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Error estimates for the Chernoff scheme to approximate a nonlocal parabolic problem

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Abstract. We study a nonlocal parabolic equation obtained from the reduction of the well-known thermistor problem. Error estimate bounds are established for a family of time discretization scheme originated by E. Magenes in *Analyse Mathématique et Applications* (Gauthier–Villars, Paris, 1988).

Key words: time discretization scheme, thermistor problem, Euler forward method, linearization technique, Chernoff scheme.

1. INTRODUCTION

The thermistor problem has received great interest from scientific community, and has been the subject of a variety of mathematical investigations in the past decade. We refer in particular to the works of Cimatti, Antontsev, and Chipot ([1–3]), as well as to the papers of many other authors (see [4–7] and the references therein). In the present work, we study the following general nonlocal initial boundary value problem which, under special simplifications, replaces the classical system of the thermistor problem (see [8,9])

$$\begin{gathered}\frac{\partial u}{\partial t} - \triangle\beta(u) = \lambda \frac{f(u)}{\left(\int_\Omega f(u)\,\mathrm{d}x\right)^2}, \text{ in } Q_T = \Omega\times]0;T[,\\ \beta(u) = 0 \quad \text{on } \partial\Omega\times]0;T[,\\ u(0) = u_0 \quad \text{in } \Omega,\end{gathered} \tag{1}$$

where $\triangle$ is the Laplacian operator with respect to the x-variables, Ω is a bounded open regular set on $\mathbb{R}^d (d \geq 2)$, with a smooth boundary $\partial\Omega$, T is a fixed positive real, and λ is a positive parameter.

The problem (1) arises, for example, in studying the heat transfer in the resistor device whose resistance $f(u)$ depends strongly on the temperature u. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a nondecreasing locally Lipschitzian continuous function with $\beta(0) = 0$, and let l_β and L_β be two constants such that $0 \leq l_\beta \leq \beta'(s) \leq L_\beta$ a.e. in $\mathbb{R}$. The standard norm in $L^2(\Omega)$ is denoted by $\|.\|$, and we denote by $\langle ., . \rangle$ the inner product in $L^2(\Omega)$ or the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

If $u_0 \in L^\infty(\Omega)$, the existence and uniqueness of the solution for (1) which satisfies

$$u \in L^\infty(Q_T) \bigcap H^1(0, T, H^{-1}(\Omega)),$$

$$\theta \in L^2(0, T, H_0^1(\Omega)),$$

$$\theta(x,t) = \beta(u(x,t)) \text{ a.e. in } Q_T,$$

can be established under the same hypotheses as the ones found in [5]:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian function;

(H2) there exist positive constants σ, c_1, c_2, and ν such that for all $\xi \in \mathbb{R}$

$$\sigma \leq f(\xi) \leq c_1 |\xi|^{\nu+1} + c_2.$$

Before proceeding we give some general notation concerning the time discretization for the problem (1). Let us denote by N a fixed positive integer and let the time step be denoted by $\tau = \frac{T}{N}$, $t^n = n\tau$, and $I_n = (t_{n-1}, t_n)$ for $n = 1, ..., N$. If z is a continuous function (respectively summable), defined in $(0, T)$ with values in $H^{-1}(\Omega)$ or $L^2(\Omega)$ or $H_0^1(\Omega)$, we define

$$z^n = z(t^n, .), \overline{z}^n = \frac{1}{\tau} \int_{I_n} z(t, .) \mathrm{d}t, \overline{z}^0 = z^0 = z(0, .).$$

Then we define the following time discretization scheme:

$$\begin{aligned} U^n - \tau \triangle \beta(U^n) &= U^{n-1} + \lambda\tau \frac{f(U^n)}{\left(\int_\Omega f(U^n)\, \mathrm{d}x\right)^2}, \text{ in } \Omega, \\ & 1 \leq n \leq N, \\ \beta(U^n) &= 0 \quad \text{on } \partial\Omega, \\ U^0 &= u_0 \quad \text{in } \Omega. \end{aligned} \tag{2}$$

