1^{\langle -1\rangle}}{x_2}, \quad u_1^{\langle -1\rangle} = \frac{x_1}{x_5}, \quad u_2^{\langle -1\rangle} = \frac{x_5}{x_3}.$$

Compute the differentials of the forward shifts of the state coordinates:

$$dx_1^{\langle 1\rangle} = u_1 u_2 dx_2 + u_2 x_2 du_1 + u_1 x_2 du_2, \quad dx_2^{\langle 1\rangle} = \frac{1}{x_5} dx_1 - \frac{x_1}{x_5^2} dx_5, \quad dx_3^{\langle 1\rangle} = dx_2,$$

$$dx_4^{\langle 1\rangle} = -\frac{u_1^2 x_5}{x_3^2} dx_3 + \frac{u_1^2}{x_3} dx_5 + \frac{2 u_1 x_5}{x_3} du_1, \quad dx_5^{\langle 1\rangle} = u_2 dx_2 + x_2 du_2.$$

Next compute recursively, according to (7)–(9), the projections of the backward shifts of $\partial/\partial u_i$, $i = 1, 2$. For $i = 1$ we get

$$\left(\frac{\partial}{\partial u_1}\right)^{\langle -1\rangle \pi} = \sum_{i=1}^n \left\langle dx_i^{\langle 1\rangle}, \frac{\partial}{\partial u_1} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_i} = (u_2 x_2)^{\langle -1\rangle} \frac{\partial}{\partial x_1} + \left(\frac{2 u_1 x_5}{x_3}\right)^{\langle -1\rangle} \frac{\partial}{\partial x_4} = x_5 \frac{\partial}{\partial x_1} + \frac{2 x_4 x_5}{x_1} \frac{\partial}{\partial x_4},$$

$$\left(\frac{\partial}{\partial u_1}\right)^{\langle -2\rangle \pi} = \sum_{i=1}^n \left\langle dx_i^{\langle 1\rangle}, \left(\frac{\partial}{\partial u_1}\right)^{\langle -1\rangle} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_i} = \sum_{i=1}^n \left\langle dx_i^{\langle 1\rangle}, x_5 \frac{\partial}{\partial x_1} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_2},$$

$$\left(\frac{\partial}{\partial u_1}\right)^{\langle -3\rangle \pi} = \sum_{i=1}^n \left\langle dx_i^{\langle 1\rangle}, \left(\frac{\partial}{\partial u_1}\right)^{\langle -2\rangle} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_i} = \sum_{i=1}^n \left\langle dx_i^{\langle 1\rangle}, \frac{\partial}{\partial x_2} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_i}$$

$$= (u_1 u_2)^{\langle -1\rangle} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + u_2^{\langle -1\rangle} \frac{\partial}{\partial x_5} = \frac{x_1}{x_3} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \frac{x_5}{x_3} \frac{\partial}{\partial x_5}.$$

Analogously, we find

$$\left(\frac{\partial}{\partial u_2}\right)^{\langle -1\rangle \pi} = \frac{x_1 x_3}{x_5} \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_5}, \quad \left(\frac{\partial}{\partial u_2}\right)^{\langle -2\rangle \pi} = \frac{x_1^2}{x_5^2} \frac{\partial}{\partial x_4}, \quad \left(\frac{\partial}{\partial u_2}\right)^{\langle -3\rangle \pi} = 0.$$

Observe, that the non-zero vector fields $(\partial/\partial u_i)^{\langle -l\rangle}$, $i = 1, 2$, $l = 1, 2, 3$, define the basis vector fields of $\text{span}_{\mathscr{K}}\{\partial/\partial x\}$ and their coefficients depend only on the variable x. Then, according to Theorem 9, the state

equations (40) are state linearizable. To find the state transformation, construct first

$$M_2 = \left[\left(\frac{\partial}{\partial u_1} \right)^{\langle -1 \rangle \pi} \left(\frac{\partial}{\partial u_2} \right)^{\langle -1 \rangle \pi} \left(\frac{\partial}{\partial u_1} \right)^{\langle -2 \rangle \pi} \left(\frac{\partial}{\partial u_2} \right)^{\langle -2 \rangle \pi} \left(\frac{\partial}{\partial u_1} \right)^{\langle -3 \rangle \pi} \right]$$

$$= \begin{pmatrix} x_5 & x_1 x_3 / x_5 & 0 & 0 & x_1/x_3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2x_4 x_5/x_1 & 0 & 0 & x_1^2/x_5^2 & 0 \\ 0 & x_3 & 0 & 0 & x_5/x_3 \end{pmatrix}.$$

