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Overview of viability results

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Abstract. Viability theorems for systems with dynamics depending on time in a measurable way, with time-dependent state constraints, are presented. We compare the results with ours using, for the first time in viability theory, generalized differential quotients. Some illustrative examples are given.

Key words: viability, generalized differential quotient (GDQ), contingent derivative, Cellina continuously approximable (CCA).

1. INTRODUCTION

Let K be a multivalued map from the interval $T = [0, a]$ to $\mathbb{R}^n$. On its graph, denoted by $Gr(K)$, another multivalued map F is defined. Its values are closed subsets of $\mathbb{R}^n$. The map F, called an *orientor field*, gives rise to a multivalued Cauchy problem

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), \quad \text{a.e. on } T, \\ y(t_0) = y_0. \end{cases} \tag{1.1}$$

The inclusion is called *viable* if for every $(t_0, y_0) \in Gr(K)$ there is a global absolutely continuous forward trajectory $y : [t_0, a] \to \mathbb{R}^n$ of this inclusion, satisfying the initial condition $y(t_0) = y_0$, where $y_0 \in K(t_0)$ and $t_0 \in [0, a[$. As the differential inclusion may come from a control system $\dot{y}(t) = f(t, y(t), u(t))$, the viability of the differential inclusion may be interpreted as a kind of controllability of the control system with time-dependent constraints. The first result on viability, by Nagumo [1], was formulated for a constant multifunction K and a single-valued time-independent F. In its full complexity, viability was studied by Aubin [2,3], Bothe [4], Deimling [5], Frankowska et al. [6], Haddad [7,8], Hu and

Papageorgiu [9,10], and many others. The obtained results give conditions on K and F that guarantee the viability of the inclusion. Besides various measurability and continuity assumptions, the essential one states that the intersection of $F(t, y)$ with some generalized derivative of K at (t, y) is nonempty for almost all $t \in [0, a]$ and all $y \in K(t)$. This requirement is a direct extension of the Nagumo [1] condition. We recall all these conditions and present our own viability criteria. As a generalized derivative we choose the generalized differential quotient (GDQ), introduced recently by Sussmann [11,12]. Since the GDQ of a multivalued function is not unique (see [11–16]), we use one, denoted by SGDQ, which seems to fit best: it contains all important directions without superfluous ones. SGDQ is the closure of the union of all minimal GDQs. Finally, we compare conditions introduced by different authors, setting a road map of the viability problem.

2. BASIC NOTATIONS AND DEFINITIONS

By a set-valued map (multifunction) we mean a triple $F = (A, B, G)$ such that A and B are sets and G is a subset of $A \times B$. The sets $A,\ B,\ G$ are, respectively, the *source*, *target*, and *graph* of F, which we write $A = So(F)$, $B = Ta(F)$ and $G = Gr(F)$. For $x \in So(F)$ we write $F(x) = \{y : (x, y) \in Gr(F)\}$, where $Gr(F) := \{(x, y) : y \in F(x)\}$ (it can happen that $F(x) = \emptyset$ for $x \in So(F)$). The sets $Do(F) = \{x \in So(F) : F(x) \neq \emptyset\}$, $Im(F) = \bigcup_{x\in So(F)} F(x)$, are, respectively, the *domain* and *image* of F. If $F = (A, B, G)$ is a set-valued map, we say that F is a set-valued map from A to B with graph G, and write $F : A \twoheadrightarrow B$. We use $SVM(A, B)$ to denote the set of all set-valued maps from A to B. We reserve capital letters for set-valued maps and small ones for ordinary (single-valued and everywhere defined) maps.

If X is a metric space supplied with a metric d, $K \subseteq X$, then we denote the *distance from x to K* by $\mathrm{dist}(x, K) := \inf_{y\in K} d(x, y)$, where we set $\mathrm{dist}(x, \emptyset) := +\infty$. The *ball of radius $\epsilon > 0$ around K in X* is denoted by $B(K, \epsilon) := \{x \in X : \mathrm{dist}(x, K) < \epsilon\}$. The balls $B(K, \epsilon)$ are neighbourhoods of K. When K is compact, each neighbourhood of K contains such a ball around K.