Investigations of the solution for (1) by this Euler forward method are rare in the literature. We refer the reader to the work [6] for the particular case $\beta = Id$, where existence and uniqueness results are obtained and also the questions of stability and error estimates are studied. Many works present diverse ways of realizing this discretization by introducing different schemes. These schemes can be explicit, implicit or semi-implicit. The difficulty of integration in time comes from the function $\theta = \beta(u)$, which can present a vanishing derivative. This yields to a degenerate parabolic problem. This nonlinearity is important in particular for pure materials where phase transition arises in an isotherm manner. In (1), we must solve, at each time step, a nonlinear elliptic equation which is a source of serious difficulties. In the nondegenerate case $l_\beta > 0$, we can solve this problem using a linearization technique and considering the following equation:

$$U^n - \tau\nabla(\beta'(U^{n-1})\nabla U^n) = U^{n-1} + \lambda\tau\frac{f(U^n)}{\left(\int_\Omega f(U^n)\,\mathrm{d}x\right)^2}, \tag{3}$$

where ∇ denotes the gradient with respect to the x-variables.

However, it is inconvenient that, for each n, the coefficients of the linear equation (3) change. This leads, after discretization in x, to the fact that the "rigidity matrix" changes at each instant n. Even in the degenerate case ($l_\beta = 0$), the choice here is fixed on a semi-implicit scheme arising in works using nonlinear semi-group theory for the resolution of solidification problems. By convention, we call it the "Chernoff scheme". The Chernoff scheme has been studied in particular by Magenes [10], Massera [11], and Verdi and Visintin [12]. This numerical method consists in the construction of a family of time discretization schemes, where at each n it is reduced to the resolution of a linear equation with coefficients independent of n, and then in a correction, which includes a calculation of a given function.

Let us explicate now the Chernoff scheme. We begin with discretizing the time derivative of Eq. (1) with the help of the implicit scheme. The equation is written as

$$\begin{gathered}\frac{U^n - U^{n-1}}{\tau} - \triangle\Theta^n = \lambda\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2},\\ \Theta^n = \beta(U^n).\end{gathered} \tag{4}$$

Now we can relax this equation by the following:

$$U^n = U^{n-1} + \mu(\Theta^n - \beta(U^{n-1})). \tag{5}$$

Finally, we substitute the relation (5) in (4), thus obtaining a semi-implicit formula for Θ^n:

$$\mu\frac{\Theta^n - \beta(U^{n-1})}{\tau} - \triangle\Theta^n = \lambda\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}. \tag{6}$$

In relations (5) and (6), μ is a scheme relaxation parameter, which satisfies the condition $0 < \mu \leq L_\beta^{-1}$.

In discrete form, the new scheme is given as follows:

$$\begin{aligned} U^0 &= u_0, \\ \Theta^n - \frac{\tau}{\mu}\triangle\Theta^n &= \beta(U^{n-1}) + \frac{\lambda\tau}{\mu}\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}, \\ U^n &= U^{n-1} + \mu(\Theta^n - \beta(U^{n-1})). \end{aligned} \tag{7}$$

The paper is structured as follows. In Section 2 we will show some stability results, and in Section 3 we will focus on the question of error estimate bounds.

In the sequel, c always denotes some generic positive constant depending on u_0, T, μ, L_β. Moreover, the values may vary from one step to the next one.

2. STABILITY RESULTS

Let $\tau = \frac{T}{N}$ be the time discretization step and define now by iteration the following scheme for $0 < \mu \leq L_\beta^{-1}$:

$$\begin{aligned} U^0 &= u_0, \\ U^n &= U^{n-1} + \mu\big(\Theta^n - \beta(U^{n-1})\big), \end{aligned} \tag{8}$$

where Θ^n is given by

$$\Theta^n = G\left(\frac{\tau}{\mu}\right)\left(\beta(U^{n-1}) + \frac{\lambda\tau}{\mu}\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}\right),$$

with $G(\frac{t}{\mu}) = (I - \frac{t}{\mu}\triangle)^{-1}$.

Writing (8) under variational formulation, we have

$$\langle U^n - U^{n-1}, \varphi\rangle + \mu\langle\beta(U^{n-1}) - \Theta^n, \varphi\rangle = 0, \forall\varphi \in L^2(\Omega). \tag{9}$$

Let us give some useful notation below. Let α be the following real value function $\alpha(s) = s - \mu\beta(s), \quad \forall s \in \mathbb{R}$. The fact that $0 < \mu \leq L_\beta^{-1}$ implies that $0 \leq \alpha'(s) \leq 1$ a.e. in $\mathbb{R}$. Denote by $\psi_g(s) = \int_0^s g(t)\mathrm{d}t,\ t \in \mathbb{R}$, the convex function for all absolutely continuous functions $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and

$$0 \leq g'(s) \leq c \quad \text{a.e. in } \mathbb{R},$$

$$\frac{1}{2c}g^2(s) \leq \psi_g(s) \leq \frac{c}{2}s^2,\ s \in \mathbb{R}.$$

We give the scheme stability in $L^\infty(\Omega)$, which plays an important role in proving error estimates.