Compute next $M_1 = M_2^{-1}$:

$$M_1 = \begin{bmatrix} \mathrm{d}X_{1,3} \\ \mathrm{d}X_{2,2} \\ \mathrm{d}X_{1,2} \\ \mathrm{d}X_{2,1} \\ \mathrm{d}X_{1,1} \end{bmatrix} = \begin{pmatrix} \dfrac{1}{x_5} & 0 & 0 & 0 & -\dfrac{x_1}{x_5^2} \\ 0 & 0 & -\dfrac{x_5}{x_3^2} & 0 & \dfrac{1}{x_3} \\ 0 & 1 & 0 & 0 & 0 \\ -\dfrac{2x_4 x_5^2}{x_1^3} & 0 & 0 & \dfrac{x_5^2}{x_1^2} & \dfrac{2x_4 x_5}{x_1^2} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

In order to define the new coordinates (33), integrate the 1-forms $\mathrm{d}X_{i,l}$, given in the form of the row vectors as the rows of matrix M_1. This leads to the state transformation $X_{1,1} = x_3$, $X_{1,2} = x_2$, $X_{1,3} = x_1/x_5$, $X_{2,1} = (x_4 x_5^2)/x_1^2$, $X_{2,2} = x_5/x_3$. The state equations read in the new coordinates $X_{1,1}^{\langle 1 \rangle} = X_{1,2}$, $X_{1,2}^{\langle 1 \rangle} = X_{1,3}$, $X_{1,3}^{\langle 1 \rangle} = u_1$, $X_{2,1}^{\langle 1 \rangle} = X_{2,2}$, $X_{2,2}^{\langle 1 \rangle} = u_2$.

5. COMPARISON WITH THE FEEDBACK LINEARIZABILITY CONDITIONS

We compare the conditions of Theorem 9 with those for static state feedback linearizability from [1], recalled in Proposition 11 below. Note that the paper [1] addresses the problem, using the same tools that are applied in this paper. The goal of the comparison is to show how *exactly* feedback allows to relax the state linearizability conditions.

A regular *static state feedback* is an analytic mapping $u = \alpha(x,v)$, $v \in \mathbb{R}^m$, such that $\mathrm{rank}_{\mathcal{K}} (\partial \alpha/\partial v) = m$. System (1) is said to be *(generically) linearizable by a regular static state feedback* if there exists a state transformation $X = \Psi(x)$ and a regular static state feedback such that the system in the new coordinates takes the Brunovsky form

$$X_{i,1}^{\langle 1 \rangle} = X_{i,2}, \quad \ldots, \quad X_{i,n_i-1}^{\langle 1 \rangle} = X_{i,n_i}, \quad X_{i,n_i}^{\langle 1 \rangle} = v_i, \quad i = 1, \ldots, m. \tag{41}$$

Note that using the Brunovsky form (41) does not lead to the loss of generality compared with the linear accessible system (18) in Theorem 9 because the equations (18) are always transformable into the form (41) by state transformation and regular static state feedback, see e.g. [9].

Recall that system (1) is (generically) static state feedback linearizable if and only if (i) $\mathscr{D}_k$, $k = 1, ..., k^\star$, are involutive and (ii) $\dim_{\mathscr{K}} \mathscr{D}_{k^\star}^\pi = n$ [1]. For easier comparison, rewrite this result as in the proposition below.

Proposition 11. *System (1) is (generically) static state feedback linearizable if and only if*
(a') $\mathscr{D}_k^\pi$, $k = 1, ..., k^\star$, *are involutive,*
(b') $\mathscr{D}_k^\pi$, $k = 1, ..., k^\star$, *are invariant under taking the Lie brackets with* $\partial / \partial z_i^{\langle -1 \rangle}$, *i.e. for* $i, j = 1, ..., m$ *and* $l_j = 1, ..., k$,

$$\left[\frac{\partial}{\partial z_i^{\langle -1 \rangle}}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] \in \mathscr{D}_k^\pi \tag{42}$$

and (c') $\dim_{\mathscr{K}} \mathscr{D}_{k^\star}^\pi = n$.

Proof. One has to show that involutivity of $\mathscr{D}_k$, $k = 1, ..., k^\star$, is equivalent to (a') and (b'). According to (10), the involutivity conditions of $\mathscr{D}_k$ can be rewritten for $i, j = 1, ..., m$ and $l_i, l_j = 1, ..., k$ as:

$$\left[\left(\frac{\partial}{\partial u_i} \right)^{\langle -l_i \rangle \pi}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] \in \mathscr{D}_k, \tag{43}$$

$$\left[\frac{\partial}{\partial z_i^{\langle -1 \rangle}}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] \in \mathscr{D}_k. \tag{44}$$

We will show first that $\mathscr{D}_k^\pi$ is involutive if and only if (43) holds. Due to (11), $\mathscr{D}_k^\pi$ is involutive if and only if

$$\left[\left(\frac{\partial}{\partial u_i} \right)^{\langle -l_i \rangle \pi}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] \in \mathscr{D}_k^\pi, \tag{45}$$

and therefore, we need to prove that (43) is equivalent to (45). Observe that the projections $(\partial / \partial u_i)^{\langle -l_i \rangle \pi} \in \text{span}_{\mathscr{K}} \{ \partial / \partial x \}$ and thus also their Lie brackets belong to this vector space. Consequently, from (43) follows (45). The inverse is obvious because $\mathscr{D}_k^\pi \subset \mathscr{D}_k$.