Let X and Y be metric spaces. We say that a set-valued map $F : X \twoheadrightarrow Y$ is *upper semicontinuous* (abbr. u.s.c.) at $\bar{x} \in Do(F)$ if and only if for any neighbourhood U of $F(x)$ there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$, $F(x) \subset U$. We say that a sequence $\{F_n\}$ of set-valued maps $F_n : X \twoheadrightarrow Y$ *graph converges* to F, and write $F_n \xrightarrow{gr} F$, if

$$\lim_{n\to\infty} \triangle(Gr(F_n), Gr(F)) = 0,$$

where

$$\triangle(A, B) = \sup\{\mathrm{dist}(q, B) : q \in A\}.$$

Let $\mathcal{T}$ be a metric space and $\{A_\tau\}_{\tau\in\mathcal{T}}$ be a family of subsets of a metric space X. The *upper limit* (lim sup) and the *lower limit* (lim inf) of $\{A_\tau\}$ at τ_0 are closed sets defined by

$$\limsup_{\tau\to\tau_0} A_\tau = \left\{ v \in X | \liminf_{\tau\to\tau_0} \operatorname{dist}(v, A_\tau) = 0 \right\},$$

$$\liminf_{\tau\to\tau_0} A_\tau = \left\{ v \in X | \limsup_{\tau\to\tau_0} \operatorname{dist}(v, A_\tau) = 0 \right\}.$$

A subset $A \subset X$ is said to be the limit of $\{A_\tau\}$ if

$$A = \limsup_{\tau\to\tau_0} A_\tau = \liminf_{\tau\to\tau_0} A_\tau =: \lim_{\tau\to\tau_0} A_\tau.$$

Throughout the paper, by $\lambda(\cdot)$ we mean the Lebesgue measure.

For $F \in SVM(\mathbb{R}^n, \mathbb{R}^m)$ we define $||F(x)|| := \sup\{||y|| : y \in F(x)\}$ if $F(x) \neq \emptyset$ and set $||\emptyset|| = -\infty$.

A set $C \subseteq \mathbb{R}^n$ is called a *cone* if $rx \in C$ for all $x \in C$ and $r \geq 0$.

Let us consider a multifunction $K : T \twoheadrightarrow \mathbb{R}^n$, $Do(K) = T = [0, a] \subseteq \mathbb{R}$. We say that K is ε–δ *upper semicontinuous from the left* (shorter: *left u.s.c.*) if for every $t_0 \in (0, a]$ and $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$K(t) \subset K(t_0) + B(0, \varepsilon) \quad \text{for all } t \in (t_0 - \delta, t_0] \cap T.$$

Let K be closed-valued. We say that K is *left absolutely continuous on* $[0, a]$ if the following property holds:

$$\forall\, \varepsilon > 0,\ \forall \text{ compact } P \subset \mathbb{R}^n,\ \exists\delta > 0,\ \forall N_0 \subset \mathbb{N}$$
$$\forall\{t_i, \tau_i : t_i < \tau_i, i \in N_0\} \text{ with } (t_i, \tau_i) \cap (t_j, \tau_j) = \emptyset \text{ for } i \neq j,$$
$$\Sigma(\tau_i - t_i) \leq \delta \Rightarrow \Sigma \triangle (K(t_i) \cap P, K(\tau_i)) \leq \varepsilon.$$

Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T = [0, a] \subseteq \mathbb{R}$, be a constraint multifunction and $F : GrK \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = GrK$, be an orientor field (i.e. multivalued vector field). Consider the multivalued Cauchy problem as follows:

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{a.e. on } T, \\ y(t_0) = y_0. \end{cases} \tag{2.1}$$

By a *viable solution* $y(\cdot)$ to (2.1) we mean an absolutely continuous function $y : [t_0, a] \to \mathbb{R}^n$ that satisfies the inclusion almost everywhere, the initial condition and $y(t) \in K(t)$ for $t \in [t_0, a]$.