Lemma 2.1. *If $u_0 \in L^\infty(\Omega)$, then for all $n \in \{1, ..., N\},\ U^n \in L^\infty(\Omega)$.*

Proof. Multiplying (7) by $|\Theta^n|^{k-1}\Theta^n,\ k \geq 1$, we have

$$\langle \Theta^n, |\Theta^n|^{k-1}\Theta^n \rangle + \langle -\triangle\Theta^n, |\Theta^n|^{k-1}\Theta^n \rangle \leq c\|\Theta^n\|_k^k.$$

In other terms,

$$\|\Theta^n\|_{k+1}^{k+1} + k\int |\nabla\Theta^n|^2|\Theta^n|^{k-1} \leq c\|\Theta^n\|_k^k.$$

Using then Holder inequality in the right-hand side and the positivity of the second term in the left-hand side, we get

$$\|\Theta^n\|_{k+1}^{k+1} \leq c\|\Theta^n\|_{k+1}^k.$$

It follows that $\|\Theta^n\|_{k+1} \leq c$. Letting $k \to +\infty$, we have $\|\Theta^n\|_\infty \leq c$, and using then the relation $U^n = U^{n-1} + \mu(\Theta^n - \beta(U^{n-1}))$, we have $U^n \in L^\infty(\Omega)$. □

Remark 2.2. The general properties of elliptic and parabolic operators that are applied in [10] for appropriate L^∞ estimate do not work here due to the presence of the nonlocal term $\lambda f(u)/\big(\int_\Omega f(u)\,\mathrm{d}x\big)^2$.

Next, we prove the following stability results.

Theorem 2.3. *Under hypotheses* (H1), (H2) *and for a fixed μ with $0 < \mu \leq L_\beta^{-1}$, there exists a positive constant c such that for any $n \in \{1, ..., N\}$*

$$\max_{1\leq n\leq N} \|\beta(U^n)\|^2 \leq c, \tag{10}$$

$$\sum_{n=1}^{N} \|U^n - U^{n-1}\|^2 \leq c, \tag{11}$$

$$\left|\sum_{n=1}^{N} \langle \beta(U^{n-1}) - \Theta^n, \Theta^n \rangle\right| \leq c, \tag{12}$$

$$\max_{1\leq n\leq N} \|(\Theta^n)\|^2 \leq c. \tag{13}$$

Proof. The proof is standard. We refer to [10] and give here a detail for sake of completeness only.

Choosing $\varphi = \Theta^n$ in (9) and summing up from 1 to m with $m \leq N$, we get

$$\sum_{n=1}^{m} \langle U^n - U^{n-1}, \Theta^n \rangle + \mu \sum_{n=1}^{m} \langle \beta(U^{n-1}) - \Theta^n, \Theta^n \rangle = 0. \tag{14}$$

With the notation above, we have

$$\begin{aligned}\Theta^n &= \frac{1}{\mu}(U^n - U^{n-1}) + \beta(U^{n-1}) \\ &= \frac{1}{2}\beta(U^n) + \frac{1}{2\mu}\big(\alpha(U^n) - \alpha(U^{n-1})\big) + \frac{1}{2\mu}U^n - \frac{1}{2\mu}\alpha(U^{n-1}).\end{aligned}$$

Using the convexity of functions ψ_α and ψ_β and with the aid of the elementary identity $2a(a-b) = a^2 - b^2 + (a-b)^2$, a and $b \in \mathbb{R}$, we obtain

$$\sum_{n=1}^{m} \int_\Omega \left\{ \big(\psi_\beta(U^n) - \psi_\beta(U^{n-1})\big) + \frac{1}{\mu}\big(\psi_\alpha(U^{n-1}) - \psi_\alpha(U^n)\big) \right\} \mathrm{d}x$$
$$+ \frac{1}{2\mu}\left(\|U^m\|^2 - \|U^0\|^2 + \sum_{n=1}^{m} \|U^n - U^{n-1}\|^2 \right) \leq 2 \sum_{n=1}^{m} \langle U^n - U^{n-1}, \Theta^n \rangle.$$