Next we will show that (42) is equivalent to (44). Since $(\partial / \partial u_j)^{\langle -l_j \rangle \pi} \in \text{span}_{\mathscr{K}} \{ \partial / \partial x \}$, it can be written as $\sum_{q=1}^n \alpha_{qjl_j} \partial / \partial x_l$. Then

$$\left[\frac{\partial}{\partial z_i^{\langle -1 \rangle}}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] = \sum_{q=1}^n \frac{\partial \alpha_{qjl_j}}{\partial z_i^{\langle -1 \rangle}} \frac{\partial}{\partial x_q} \in \text{span}_{\mathscr{K}} \left\{ \frac{\partial}{\partial x} \right\}.$$

Therefore, if (44) is satisfied, (42) also holds. The inverse is obvious. □

Compare now the conditions of state linearizability given in Theorem 9 and those from Proposition 11. We suppose that the static state feedback linearizability conditions from Proposition 11 hold and examine which additional restrictions are necessary for system (1) to be state linearizable. First, due to Remark 2, the condition (a) of Theorem 9 is equivalent to the condition (c') of Proposition 11.

Though the conditions of Theorem 9 are easy to check (no need to compute the Lie brackets of vector fields), for comparison with the static state feedback linearizability conditions it is important to recall from the proof of the theorem that the condition (b) is equivalent to the condition (28), and the latter implies, by Lemma 6, the condition (29).

Compare now the condition (b') of Proposition 11 with (28). Suppose that (b'), given by (42), holds and show which additional restrictions are required when the feedback is not allowed. From (b') follows

that the distributions $\mathscr{D}_k^\pi$, $k = 1, ..., k^\star$, are invariant under taking the Lie brackets with respect to $\partial/\partial z^{\langle -1 \rangle}$ (recall that $k^\star = \max\{n_i\}$). Since by (9) the vector fields $(\partial/\partial u_i)^{\langle -l_i \rangle}$, $i = 1, ..., m$, $l_i = 1, ..., n_i + 1$, are the basis vector fields of $\mathscr{D}_k^\pi$, $k = 1, ..., k^\star + 1$, the corresponding state linearizability condition (28) has two supplementary restrictions. First, it requires that the above condition holds also for $k = k^\star + 1$. Second, (28) requires that the Lie brackets of the vector fields $(\partial/\partial u_i)^{\langle -l_i \rangle}$ with $\partial/\partial z^{\langle -1 \rangle}$ must be identically equal to zero, i.e. the coefficients of $(\partial/\partial u_i)^{\langle -l_i \rangle}$ depend only on x, which is not the case for (b′).

Next consider the condition (a′) from Proposition 11, given by (45) and requiring that the distributions $\mathscr{D}_k^\pi$, $k = 1, ..., k^\star$, must be involutive. The corresponding condition in the case of a state linearizable system is (29), which is also more restrictive than (45), having two additional requirements. First, it also holds again for $k = k^\star + 1$ and second, it requires not only the involutivity of the distributions $\mathscr{D}_k^\pi$, $k = 1, ..., k^\star + 1$, but also the commutativity of all the basis vector fields $(\partial/\partial u_i)^{\langle -l_i \rangle \pi}$, $i = 1, ..., m$, $l_i = 1, ..., n_i + 1$, of these distributions. However, recall that, due to Lemma 6, if (28) holds, then (29) is automatically satisfied. For a static state feedback linearizable system the analogous condition (45) does not follow from (42) and must be checked separately. Nevertheless, if (28) holds, (45) is also satisfied.

To resume, if we do not allow the feedback, then two additional restrictions are required: for $i, j = 1, ..., m$, $l_i = 1, ..., n_i + 1$, $l_j = 1, ..., n_j + 1$,

$$(1) \quad \left[\left(\frac{\partial}{\partial u_i} \right)^{\langle -l_i \rangle \pi}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] \equiv 0, \tag{46}$$

and

$$(2) \quad \left[\frac{\partial}{\partial z_i^{\langle -1 \rangle}}, \left(\frac{\partial}{\partial u_j} \right)^{\langle -l_j \rangle \pi} \right] \equiv 0. \tag{47}$$

Example 12. Consider the dynamical system

$$x_1^{\langle 1 \rangle} = \frac{x_2}{x_1}, \quad x_2^{\langle 1 \rangle} = \frac{u_1 x_2}{x_1}, \quad x_3^{\langle 1 \rangle} = \frac{u_2 x_2 x_3}{x_1^2}.$$

Defining $z_1 = x_1$, $z_2 = u_2$, we get the extended state equations, resulting in the backward shift equations

$$x_1^{\langle -1 \rangle} = z_1^{\langle -1 \rangle}, \quad x_2^{\langle -1 \rangle} = z_1^{\langle -1 \rangle} x_1, \quad x_3^{\langle -1 \rangle} = \frac{z_1^{\langle -1 \rangle} x_3}{x_1 z_2^{\langle -1 \rangle}}.$$