Definition 2.1. [17] *Let X, Y be normed spaces and $F : X \twoheadrightarrow Y$ be a set-valued map*. The contingent derivative $DF(x_0, y_0)$ *at $x_0 \in X$ and $y_0 \in F(x_0)$ is the set-valued map from X to Y defined by*

$$Gr(DF(x_0, y_0)) := T_{Gr(F)}(x_0, y_0),$$

where $T_C(x)$ *is the* contingent cone (*the "Bouligand cone"*) *to* C *at* x *and is defined by*

$$T_C(x) = \left\{ w \in X : \liminf_{t \downarrow 0} \frac{\text{dist}(x + tw, C)}{t} = 0 \right\}.$$

In other words,

$$v_0 \in DF(x_0, y_0)(u_0) \iff (u_0, v_0) \in T_{Gr(F)}(x_0, y_0).$$

Equivalently, we can write:

$$v_0 \in DF(x_0, y_0)(u_0) \iff \liminf_{h \to 0^+ u \to u_0} \text{dist} \left(v_0, \frac{F(x_0 + hu) - y_0}{h} \right) = 0.$$

When $F := f$ is single-valued, we set $Df(x) := Df(x, f(x))$.

For F locally Lipschitz, the definition of the contingent derivative reduces to the following (see, e.g., [[18]]):

$$v_0 \in DF(x_0, y_0)(u_0) \iff \liminf_{h \to 0^+} \text{dist} \left(v_0, \frac{F(x_0 + hu_0) - y_0}{h} \right) = 0.$$

Definition 2.2. [[12]] *Let* X *and* Y *be metric spaces. A set-valued map* $F : X \twoheadrightarrow Y$ *is* Cellina continuously approximable (abbreviated CCA) *if for every compact subset* K *of* X

(1) $Gr(F\,|_K)$ *is compact*;

(2) *there exists a sequence* $\{f_j\}_{j=1}^{\infty}$ *of single-valued continuous maps* $f_j : K \to Y$ *such that* $f_j \xrightarrow{gr} F\,|_K$.

We use $CCA(X, Y)$ to denote the set of all CCA set-valued maps from X to Y.

When $f : X \to Y$ is a single-valued map, then f belongs to $CCA(X, Y)$ if and only if f is continuous.

The CCA property of set-valued maps is strongly related to the following definition of directional generalized differential quotients (abbr. GDQs).

Definition 2.3. [[11]] *Let* $m, n \in \mathbb{N}$, $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ *be a set-valued map*, $\overline{x} \in \mathbb{R}^m$, $\overline{y} \in \mathbb{R}^n$, $\overline{y} \in F(\overline{x})$ *and let* Λ *be a nonempty compact subset of* $\mathbb{R}^{n \times m}$ (*then an element of* Λ *is an* $n \times m$ *matrix*). *Let* S *be a subset of* $\mathbb{R}^m$. *We say that* Λ *is a* generalized differential quotient (GDQ) of F at $(\overline{x}, \overline{y})$ in the direction S, *and write* $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ *if for every positive real number* δ *there exist* U, G *such that*

(1) U *is a compact neighbourhood of 0 in* $\mathbb{R}^m$ *and* $U \cap S$ *is compact*;

(2) G *is a CCA set-valued map from* $\overline{x} + U \cap S$ *to the* δ*-neighbourhood* Λ^δ *of* Λ *in* $\mathbb{R}^{n \times m}$;

(3) $G(x) \cdot (x - \overline{x}) \subseteq F(x) - \overline{y}$ *for every* $x - \overline{x} \in U \cap S$.

The concept of GDQs is a generalization of the classical derivative. In particular, if $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\overline{x} \in \mathbb{R}^m$, then $f'(x) \in GDQ(f; \overline{x}, f(\overline{x}); \mathbb{R}^m)$.

Observe that GDQs are not unique. If $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$, then for any compact overset Λ' of Λ also $\Lambda' \in GDQ(F; \overline{x}, \overline{y}; S)$.