Due to the fact that $0 \leq \alpha' \leq 1$ and proprieties of ψ_α and ψ_β, we get

$$\sum_{n=1}^{m} \langle U^n - U^{n-1}, \Theta^n \rangle \geq -c + c\|\beta(U^m)\|^2 + \frac{1}{4\mu} \sum_{n=1}^{m} \|U^n - U^{n-1}\|^2.$$

From (14) we have

$$\max_{1 \leq n \leq N} \|\beta(U^n)\|^2 + \sum_{n=1}^{N} \|U^n - U^{n-1}\|^2 + \sum_{n=1}^{N} \langle \beta(U^{n-1}) - \Theta^n, \Theta^n \rangle \leq c.$$

Then

$$\left| \sum_{n=1}^{N} \langle \beta(U^{n-1}) - \Theta^n, \Theta^n \rangle \right| \leq c,$$

and (10)–(12) hold.

On the other hand, from (8) we have $\Theta^n = \frac{1}{\mu}(U^n - U^{n-1}) + \beta(U^{n-1})$. Using then (10), (11), we get (13). □

3. ERROR ESTIMATES

Let $(-\triangle)^{-1}$ be the Green operator satisfying

$$\langle \nabla(-\triangle)^{-1}v, \nabla w\rangle = \langle v, w\rangle \quad \forall v \in H_0^1(\Omega), \quad w \in H^{-1}(\Omega).$$

For $n = 1, ..., N$, there exists a unique solution Θ_*^n for the problem

$$\begin{aligned} \Theta_*^n \in & H_0^1(\Omega), \\ \tau\triangle\Theta_*^n = \mu(\Theta^n - \beta(U^{n-1})) - \lambda\tau & \frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}, \end{aligned} \tag{15}$$

which verifies

$$\|\Theta_*^n - \Theta^n\| \le \|\beta(U^{n-1}) - \Theta^n\|. \tag{16}$$

Further, we can construct a function $\Theta^n : I_n \to H_0^1(\Omega)$ such that

$$\Theta^n(t_n) = \Theta^n, \tag{17}$$

$$\overline{\Theta^n} = \frac{1}{\tau}\int_{I_n} \Theta^n(t)\mathrm{d}t = \Theta_*^n, \tag{18}$$

$$\|\Theta^n(t) - \Theta^n\| \le \|\beta(U^{n-1}) - \Theta^n\|, \quad t \in I_n. \tag{19}$$

It suffices, for example, to consider $\Theta^n(t_n) = \Theta^n$, $\Theta^n(t) = \Theta_*^n$ for $t_{n-1} < t < t_n$. By (15), (17), and (18) we have

$$U^n - U^{n-1} = \mu(\Theta^n(t_n) - \beta(U^{n-1})) = \tau\triangle\overline{\Theta^n} + \lambda\tau\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}.$$

Then, applying (11), we obtain

$$\mu^2\sum_{n=1}^N \|\Theta^n(t_n) - \beta(U^{n-1})\|^2 = \tau^2\sum_{n=1}^N \|\triangle\overline{\Theta^n} + \lambda\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}\|^2 \le c, \tag{20}$$

which gives, by hypotheses and L^∞ estimate of U^n, that

$$\tau^2\sum_{n=1}^N \|\triangle\overline{\Theta^n}\|^2 \le c. \tag{21}$$

We also have by (12) that

$$\tau\sum_{n=1}^N \left\langle -\triangle\overline{\Theta^n} - \lambda\frac{f(U^{n-1})}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}, \Theta^n(t_n)\right\rangle \le c, \tag{22}$$

or

$$\sum_{n=1}^{N}\|\Theta^n(t_n)\|^2 \le 2\sum_{n=1}^{N}\|\Theta^n(t_n)-\beta(U^{n-1})\|^2 + 2\sum_{n=1}^{N}\|\beta(U^{n-1})\|^2 \le c.$$

We deduce from (22) that

$$\tau\sum_{n=1}^{N}\langle\nabla\overline{\Theta^n}, \nabla\Theta^n(t_n)\rangle \le c, \tag{23}$$

and by (19)–(21) we obtain

$$\sum_{n=1}^{N}\|\Theta^n(t)-\Theta^n(t_n)\|^2 \le c, \qquad \sum_{n=1}^{N}\|\overline{\Theta^n}-\Theta^n(t_n)\|^2 \le c. \tag{24}$$