Compute the differentials of the forward shifts of the state coordinates

$$
\begin{aligned}
dx_1^{\langle 1 \rangle} &= -\frac{x_2}{x_1^2} dx_1 + \frac{1}{x_1} dx_2, \\
dx_2^{\langle 1 \rangle} &= -\frac{u_1 x_2}{x_1^2} dx_1 + \frac{u_1}{x_1} dx_2 + \frac{x_2}{x_1} du_1, \\
dx_3^{\langle 1 \rangle} &= -\frac{2 u_2 x_2 x_3}{x_1^3} dx_1 + \frac{u_2 x_3}{x_1^2} dx_2 + \frac{u_2 x_2}{x_1^2} dx_3 + \frac{x_2 x_3}{x_1^2} du_2,
\end{aligned}
\tag{48}
$$

and the projections of the backward shifts of the vector fields $\partial/\partial u_i$ according to (7), (8) and (9), taking into account also (48):

$$
\left(\frac{\partial}{\partial u_1}\right)^{\langle-1\rangle\pi} = \sum_{i=1}^{3}\left\langle dx_i^{\langle1\rangle}, \frac{\partial}{\partial u_1}\right\rangle^{\langle-1\rangle} \frac{\partial}{\partial x_i} = \left(\frac{x_2}{x_1}\right)^{\langle-1\rangle}\frac{\partial}{\partial x_2} = x_1\frac{\partial}{\partial x_2},
$$

$$
\left(\frac{\partial}{\partial u_2}\right)^{\langle-1\rangle\pi} = \sum_{i=1}^{3}\left\langle dx_i^{\langle1\rangle}, \frac{\partial}{\partial u_2}\right\rangle^{\langle-1\rangle} \frac{\partial}{\partial x_i} = \left(\frac{x_2 x_3}{x_1^2}\right)^{\langle-1\rangle}\frac{\partial}{\partial x_3} = \frac{x_3}{z_2^{\langle-1\rangle}}\frac{\partial}{\partial x_3},
$$

$$
\left(\frac{\partial}{\partial u_1}\right)^{\langle-2\rangle\pi} = \sum_{i=1}^{3}\left\langle dx_i^{\langle1\rangle}, \left(\frac{\partial}{\partial u_1}\right)^{\langle-1\rangle}\right\rangle^{\langle-1\rangle} \frac{\partial}{\partial x_i}
$$

$$
= \frac{\partial}{\partial x_1} + u_1^{\langle-1\rangle}\frac{\partial}{\partial x_2} + \left(\frac{u_2 x_3}{x_1^2}\right)^{\langle-1\rangle}\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_1} + \frac{x_2}{x_1}\frac{\partial}{\partial x_2} + \frac{x_3}{x_1}\frac{\partial}{\partial x_3}.
$$

(49)

Since the vector fields (49) already span the entire space $\mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$, further computations do not yield additional linearly indepent vector fields. According to (11) we get the distributions

$$
\mathscr{D}_1^\pi = \mathrm{span}_{\mathscr{K}}\left\{x_1\frac{\partial}{\partial x_2}, \frac{x_3}{z_2^{\langle-1\rangle}}\frac{\partial}{\partial x_3}\right\},
$$

$$
\mathscr{D}_2^\pi = \mathrm{span}_{\mathscr{K}}\left\{x_1\frac{\partial}{\partial x_2}, \frac{x_3}{z_2^{\langle-1\rangle}}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} + \frac{x_2}{x_1}\frac{\partial}{\partial x_2} + \frac{x_3}{x_1}\frac{\partial}{\partial x_3}\right\}.
$$

(50)

Check first the static state feedback linearizability conditions from Proposition 11. One can easily see that the Lie brackets of the basis vector fields of $\mathscr{D}_1^\pi$ ($\mathscr{D}_2^\pi$) are either zero or belong to $\mathscr{D}_1^\pi$ ($\mathscr{D}_2^\pi$). Consequently, the condition (a′) holds for $\mathscr{D}_1^\pi$ and $\mathscr{D}_2^\pi$. Computing the Lie brackets of the basis vector fields of $\mathscr{D}_1^\pi$ and $\mathscr{D}_2^\pi$ with respect to $\partial/\partial z_1^{\langle-1\rangle}$ shows that (b′) is satisfied. Next, $\dim_{\mathscr{K}}\mathscr{D}_2^\pi = 3$, and therefore, (c′) is also true. The system is static state feedback linearizable. Note that the application of the state transformation $X_{1,1} = x_1$, $X_{1,2} = x_2/x_1$, $X_{2,1} = x_3$, and the state feedback $u_1 = v_1$, $u_2 = v_2 x_1^2/(x_2 x_3)$ results in the linear equations

$$
X_{1,1}^{\langle1\rangle} = X_{1,2}, \quad X_{1,2}^{\langle1\rangle} = v_1, \quad X_{2,1}^{\langle1\rangle} = v_2.
$$