We say that $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ is *GDQ-differentiable at* $(\overline{x}, \overline{y})$ in the direction S if there exists at least one $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$.

Definition 2.4. [16] *Let F be GDQ-differentiable at $(\overline{x}, \overline{y})$ in the direction S. A* minimal GDQ *of F at $(\overline{x}, \overline{y})$ in the direction S is a minimal element of the set $GDQ(F; \overline{x}, \overline{y}; S)$ (minimal in the sense of inclusions of sets).*

Theorem 2.5. [14] *If the set of all GDQs of a set-valued map F at $(\overline{x}, \overline{y})$ in the direction S is not empty, then there exists in this set at least one minimal GDQ at the same point and in the same direction.*

As we can have more than one minimal GDQ, we introduce the following concept of SGDQ. Let $SGDQ(K; t, y; S)$ denote the closure of the union of all minimal GDQs of K at $(t, y) \in GrK$ in the direction S.

Example 2.6. Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map such that

$$F(x) = \begin{cases} [-|x|, |x|] & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

One can show that any singleton $\{a\}$ for $a \in [-1, 1]$ is a minimal GDQ of F at $(0, 0)$. Then we compute $SGDQ(K; 0, 0; \mathbb{R}) = [-1, 1]$.

In order to give an idea what are relations between SGDQ and the contingent derivative, we present the following results.

Theorem 2.7. [15] *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$, $Do(F) = T \subseteq \mathbb{R}$, be a set-valued map and $y \in F(t)$. Then*

$$\Lambda \in \min GDQ(F; t, y; \mathbb{R}_+) \ \Rightarrow \ \Lambda \subseteq DF(t, y)(1).$$

Remark 2.8. Similarly, one can show that

$$\Lambda \in \min GDQ(F; t, y; \mathbb{R}_-) \ \Rightarrow \ \Lambda \subseteq DF(t, y)(-1).$$

Corollary 2.9. *Under assumptions of Theorem* 2.7 *we have the following inclusion*:

$$SGDQ(F; t, y; \mathbb{R}_+) \subseteq DF(t, y)(1).$$

Corollary 2.10. *Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$. If F is GDQ-differentiable at the point (x, y) in the direction of $\mathbb{R}_+$ ($\mathbb{R}_-$), then there exists the contingent derivative $DF(t, y)(1)$ ($DF(t, y)(-1)$).*

The next example shows that the contingent derivative in general is larger than the closure of the union of minimal GDQs.

Example 2.11. Consider a set-valued map $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined as follows:

$$K(t) = \begin{cases} \{t - \frac{1}{n} \mid n \in \mathbb{N},\ n > \frac{1}{t}\} \cup \{t\} & \text{if } t \neq 0, \\ \{0\} & \text{if } t = 0. \end{cases}$$

Then $DK(0, 0)(1) = [0, 1]$, while $SGDQ(K; 0, 0; \mathbb{R}_+) = \{1\}$.

3. A COMPARISON OF SOME THEOREMS ON VIABILITY

In this section we present and compare some theorems on the viability of differential inclusions. The first viability result is due to Nagumo [1]. His theorem concerns single-valued continuous, time-independent f and closed time-independent constraints $K(t) \equiv K_0$, and gives a necessary and sufficient condition for the existence of a solution to the problem. Namely,

Theorem 3.1. [1] *Let K_0 be a closed subset of $\mathbb{R}^n$ and let $f : K_0 \to \mathbb{R}^n$ be a continuous, bounded map. A necessary and sufficient condition for a differential equation $\dot{y} = f(y)$ to have a viable solution for any initial condition $y_0 \in K_0$ is*

$$\forall\, y \in K_0, \quad f(y) \in T_{K_0}(y).$$

An easy generalization of the Nagumo result, in the case of a time-independent set-valued map, i.e. $F(t, y) = F(y)$, is the following theorem which can be found in [18].