On the other hand, using (23), (24), and Young inequality, we get

$$\begin{aligned}
\tau\|\nabla\overline{\Theta^n}\|^2 =& -\tau\sum_{n=1}^{N}\langle\triangle\overline{\Theta^n}, \overline{\Theta^n}\rangle \\
=& -\tau\sum_{n=1}^{N}\langle\triangle\overline{\Theta^n}, \Theta^n(t_n)\rangle - \tau\sum_{n=1}^{N}\langle\triangle\overline{\theta^n}, \overline{\Theta^n}-\Theta^n(t_n)\rangle \\
\le& -\tau\sum_{n=1}^{N}\langle\nabla\overline{\Theta^n}, \nabla\Theta^n(t_n)\rangle + \frac{1}{2}\sum_{n=1}^{N}\tau^2\|\triangle\overline{\Theta^n}\|^2 \\
&+ \frac{1}{2}\sum_{n=1}^{N}\|\overline{\Theta^n}-\Theta^n(t_n)\|^2 \le c.
\end{aligned} \tag{25}$$

We are now ready to give the main theorem of error estimates. To this end, we introduce the errors e_θ and $\widetilde{e_\theta}$ for $t \in I_n$, defined by $e_\theta = \theta(t) - \Theta^n(t)$, $\widetilde{e_\theta} = \theta(t) - \overline{\Theta^n}$, and $e_u = u(t) - U^n$.

Theorem 3.1. *Under hypotheses* $(H1)$, $(H2)$ *and for all fixed* μ *such that* $0 < \mu \le L_\beta^{-1}$, *we have*

$$\|e_\theta\|_{L^2(Q_T)} + \left\|\int_0^t e_\theta \mathrm{d}s\right\|_{L^\infty(0,T,H^1(\Omega))} \le c\tau^{1/4}.$$

Proof. Using (15) and (18), we reformulate (9) as

$$\langle U^n - U^{n-1}, \varphi\rangle + \lambda\langle\nabla\overline{\Theta^n}, \nabla\varphi\rangle = \frac{\lambda\tau}{\left(\int_\Omega f(U^{n-1})\,\mathrm{d}x\right)^2}\langle f(U^{n-1}), \varphi\rangle, \forall\varphi \in H_0^1(\Omega). \tag{26}$$

Integrating the continuous problem (1) over I_n, we get, under the notation given in the introduction, that

$$\langle u^n - u^{n-1}, \varphi\rangle + \tau\langle \nabla\overline{\theta}^n, \nabla\varphi\rangle = \lambda \int_{I_n} \frac{1}{\left(\int_\Omega f(u)\,\mathrm{d}x\right)^2}\langle f(u), \varphi\rangle, \quad \forall\,\varphi \in H_0^1(\Omega). \tag{27}$$

Subtracting (27) from (26) and summing up from $n = 1, .., m$ with $m \leq N$, we deduce

$$\begin{aligned}\langle u^m - U^m, \varphi\rangle &+ \tau\sum_{n=1}^m \langle \nabla\left(\overline{\theta}^n - \overline{\Theta}^n\right), \nabla\varphi\rangle \\ &\leq c\,\tau\left|\sum_{n=1}^m \langle \overline{f(u)}^n - f(U^{n-1}), \varphi\rangle\right| + c\,\tau\left|\sum_{n=1}^m \langle f(U^{n-1}), \varphi\rangle\right|. \end{aligned}\tag{28}$$

Choosing $\varphi = \tau\sum_{n=1}^m \left(\overline{\theta}^n - \overline{\Theta}^n\right)$ in (28), we get

$$\left\langle u^m - U^m, \tau\sum_{n=1}^m \left(\overline{\theta}^n - \overline{\Theta}^n\right)\right\rangle + \tau^2\left\|\sum_{n=1}^m \nabla\left(\overline{\theta}^n - \overline{\Theta}^n\right)\right\|^2 \leq I_1 + I_2, \tag{29}$$

where

$$I_1 = c\tau^2\left|\sum_{n=1}^m \left\langle \overline{f(u)}^n - f(U^{n-1}), \sum_{n=1}^m \left(\overline{\theta}^n - \overline{\Theta}^n\right)\right\rangle\right|$$

and

$$I_2 = c\,\tau^2\left|\sum_{n=1}^m \left\langle f(U^{n-1}), \sum_{n=1}^m \left(\overline{\theta}^n - \overline{\Theta}^n\right)\right\rangle\right|.$$