Next check state linearizability. The condition (a) of Theorem 9 is satisfied with $n_1 = 2$, $n_2 = 1$. Although the distributions $\mathscr{D}_1^\pi$ and $\mathscr{D}_2^\pi$ are invariant under taking the Lie derivatives with respect to $\partial/\partial z^{\langle-1\rangle}$, for state linearizability the Lie brackets of the vector fields (49) with $\partial/\partial z^{\langle-1\rangle}$ must be identically equal to zero, see (47). However, for the vector field $(\partial/\partial u_2)^{\langle-1\rangle\pi}$, given by (49), this is obviously not the case. Therefore, the system is not state linearizable.

Finally, note that static state feedback linearization as an iconic research problem has been studied, using different approaches for different discrete-time nonlinear system classes. In [10] and [11] the problem is addressed by applying another algebraic approach, the so-called functions' algebra (FA), by working directly with functions, defined by system equations and not with the vector spaces of the 1-forms or vector fields like in [2] and [1], respectively. The solvability conditions based on FA were compared with those in [2] for analytic systems in [10]. Moreover, the conditions of [2] were compared with those recalled in this paper in [1]. Thus, by combining these two comparisons, one may conclude that at least for analytic systems addressed in this paper, the FA approach provides no advantages. For this reason it is probably not important to find the analogues of the additional conditions (46) and (47) in this paper by means of the FA approach, though, in principle, it is possible. In [12] it is shown how to use the vector spaces of differential 1-forms to formalize and simplify the computations in the smooth case. Of course, the FA approach allows to address also non-smooth systems, though the computations in the non-smooth case are difficult, see [12]. In a similar manner (but for continuous-time case) the distributions of vector fields have been used to assist the computations, see for instance [13]. It would be more interesting to compare the solvability conditions from [10] with the conditions when state feedback is not allowed within the FA formalism to address the non-smooth case. However, as shown in [12], computations in the non-smooth case are difficult to handle.

6. CONCLUSION

The present paper is devoted to the linearizability problem of discrete-time nonlinear control systems via state transformation. The necessary and sufficient conditions for the existence of such state transformation are formulated in terms of the backward shifts of the vector fields, defined by system dynamics, and the rule for finding the corresponding state transformation is given. Next, the relationship of these conditions with static state feedback linearizability conditions, which are also given in terms of backward shifts of vector fields, is examined.

Since our linearizability conditions are formulated in a form that allows, by direct inspection, to see how the conditions will be relaxed if also state feedback is allowed, we believe that the conditions will be helpful in finding the conditions under which one can transform the state equations directly into the classical or extended observer forms without finding the input-output equations, as carried out, for example, in [14]. It should be noted that such conditions are still missing in case the state equations are not state reversible. It is important to recall that such observer forms consist of linear equations up to nonlinear input/output injection terms, which depend only on inputs and outputs in the case of the classical observer form, but also on their backward shifts in the case of the extended observer form.

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REFERENCES

1. Mullari, T., Kotta, Ü., Bartosiewicz, Z., Pawluszewicz, E., and Moog, C. H. Forward and backward shifts of vector fields: towards the dual algebraic framework. *IEEE Trans. Automat. Contr.*, 2017, **62**(6), 3029–3033.
2. Aranda-Bricaire, E., Kotta, Ü., and Moog, C. H. Linearization of discrete-time systems. *SIAM J. Control Optim.*, 1996, **34**(6), 1999–2023.
3. Belikov, J., Kaldmäe, A., and Kotta, Ü. Global linearization approach to nonlinear control systems: a brief tutorial overview. *Proc. Estonian Acad. Sci.*, 2017, **66**(3), 243–263.
4. Lee, A. G., Arapostathis, A., and Marcus, S. I. Linearization of discrete-time systems. *Int. J. Control*, 1987, **45**(5), 1803–1822.
5. Grizzle, J. V. Feedback linearization of discrete-time systems. *Lect. Notes Control Inf. Sci.*, 1986, **83**, 273–281.
6. Nam, K. Linearization of discrete-time systems and a canonical structure. *IEEE Trans. Automat. Contr.*, 1989, **34**(1), 119–122.
7. Conte, G., Moog, C. H., and Perdon, A. M. *Nonlinear Control Systems: an Algebraic Setting.* Springer, London, 1999.
8. Nijmeijer, H. and van der Schaft, A. J. *Nonlinear Dynamical Control Systems,* Springer, New York, 1990.
9. Kailath, T. *Linear Systems,* Prentice-Hall, Inc., Upper Saddle River, 1980.
10. Kotta, Ü., Tõnso, M., Shumsky, A., and Zhirabok, A. Feedback linearization and lattice theory. *Syst. Control Lett.*, 2013, **62**(3), 248–255.
11. Kaldmäe, A., Kotta, Ü., Shumsky, A., and Zhirabok, A. Feedback linearization of possibly non-smooth systems. *Proc. Estonian Acad. Sci.*, 2017, **66**(2), 109–123.
12. Belikov, J., Kaldmäe, A., Kaparin, V., Shumsky, A., Kotta, Ü., Tõnso, M., and Zhirabok, A. Functions' algebra in nonlinear control: computational aspects and software. *Proc. Estonian Acad. Sci.*, 2017, **66**(1), 89–107.
13. Kotta, Ü. and Mullari, T. Discussion on: "Unified approach to the problem of full decoupling via output feedback". *Eur. J. Control*, 2010, **16**(4), 326–328.
14. Kaparin, V. and Kotta, Ü. Transformation of nonlinear MIMO discrete-time systems into the extended observer form. *Asian J. Control*, 2019, **21**(5), 2202–2217.