Theorem 3.2. [18] *Let K_0 be a closed subset of $\mathbb{R}^n$ and let a multifunction $F : K_0 \twoheadrightarrow \mathbb{R}^n$ be bounded, u.s.c. with closed convex values. Then the Cauchy problem*

$$\begin{cases} \dot{y} \in F(y), \\ y(0) = y_0, \quad y_0 \in K_0 \end{cases} \tag{3.1}$$

has a solution on $\mathbb{R}_+$ for every $y_0 \in K_0$ if and only if

$$F(y) \cap T_{K_0}(y) \neq \emptyset \text{ on } K_0.$$

The above theorem generalizes Nagumo's result replacing a differential equation by an inclusion. Thus, if F is single-valued, we get Nagumo Theorem as a consequence. One can see that the tangential condition from Nagumo Theorem

$$\forall\, x \in K_0,\ \ f(x) \in T_{K_0}(x)$$

is replaced in the last theorem by the following:

$$F(y) \cap T_{K_0}(y) \neq \emptyset \text{ on } K_0$$

(it is due to convex values of F; in a nonconvex case one has to assume $F(y) \subset T_{K_0}(y)$ on K_0, see, e.g., [[18]]). We call the last condition *the tangential condition*. The tangential condition will also appear in other theorems in the sequel (it may be slightly changed) with an exception of our theorem, where we use GDQs instead of the contingent derivative.

Going further, one wants to give sufficient conditions guaranteeing the existence of a trajectory of an orientor field remaining in time-dependent constraints $K(t)$. This leads to a generalization of Theorem 3.2 by Bothe [[4]] (in 1992). He gives sufficient conditions assuming that for all t and all $y \in K(t)$ the contingent derivative $DK(t,y)(1)$ is nonempty and contains $F(t,y)$ for almost all t, where K is left u.s.c. and F is measurable with respect to t and u.s.c. with respect to y. Under these conditions there exists a viable solution to the Cauchy problem (2.1).

Theorem 3.3. [[4]] *Let $T = [0,a] \subset \mathbb{R}$ and $K : T \twoheadrightarrow \mathbb{R}^n$, $Do(K) = T$, be a left u.s.c. set-valued map with closed convex values such that the interior of $K(t)$ is not empty a.e. in T. Let $F : GrK \twoheadrightarrow \mathbb{R}^n$, $Do(F) = GrK$ have closed convex values, $F(\cdot,y)$ be measurable, $F(t,\cdot)$ be u.s.c., and $||F(t,y)|| \leq \alpha(t)(1+|y|)$ on GrK with $\alpha \in L^1(T)$. Finally, let*

$$\begin{aligned} &(\{1\} \times F(t,y)) \subset T_{GrK}(t,y) \quad \text{for all } t \in [0,a)\backslash N, y \in K(t),\\ &(\{1\} \times \mathbb{R}^n) \cap T_{GrK}(t,y) \neq \emptyset \quad \text{for all } t \in N, y \in K(t), \end{aligned} \tag{3.2}$$

where $N \subset T$ and $\lambda(N) = 0$. Then (2.1) *has a viable solution.*

Another work that concerns the same problem is the article, from 1995, of Frankowska et al. [[6]]. Changing conditions on K and the tangential condition, the authors give sufficient conditions guaranteeing that Cauchy problem (2.1) has a viable solution.

Theorem 3.4. [[6]] *Let K be a left absolutely continuous multifunction with closed values on $T = [0,a]$, $F : GrK \twoheadrightarrow \mathbb{R}^n$ have closed, convex values and let F satisfy $||F(t,y)|| \leq \alpha(t)$. Let $F(\cdot,y)$ be measurable for any $y \in Im(K)$ and let for almost all $t \in T$ and for every $y \in K(t)$*

$$\forall \beta > 0, \quad DK(t,y)(1) \cap \liminf_{h\to 0^+} \frac{1}{h}\int_t^{t+h} F(s, y+\beta B)\mathrm{d}s \neq \emptyset. \tag{3.3}$$

Then for every $t_0 \in T$ *and* $y_0 \in K(t_0)$ *there exists a solution* $y(\cdot)$ *to* (2.1) *defined on* $[t_0, a]$ *and satisfying* $y(t) \in K(t)$ *for every* $t \in [t_0, a]$.