We have

$$\tau^2\left\|\sum_{n=1}^m \nabla\left(\overline{\theta}^n - \overline{\Theta}^n\right)\right\|^2 = \left\|\sum_{n=1}^m \nabla\int_{I_n}\left(\theta(t) - \overline{\Theta}^n\right)\right\|^2 = \left\|\nabla\int_0^{t_m} e_\theta \mathrm{d}t\right\|^2. \tag{30}$$

On the other hand,

$$
\begin{aligned}
&\tau \left\langle u^m - U^m, \sum_{n=1}^{m} \left(\overline{\theta}^n - \overline{\Theta}^n\right) \right\rangle = \tau \sum_{n=1}^{m} \langle u^m - U^m, (\overline{\theta}^n - \overline{\Theta}^n) \rangle \\
&= \sum_{n=1}^{m} \int_{I_n} \langle u^m - U^m, \theta(t) - \overline{\theta}^n \rangle \\
&= \sum_{n=1}^{m} \int_{I_n} \langle u^m - U^m, \widetilde{e_\theta} \rangle \\
&= \sum_{n=1}^{m} \int_{I_n} \langle u(t) - U^m, \widetilde{e_\theta} \rangle \mathrm{d}t + \sum_{n=1}^{m} \int_{I_n} \langle u^m - u(t), \widetilde{e_\theta} \rangle \mathrm{d}t \\
&= \sum_{n=1}^{m} \int_{I_n} \langle e_u, \widetilde{e_\theta} \rangle \mathrm{d}t + \sum_{n=1}^{m} \int_{I_n} \langle u^m - u(t), \widetilde{e_\theta} \rangle \mathrm{d}t \\
&= I_3 + I_4.
\end{aligned}
$$

We estimate I_4:

$$
\begin{aligned}
I_4 &= \sum_{n=1}^{m} \int_{I_n} \langle u^m - u(t), \widetilde{e_\theta} \rangle \mathrm{d}t \\
&= \sum_{n=1}^{m} \int_{I_n} \left\langle \int_t^{t_m} \frac{\partial u}{\partial s}, \widetilde{e_\theta} \right\rangle \mathrm{d}t \\
&\leq \sum_{n=1}^{m} \int_{I_n} \left(\int_t^{t_m} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}(\Omega)} \mathrm{d}s \right) \|\widetilde{e_\theta}\|_{H^1(\Omega)} \mathrm{d}t \\
&\leq \tau \left\| \frac{\partial u}{\partial s} \right\|_{L^2(0,t_m,H^{-1}(\Omega))} \|\widetilde{e_\theta}\|_{L^2(0,t_m,H^1(\Omega))}.
\end{aligned}
$$

But, using (25), we obtain

$$
\begin{aligned}
\|\widetilde{e_\theta}\|^2_{L^2(0,t_m,H^1(\Omega))} &= \sum_{n=1}^{m} \int_{I_n} \|\nabla\theta(t) - \nabla\overline{\Theta^n}\|^2 \mathrm{d}t \\
&\leq 2 \sum_{n=1}^{m} \int_{I_n} \|\nabla\theta(t)\|^2 \mathrm{d}t + 2 \sum_{n=1}^{m} \int_{I_n} \|\nabla\overline{\Theta^n}\|^2 \mathrm{d}t \\
&\leq 2\|\theta\|^2_{L^2(0,t_m,H^1(\Omega))} + 2 \sum_{n=1}^{m} \tau \|\nabla\overline{\Theta^n}\|^2 \mathrm{d}t \leq c,
\end{aligned}
$$