THE PROOF OF LEMMA 3 APPENDIX A

Define n linearly independent 1-forms $\omega_i = \sum_{l=1}^{n} \alpha_{il}(x)\mathrm{d}x_l$, $i = 1,...,n$, such that

$$\left\langle \omega_i, \Xi_j \right\rangle \equiv \delta_{ij}. \tag{51}$$

This is possible since the vector fields Ξ_j, $j = 1,...,n$, being linearly independent, can be understood as column vectors of a non-singular matrix. The rows of its inverse matrix define the 1-forms in (51). We will prove that under the commutativity assumption of Ξ_j, all ω_i are *exact*, i.e. there exist n independent functions $\Psi_i(x)$ such that $\omega_i = \mathrm{d}\Psi_i$, or equivalently, the exterior differentials $\mathrm{d}\omega_i \equiv 0$. Recall that $\mathrm{d}\omega_i$ is the 2-form

$$\mathrm{d}\omega_i = \sum_{j=1}^{l} \sum_{l=2}^{n} \left(\frac{\partial \alpha_{il}}{\partial x_j} - \frac{\partial \alpha_{ij}}{\partial x_l} \right) \mathrm{d}x_j \wedge \mathrm{d}x_l, \tag{52}$$

and so from $\mathrm{d}\omega_i \equiv 0$ we get

$$\frac{\partial \alpha_{il}}{\partial x_j} - \frac{\partial \alpha_{ij}}{\partial x_l} \equiv 0, \quad i,j,l = 1,...,n. \tag{53}$$

The scalar product (51) can be rewritten as

$$\sum_{l=1}^{n} \alpha_{il}(x)\xi_{lj}(x) \equiv \delta_{ij}. \tag{54}$$

Taking the partial derivative of this equality with respect to x_k, we obtain

$$\sum_{l=1}^{n} \frac{\partial \alpha_{il}}{\partial x_k} \xi_{lj} = -\sum_{l=1}^{n} \alpha_{il} \frac{\partial \xi_{lj}}{\partial x_k}. \tag{55}$$

Recall next that the Lie derivative of a scalar function $\phi(x)$ with respect to the vector field $\Xi = \sum_{i=1}^{n} \xi_i(x)$ $(\partial/\partial x_i)$ is defined as

$$L_\Xi \phi = \sum_{i=1}^{n} \frac{\partial \phi(x)}{\partial x_i} \xi_i(x). \tag{56}$$

Compute, according to (56), the Lie derivative of the scalar product (54) with respect to the vector field Ξ_k. This gives the first and the second terms in (57) below. Next we add and subtract the third and the fourth term which are equal, but with opposite signs, which allows to rewrite later the Lie derivative in a suitable form

$$L_{\Xi_k} \left\langle \omega_i, \Xi_j \right\rangle = \sum_{l,q=1}^{n} \left(\frac{\partial \alpha_{il}}{\partial x_q} \xi_{lj} \xi_{qk} + \alpha_{il} \frac{\partial \xi_{lj}}{\partial x_q} \xi_{qk} + \alpha_{il} \frac{\partial \alpha_{lk}}{\partial x_q} \xi_{qj} - \alpha_{il} \frac{\partial \alpha_{lk}}{\partial x_q} \xi_{qj} \right) \equiv 0. \tag{57}$$

Due to the definition of the Lie brackets

$$[\Xi_k, \Xi_j] = \sum_{l,q=1}^{n} \left(\frac{\partial \xi_{lj}}{\partial x_q} \xi_{qk} - \frac{\partial \xi_{lk}}{\partial x_q} \xi_{qj} \right) \frac{\partial}{\partial x_q} \tag{58}$$

and from the definition of the scalar product we obtain

$$\left\langle \omega_i, [\Xi_k, \Xi_j] \right\rangle = \sum_{l,q=1}^{n} \alpha_{il} \left(\frac{\partial \xi_{lj}}{\partial x_q} \xi_{qk} - \frac{\partial \xi_{lk}}{\partial x_q} \xi_{qj} \right). \tag{59}$$