The next result which deals with problem (2.1) is the theorem presented by Hu and Papageorgiou [10]. The authors give sufficient conditions assuming that K is u.s.c. and the orientor field F defined on the graph of K is jointly measurable and u.s.c. w.r. to y.

Theorem 3.5. [10] *Let* $K : T \twoheadrightarrow \mathbb{R}^n$ *with nonempty closed values be an u.s.c. set-valued map such that for almost all* $s \in (0, a)$ *and for all* $y \in K(s)$, *there is a continuous map* $t \to y(t)$ *on* $[0, s]$ *or* $[s, a]$ *such that* $y(s) = y$, $DK(\cdot, y(\cdot))$ *is closed at* s. *Let* $F : GrK \twoheadrightarrow \mathbb{R}^n$ *with closed convex nonempty values satisfy*
(i) F *is jointly measurable*;
(ii) $||F(t, y)|| \leq \alpha(t)(1 + ||y||)$ *a.e. on* T *with* $\alpha \in L^1(T)$;
(iii) $y \mapsto F(t, y)$ *is u.s.c.*
Finally, let

$$F(t, y) \cap DK(t, y)(1) \neq \emptyset$$

for every $[t, y] \in GrK$. *Then for every* $(t_0, y_0) \in GrK$ *the multivalued Cauchy problem* (2.1) *has a viable solution* $y : [t_0, a] \to \mathbb{R}^n$ *which is an absolutely continuous function.*

All above theorems on the viability of differential inclusions use the contingent derivative as a main tool in the tangential condition. However, only its value at $t = 1$ is used in this condition. Our idea was to use for the first time in viability theory another tool of differentiation: GDQs. One of the advantages of GDQs over the contingent derivative is that, as Example 2.6 shows, SGDQ contains all important directions while the contingent derivative has, besides these directions, some superfluous elements.

Theorem 3.6. [15] *Consider the multivalued Cauchy problem* (2.1). *Assume that* $K : T \twoheadrightarrow \mathbb{R}^n$, *where* $T = [0, a]$, *is a left u.s.c. multifunction with nonempty closed values such that for all* $(t, y) \in GrK$, *where* $t \in [0, a)$, K *is GDQ differentiable at* (t, y) *in the direction of* $\mathbb{R}_+$ *and for every* $\varepsilon > 0$ *there exists* $T_\varepsilon \subseteq T$ *such that* $\lambda(T \backslash T_\varepsilon) < \varepsilon$ *and the map* $(t, y) \mapsto SGDQ(K; t, y; \mathbb{R}_+)$ *is u.s.c. on* $(T_\varepsilon \times \mathbb{R}^n) \cap GrK$. *Let* $F : GrK \twoheadrightarrow \mathbb{R}^n$ *with nonempty closed convex values satisfy*
(a) $\forall\, \gamma(\cdot)$*-measurable* $t \mapsto F(t, \gamma(t))$ *is measurable*;
(b) $y \mapsto F(t, y)$ *is u.s.c. for every* $t \in [0, a]$;
(c) $||F(t, y)|| \leq \alpha(t)(1 + ||y||)$ *a.e. on* T *with* $\alpha \in L^1(T)$.
Additionally, assume that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ *for almost every* t, $(t, y) \in GrK$. *Then for* $y_0 \in K(t_0)$, *problem* (2.1) *has a solution.*

Proof. For the proof see [15]. □

Remark 3.7. In the above theorem the assumption on K to be GDQ-differentiable at every $(t, y) \in (T \times \mathbb{R}^n) \cap GrK$ is important. Indeed, let $T = [0, 1]$ and $y : T \to \mathbb{R}$ be the Cantor function. Thus y is continuous, nondecreasing, $\dot{y}(t) = 0$ for almost every $t \in T$, $y(T) = T$, and $y(\cdot)$ is not absolutely continuous. Let $K(t) = \{y(t)\}$ and $F(t, y) = \{0\}$. Then the tangential condition is satisfied for all $t \in T \backslash N$, $\lambda(N) = 0$, such that $\dot{y}(t) = 0$. However, problem (2.1) has no solution, since $0 \notin K(t)$ for $t > 0$.