and we then deduce

$$
|I_4| \leq c\tau.
$$

By (25) and L^∞ estimates of $u(t)$ and U^n we have

$$
\begin{aligned}
I_1 &= c\tau^2 \left| \sum_{n=1}^{m} \left\langle \overline{f(u)}^n - f(U^{n-1}), \sum_{n=1}^{m} \left(\overline{\theta}^n - \overline{\Theta}^n\right) \right\rangle \right| \\
&\le c \left(\int_\Omega \Big(\sum_{n=1}^{m} \int_{I_n} (f(u) - f(U^{n-1}))\mathrm{d}t \Big)^2 \mathrm{d}x \right)^{1/2} \\
&\quad \times \left(\int_\Omega \Big(\sum_{n=1}^{m} \int_{I_n} (\theta(t) - \overline{\Theta^n})\mathrm{d}t \Big)^2 \mathrm{d}x \right)^{1/2} \\
&\le c \left(\sum_{n=1}^{m} \int_0^{I_n} \|f(u) - f(U^{n-1})\|_2^2 \mathrm{d}t \right)^{1/2} \times \left(\sum_{n=1}^{m} \int_0^{I_n} \|\theta(t) - \overline{\Theta^n}\|_2^2 \mathrm{d}t \right)^{1/2} \\
&\le c \left(\sum_{n=1}^{m} \int_0^{I_n} \|f(u) - f(U^{n-1})\|_2^2 \mathrm{d}t \right)^{1/2} \left(\int_0^{t_m} \|\widetilde{e_\theta}\|_2^2 \mathrm{d}t \right)^{1/2} \\
&\le c \left(\sum_{n=1}^{m} \int_0^{I_n} \|f(u) - f(U^{n-1})\|_2^2 \mathrm{d}t \right)^{1/2} \|\widetilde{e_\theta}\|_{L^2(0,t_m,L^2(\Omega))} \\
&\le c \left(\sum_{n=1}^{m} \int_0^{I_n} \|f(u) - f(U^{n-1})\|_2^2 \mathrm{d}t \right)^{1/2} \|\widetilde{e_\theta}\|_{L^2(0,t_m,H^1(\Omega))} \\
&\le c\tau^{1/2}.
\end{aligned}
$$

The proof is complete. □

As a consequence of the theorem, we have the following corollary.

Corollary 3.2. *For a τ small enough, the inequality*

$$\|e_u\|_{L^\infty(0,T,H^{-1}(\Omega))} \le c\tau^{1/4}$$

holds.

Proof. Coming back to (28) and choosing $\varphi = -\triangle^{-1}(u^m - U^m)$ yields

$$
\begin{aligned}
&\langle u^m - U^m, -\triangle^{-1}(u^m - U^m)\rangle \\
&\quad - \tau \sum_{n=1}^{m} \langle \nabla\big(\overline{\theta}^n - \overline{\Theta}^n\big), \nabla\triangle^{-1}(u^m - U^m)\rangle \\
&\le c\,\tau \left| \sum_{n=1}^{m} \langle \overline{f(u)}^n - f(U^{n-1}), \triangle^{-1}(u^m - U^m)\rangle \right| \\
&\quad + c\,\tau \left| \sum_{n=1}^{m} \langle f(U^{n-1}), \triangle^{-1}(u^m - U^m)\rangle \right|.
\end{aligned}
$$

Using the properties of the Green operator $\triangle^{-1}$, we obtain that

$$\|u^m - U^m\|^2_{H^{-1}(\Omega)} - \tau \sum_{n=1}^{m} \langle (\overline{\theta}^n - \overline{\Theta}^n), u^m - U^m \rangle$$
$$\leq c\,\tau \left| \sum_{n=1}^{m} \langle \overline{f(u)}^n - f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|$$
$$+ c\,\tau \left| \sum_{n=1}^{m} \langle f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|.$$

Moreover, we have

$$\|u^m - U^m\|^2_{H^{-1}(\Omega)}$$
$$\leq \|u^m - U^m\|_{H^{-1}(\Omega)} \left\| \nabla \int_0^{t_m} e_\theta \mathrm{d}t \right\|$$
$$+\ c\,\tau \left| \sum_{n=1}^{m} \langle \overline{f(u)}^n - f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|$$
$$+\ c\,\tau \left| \sum_{n=1}^{m} \langle f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|.$$

From Young inequality and the previous theorem, we get

$$\|u^m - U^m\|^2_{H^{-1}(\Omega)}$$
$$\leq c\tau^{1/2} + c\,\tau \left| \sum_{n=1}^{m} \langle \overline{f(u)}^n - f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|$$
$$+ c\,\tau \left| \sum_{n=1}^{m} \langle f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|. \tag{31}$$