The left hand side of (59) is identically equal to zero according to the commutativity of Ξ_i and Ξ_j. Observe that the right hand side of (59) is exactly the sum of the second and the fourth terms in (57). Therefore, (57) takes the simpler form

$$\sum_{l,q=1}^{n} \left(\frac{\partial \alpha_{il}}{\partial x_q} \xi_{lj} \xi_{qk} + \alpha_{il} \frac{\partial \xi_{lk}}{\partial x_q} \xi_{qj} \right) \equiv 0. \tag{60}$$

The second term on the left hand side of (60) can be rewritten, according to (55), as

$$\sum_{l,q=1}^{n} \alpha_{il} \frac{\partial \xi_{lk}}{\partial x_q} \xi_{qj} = - \sum_{l,q=1}^{n} \frac{\partial \alpha_{il}}{\partial x_q} \xi_{lk} \xi_{qj}. \tag{61}$$

Substituting (61) into (60) and exchanging the summation indices l and q in the second term, we obtain

$$\sum_{l,q=1}^{n} \left(\frac{\partial \alpha_{il}}{\partial x_q} - \frac{\partial \alpha_{iq}}{\partial x_l} \right) \xi_{lj} \xi_{qk} \equiv 0, \quad i,j,k=1,...n. \tag{62}$$

The above equality can be interpreted as the product of three $(n \times n)$-dimensional matrices

$$P^T Q_i P \equiv 0_{n,n}, \tag{63}$$

where $0_{n,n}$ is the $(n \times n)$-dimensional zero matrix and the matrices P and Q_i are defined as

$$P_{lj} = \xi_{lj}, \quad Q_{i,lq} = \frac{\partial \alpha_{il}}{\partial x_q} - \frac{\partial \alpha_{iq}}{\partial x_l}. \tag{64}$$

We show next that all elements of Q_i are identically zero. The matrix $P^T Q_i P$ in (62) is a zero matrix and therefore $\text{rank}_{\mathscr{K}}(P^T Q_i P) \equiv 0$. Due to the Sylvester formula, $\text{rank}_{\mathscr{K}}(P^T Q_i P) = \min(\text{rank}_{\mathscr{K}} Q_i, \text{rank}_{\mathscr{K}} P)$. As $\text{rank}_{\mathscr{K}} P = n$ due to linear independence of Ξ_j, we get $\text{rank}_{\mathscr{K}} Q_i = 0$. Since only the zero matrix has the rank zero, (53) holds for all i,j,l, and so ω_i, $i=1,...,n$, are exact. By (16) $\Psi_i(x)$, $i=1,...,n$, are independent functions and therefore they define the coordinate transformation $X_i = \Psi_i(x)$.

Next, let us express the scalar product (16) in terms of the new coordinates X, taking into account that 1) due to (14) $d\Psi_i = dX_i$, 2) the vector field Ξ_i transforms into the vector field $\Psi_* \Xi_i$ according to (15), and 3) the coordinate transformation does not change the value of a constant. We get, in the new coordinates,

$$\langle dX_i, \Psi_* \Xi_j \rangle \equiv \delta_{ij}, \quad i,j=1,...,n. \tag{65}$$

Comparing (65) with the obvious equality $\langle dX_i, \partial/\partial X_j \rangle \equiv \delta_{ij}$, we obtain (17).

THE PROOF OF LEMMA 4   **APPENDIX B**

In principle, the proof of this lemma can be performed by induction. Since the k-th step is technically extremely complicated to follow, we prove the first and the second steps instead. The continuation is obvious.

We first examine the case $l=1$. Observe that by (7) and (8) the vector field $(\partial/\partial u_i)^{\langle -1 \rangle}$ has no components in the directions of $\partial/\partial u^{\langle k \rangle}$, $k \geq 0$, and $\partial/\partial z^{\langle -l \rangle}$, $l > 1$. Really, all the coefficients b_{jk} in (8) are zero because $\langle du_j^{\langle k+1 \rangle}, \partial/\partial u_i \rangle \equiv 0$ for all $j=1,...,m$ and $k \geq 0$. Moreover, also the coefficients $c_{ql} \equiv 0$ for $l \geq 2$, because $\langle dz_q^{\langle -l+1 \rangle}, \partial/\partial u_i \rangle \equiv 0$ for $l \geq 2$. Therefore, according to (7), (8), (1) and (2)

$$\left(\frac{\partial}{\partial u_i} \right)^{\langle -1 \rangle} = \sum_{j_1=1}^{n} \left(\frac{\partial \overline{\Phi}_{j_1}}{\partial u_i} \right)^{\langle -1 \rangle} \frac{\partial}{\partial x_{j_1}} + \sum_{p_1=1}^{m} \left(\frac{\partial \chi_{p_1}}{\partial u_i} \right)^{\langle -1 \rangle} \frac{\partial}{\partial z_{p_1}^{\langle -1 \rangle}}. \tag{66}$$

From (66) follows that for $l = 1$ the property (i) is satisfied. Next, since the functions $\overline{\Phi}$ and χ in (1) and (2) depend only on the variables x and u, their partial derivatives are also the functions of x and u only. By (3), the backward shifts of partial derivatives depend only on x and $z^{\langle -1 \rangle}$. Hence, the coefficients of the vector field (66) are the functions of x and $z^{\langle -1 \rangle}$ only and (ii) holds.