The main goal of this section, besides presenting a few viability theorems, is to compare them with our Theorem 3.6. We compare only those theorems that deal with a time-dependent constrain multifunction.

3.1. Tangential condition

The main difference between Theorem 3.6 and others is, as it was mentioned before, that we use GDQs theory instead of the contingent derivative to formulate the tangential condition for problem (2.1). Namely,

$$F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset \text{ for almost every } t, (t, y) \in GrK. \tag{3.4}$$

In other theorems this condition is formulated as follows. In Theorem 3.3 there is the strongest tangential condition,

$$\begin{aligned} &(\{1\} \times F(t, y)) \subset T_{GrK}(t, y) \quad \text{for all } t \in [0, a) \backslash N, y \in K(t), \\ &(\{1\} \times \mathbb{R}^n) \cap T_{GrK}(t, y) \neq \emptyset \quad \text{for all } t \in N, y \in K(t). \end{aligned} \tag{3.5}$$

In Theorem 3.5 the authors put the “classical” tangential condition,

$$\begin{gathered} \text{“for almost every } t \in [0, a] \text{ and for every } y \in K(t), \\ F(t, y) \cap DK(t, y)(1) \neq \emptyset.\text{”} \end{gathered} \tag{3.6}$$

Finally, in Theorem 3.4 the authors assume the following:

$$\begin{gathered} \text{“for almost all } t \in T, \text{ for every } y \in K(t) \\ \forall \beta > 0, \quad DK(t, y)(1) \cap \liminf_{h \to 0^+} \tfrac{1}{h} \int_t^{t+h} F(s, y + \beta B) ds \neq \emptyset.\text{”} \end{gathered} \tag{3.7}$$

Remark 3.8. There are the following relations among the above conditions:

$$\begin{aligned} (3.4) &\Rightarrow (3.6), \\ (3.5) &\Rightarrow (3.6), \\ (3.7) &\Rightarrow (3.6). \end{aligned}$$

Indeed, in (3.4) we intersect F with, possibly smaller than the contingent derivative, the closed union of minimal GDQs of K. In (3.7), the authors intersect $DF(t, y)(1)$

with, smaller than $F(t,y)$, the set $\liminf_{h\to 0^+}\frac{1}{h}\int_t^{t+h} F(s, y+\beta B)\mathrm{d}s$. The conditions (3.4), (3.5), and (3.7) are not comparable.

Remark 3.9. The authors of Theorem 3.4 proved also another viability theorem in [6]. They showed that for $K : [0,a] \twoheadrightarrow \mathbb{R}^n$ left absolutely continuous and $F : [0,a]\times\mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ with closed convex values, continuous w.r. to y, measurable w.r. to t and bounded as in Theorem 3.5, the tangential condition (3.6) is equivalent to the existence of a viable solution to problem (2.1). As we can see, they weaken the tangential condition making stronger the continuity assumption on F.

3.2. Assumptions on K

There are different assumptions on a constraint multifunction K in theorems presented above. In Theorem 3.3, the author requires K to be *a left u.s.c. set-valued map with closed convex values such that the interior of $K(t)$ is not empty a.e. in T*; in Theorem 3.4, K has to be *a left absolutely continuous multifunction with closed values on $T=[0,a]$*; in Theorem 3.5 the authors want K to be *with nonempty closed values; be an u.s.c. set-valued map such that for almost all $s\in(0,a)$ and for all $y\in K(s)$, there is a continuous map $t\to y(t)$ on $[0,s]$ or $[s,a]$ such that $y(s)=y$, $DK(\cdot,y(\cdot))$ is closed at s*; and in our Theorem 3.6, K is required to be *a left u.s.c. multifunction with nonempty closed values such that for all $(t,y)\in GrK$, where $t\in[0,a)$, K is GDQ differentiable at (t,y) in the direction of $\mathbb{R}_+$*. The weakest continuity assumption is left upper semicontinuity. Indeed, it is obvious that every u.s.c. set-valued map with compact values is left u.s.c., but the converse is not true as the following example shows.