Using again Holder and Young inequalities yields

$$\tau \left| \sum_{n=1}^{m} \int_{I_n} \langle \overline{f(u)}^n - f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right|$$
$$= \left| \sum_{n=1}^{m} \langle (f(u) - f(U^{n-1}))\mathrm{d}t, \triangle^{-1}(u^m - U^m) \rangle \right|$$
$$\leq c\tau^{1/2} \left(\sum_{n=1}^{m} \int_{I_n} \|(f(u) - f(U^{n-1}))\|_2^2 \mathrm{d}t \right)^{1/2} \times \|u^m - U^m\|_{H^{-1}(\Omega)}$$
$$\leq c\tau^{1/2} \|u^m - U^m\|_{H^{-1}(\Omega)} = c\tau^{1/4}\tau^{1/4}\|u^m - U^m\|_{H^{-1}(\Omega)}$$
$$\leq c\tau^{1/2} + c\tau^{1/2}\|u^m - U^m\|^2_{H^{-1}(\Omega)}. \tag{32}$$

In the same way, we have

$$\tau \left| \sum_{n=1}^{m} \langle f(U^{n-1}), \triangle^{-1}(u^m - U^m) \rangle \right| \leq c\tau^{1/2} + c\tau^{1/2} \|u^m - U^m\|^2_{H^{-1}(\Omega)}. \quad (33)$$

From (31)–(33) we get that

$$(1 - c\tau^{1/2}) \|u^m - U^m\|^2_{H^{-1}(\Omega)} \leq c\tau^{1/2}.$$

Then we have for a small τ

$$\|u^m - U^m\|^2_{H^{-1}(\Omega)} \leq c\tau^{1/2}.$$

In other terms, we have

$$\max_{1 \leq m \leq N} \|u^m - U^m\|_{L^\infty(0,T,H^{-1}(\Omega))} \leq c\tau^{1/4}.$$

We use finally the fact that $\frac{\partial u}{\partial t} \in L^2(0, T, H^{-1}(\Omega))$ to conclude with the intended result. □

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REFERENCES

1. Cimatti, G. A remark on the thermistor problem with rapidly growing conductivity. *Appl. Anal.*, 2001, **1–2**, 133–140.
2. Antontsev, S. N. and Chipot, M. The thermistor problem: existence, smoothness uniqueness, blowup. *SIAM J. Math. Anal.*, 1994, **4**, 1128–1156.
3. Chipot, M. and Cimatti, G. A uniqueness result for the thermistor problem. *European J. Appl. Math.*, 1991, **2**, 97–103.
4. Allegretto, A., Yanping Lin and Shunqing Ma. On the time periodic thermistor problem. *European J. Appl. Math.*, 2004, **15**, 55–77.
5. El Hachimi, A. and Sidi Ammi, M. R. Existence of global solution for a nonlocal parabolic problem. *Electron. J. Qual. Theory Differ. Equations*, 2005, 1, 1–9.
6. El Hachimi, A. and Sidi Ammi, M. R. Semi-discretization for a nonlocal parabolic problem. *Int. J. Math. Math. Sci.*, 2005, **10**, 1655–1664.
7. Hyung-Chun Lee and Timofey Shi Lkin. Analysis of optimal control problems for the two-dimensional thermistor system. *Siam. J. Control. Optimal*, 2005, **1**, 268–282.

8. Lacey, A. A. Thermal runway in a non-local problem modelling ohmic heating, part I: model derivation and some special cases. *European J. Appl. Math.*, 1995, **6**, 127–144.
9. Lacey, A. A. Thermal runway in a non-local problem modelling ohmic heating, part II: general proof of blow-up and asymptotics runways. *European J. Appl. Math.*, 1995, **6**, 201–224.
10. Magenes, E. Remarques sur l'approximation des problèmes paraboliques non linéaires. In *Analyse Mathématique et Applications*. Gauthier–Villars, Paris, 1988, 1–21.
11. Massera, A. *Optimisation et simulation numérique du chauffage par induction pour le procédé de thixoformage.* Thèse soutenue à L'École Polytechnique Féderale de Lausane, 2003.
12. Verdi, C. and Visintin, A. Error estimates for a semi-explicit numerical scheme for stefan-type problem. *Numer. Math.*, 1988, **52**, 165–195.

Chernoffi skeemi veahinnangud mittelokaalse paraboolse ülesande aproksimeerimisel

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On uuritud termistori probleemi lähendamisest saadud mittelokaalset paraboolset võrrandit. On leitud veahinnangute piirid Magenes'i poolt 1988. aastal loodud aja diskretiseerimise skeemile.