We show next that the lemma is valid for $l = 2$. Again, one can easily show that $(\partial / \partial u_i)^{\langle -2 \rangle}$ has no components in the directions of the vector fields $\partial / \partial u_j^{\langle k \rangle}$, $k \geq 0$, and $\partial / \partial z_q^{\langle -l \rangle}$, $l > 2$. Namely, taking $\Xi = (\partial / \partial u_i)^{\langle -1 \rangle}$ from (66) and substituting it into (8), all the coefficients b_{ik} and c_{ql}, $l > 2$, are identically equal to zero. Shift the vector field $(\partial / \partial u_i)^{\langle -1 \rangle}$ backward, using (7), (8) and (66) to get

$$
\left(\frac{\partial}{\partial u_i} \right)^{\langle -2 \rangle} = \sum_{j_2=1}^{n} \left\langle \mathrm{d}x_{j_2}^{\langle 1 \rangle}, \left(\frac{\partial}{\partial u_i} \right)^{\langle -1 \rangle} \right\rangle^{\langle -1 \rangle} \frac{\partial}{\partial x_{j_2}} + \sum_{p_2=1}^{m} \left\langle \mathrm{d}z_{p_1}, \left(\frac{\partial}{\partial u_i} \right)^{\langle -1 \rangle} \right\rangle^{\langle -1 \rangle} \frac{\partial}{\partial z_{p_2}^{\langle -1 \rangle}}
$$
$$
+ \sum_{p_2=1}^{m} \left\langle \mathrm{d}z_{p_1}, \left(\frac{\partial}{\partial u_i} \right) \right\rangle^{\langle -2 \rangle} \frac{\partial}{\partial z_{p_2}^{\langle -2 \rangle}}. \tag{67}
$$

Substituting the vector field $(\partial / \partial u_i)^{\langle -1 \rangle}$ from (66) into (67), we get

$$
\left(\frac{\partial}{\partial u_i} \right)^{\langle -2 \rangle} = \sum_{j_1,j_2=1}^{n} \left(\frac{\partial \overline{\Phi}_{j_2}}{\partial x_{j_1}} \right)^{\langle -1 \rangle} \left(\frac{\partial \overline{\Phi}_{j_1}}{\partial u_i} \right)^{\langle -2 \rangle} \frac{\partial}{\partial x_{j_2}}
$$
$$
+ \sum_{j_1=1}^{n} \sum_{p_2=1}^{m} \left(\frac{\partial \chi_{p_2}}{\partial x_{j_1}} \right)^{\langle -1 \rangle} \left(\frac{\partial \overline{\Phi}_{j_1}}{\partial u_i} \right)^{\langle -2 \rangle} \frac{\partial}{\partial z_{p_2}^{\langle -1 \rangle}} + \sum_{p_1=1}^{m} \left(\frac{\partial \chi_{p_1}}{\partial u_i} \right)^{\langle -2 \rangle} \frac{\partial}{\partial z_{p_1}^{\langle -2 \rangle}}. \tag{68}
$$

From (68) follows that

$$
\left(\frac{\partial}{\partial u_i} \right)^{\langle -2 \rangle} \in \mathrm{span}_{\mathscr{K}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial z^{\langle -1 \rangle}}, \frac{\partial}{\partial z^{\langle -2 \rangle}} \right\},
$$

i.e. for $l = 2$ (i) is satisfied. Moreover, the coefficients of $(\partial / \partial u_i)^{\langle -2 \rangle}$ contain only the backward shifts of the partial derivatives of $\overline{\Phi}$ and χ up to order 2, i.e. depend only on the variables x, $z^{\langle -1 \rangle}$ and $z^{\langle -2 \rangle}$. Consequently, (ii) is also satisfied.

Continuing in the similar manner, one can prove that the lemma holds for all $l = 1, ..., k^\star + 1$.

Lineariseeritavus olekuteisenduse abil

Tanel Mullari ja Ülle Kotta

On esitatud tarvilikud ja piisavad tingimused diskreetaja mittelineaarse mitme sisendiga dünaamilise süsteemi lineariseeritavuseks olekuteisenduste abil. Tingimused on kirjeldatud süsteemi dünaamika poolt defineeritud vektorväljade tagasinihete abil. Samuti on näidatud, kuidas leida olekuteisendust juhul, kui tingimused on rahuldatud. Artiklis toodud tingimusi on võrreldud varasemalt sama aparatuuri abil leitud tingimustega, kus lisaks olekuteisendusele on lubatud ka staatiline olekutagasiside: on näidatud, millised tingimused lisanduvad, kui olekutagasisidet ei saa kasutada. Teoreetilisi tulemusi on illustreeritud näitega.