Example 3.10. Consider the set-valued map $F:\mathbb{R}\twoheadrightarrow\mathbb{R}$ defined as follows:

$$F(x)=\begin{cases}0 & \text{if } x\le 0,\\ [0,1] & \text{if } x>0.\end{cases}$$

Then it is left u.s.c. at 0, but not u.s.c. at this point.

One can also observe that

Remark 3.11. If a set-valued map $K:T\twoheadrightarrow\mathbb{R}^n$ with values contained in a compact set C is left absolutely continuous on T, then it is left upper semicontinuous on T. Indeed, it is enough to consider for every $\tau_0\in T$ one of intervals from the definition of left absolutely continuity instead of the union of intervals, i.e.:

$$\tau_0-t_0\le\delta \;\Rightarrow\; \triangle(K(t_0)\cap P, K(\tau_0))\le\epsilon,$$

which implies left upper semicontinuity of K at every $\tau_0\in T$ if we put $P=C$.

We also assume that a constraint multifunction K is GDQ-differentiable at every $(t,y)\in GrK$. In [13] the following proposition is proved.

Proposition 3.12. *If K is GDQ-differentiable at (s, y) in the direction $\mathbb{R}_+$, then there exists a map $\gamma : [s, s + \delta] \rightarrow \mathbb{R}^n$ such that $\gamma(t) \in K(t)$ for $t \in [s, s + \delta]$, $\gamma(s) = y$ and γ is measurable and continuous at s.*

By the above proposition, if we assume GDQ-differentiability of K at every point (t, y), there exists a measurable map $\gamma(\cdot)$ starting at this point. Hence we do not assume additionally the existence of such a map, as it is done in Theorem 3.5.

3.3. Assumptions on F

In all theorems presented above F is closed, convex-valued. But there are different hypotheses on the measurability and bound of F. In Theorem 3.3 and Theorem 3.4, F is required to be measurable with respect to t. In Theorem 3.5, F has to be jointly measurable and in our Theorem 3.6 the following is required to be fulfilled: $\forall\ \gamma(\cdot)$*-measurable*, $t \mapsto\!\!\!\rightarrow F(t, \gamma(t))$ *is measurable*. The strongest assumption is joint measurability of F in Theorem 3.5. This assumption implies ours. In turn, our assumption implies the weakest one, i.e. the measurability of F w.r. to t as it is requested in Theorem 3.3 and Theorem 3.4. The price of making weaker assumptions on the measurability of F is the necessity of putting stronger assumptions on bound of F (or on something else). And so, in Theorem 3.4, one can find the strongest bound hypothesis, $||F(t, y)|| \leq \alpha(t)$ with $\alpha \in L^1(T)$, in Theorem 3.3 there is the strongest tangential condition and K is required to have nonempty interior. Besides Theorem 3.4, in all other presented theorems there is a weaker hypothesis on a bound of F, namely $||F(t, y)|| \leq \alpha(t)(1 + ||y||)$ a.e. on T with $\alpha \in L^1(T)$. To see that these two assumptions are not equivalent, let us show the following example:

Example 3.13. Consider the set-valued map $F : [0, 1] \times \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined as $F(t, y) = [-\frac{1}{\sqrt{t}}(1 + y), \frac{1}{\sqrt{t}}(1 + y)]$. Then it is easy to see that $||F(t, y)|| \leq \alpha(t)(1 + ||y||)$, where $\alpha(t) = \frac{1}{\sqrt{t}}$ and so $\alpha \in L^1(T)$, but there is no such a map $\alpha \in L^1(T)$ that $||F(t, y)|| \leq \alpha(t)$.

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Ülevaade vitaalsustulemustest

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Saadud tulemusi, milles on esmakordselt kasutatud vitaalsusteooria üldistatud diferentsiaalfaktoreid, on võrreldud varasematega. Võrdlust on illustreeritud näidetega.