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Properties of 2-dimensional time-like ruled surfaces in the Minkowski space $\mathbb{R}_1^n$

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Abstract. Some results, which are well known for the ruled surfaces in the Euclidean space $\mathbb{R}^n$, are generalized here to the case of $\mathbb{R}_1^n$. In particular, it is shown that a time-like ruled surface in $\mathbb{R}_1^n$ is developable if and only if it has zero Gaussian curvature; moreover, it is then minimal if and only if it is totally geodesic.

Key words: ruled surfaces, Minkowski spaces.

1. INTRODUCTION

We shall assume throughout the paper that all manifolds, maps, vector fields, etc. are differentiable of class C^∞. First of all, we give some properties of a general submanifold M of the Minkowski n-space $\mathbb{R}_1^n$. Suppose that $\bar{D}$ is the Levi-Civita connection of $\mathbb{R}_1^n$ and D is the Levi-Civita connection of M. Then, if X, Y are the vector fields of M and if V is the second fundamental tensor of M, we may decompose $\bar{D}_X Y$ into a tangential and a normal component:

$$\bar{D}_X Y = D_X Y + V(X, Y). \tag{1}$$

Equation (1) is called Gauss equation [1]. If ξ is any normal vector field on M, we find the Weingarten equation by decomposing $\bar{D}_X \xi$ into a tangential and a normal component:

$$\bar{D}_X \xi = -A_\xi(X) + D_X^\perp \xi, \tag{2}$$

where A_ξ determines at each point a self-adjoint linear map and $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. We use the same notation A_ξ for the linear map and the matrix of the linear map [2].

A normal vector field ξ is called parallel in the normal bundle $\chi^\perp(M)$ if $D_X^\perp \xi = 0$ for each vector field X. If η is a normal unit vector at the point $p \in M$, then

$$G(p, \eta) = \det A_\eta$$

is the Lipschitz–Killing curvature of M at p in the direction η [3].

Let V be the second fundamental tensor of M. If

$$V(X, X) = 0$$

for X in the tangent bundle $\chi(M)$, then X is called an asymptotic vector field on M. If

$$V(X, Y) = 0$$

for all $X, Y \in \chi(M)$, then M is totally geodesic [4].

Suppose that $X, Y \in \chi(M)$, while $\xi \in \chi^\perp(M)$. If the standard metric tensor of $\mathbb{R}_1^n$ is denoted by $\langle , \rangle$, then we have

$$< \bar{D}_X Y, \xi > = < V(X, Y), \xi >$$

and

$$< \bar{D}_X Y, \xi > = < A_\xi(X), Y > .$$

From the above equations we obtain

$$< V(X, Y), \xi > = < A_\xi(X), Y > .$$

If $\xi_1, \xi_2, \ldots, \xi_{n-2}$ constitute an orthonormal base field of the normal bundle $\chi^\perp(M)$, then we set

$$< V(X, Y), \xi_j > = V_j(X, Y)$$

or

$$V(X, Y) = \sum_{j=1}^{n-2} V_j(X, Y) \xi_j .$$

The mean curvature vector H of M at the point p is given by

$$H = \sum_{j=1}^{n-2} \frac{\operatorname{tr} A_{\xi_j}}{2} \xi_j .$$

Here $\|H\|$ is the mean curvature. If $H = 0$ at each point p of M, then M is said to be minimal [5].

2. TWO-DIMENSIONAL TIME-LIKE RULED SURFACE IN $\mathbb{R}_1^n$

A time-like ruled surface M in $\mathbb{R}_1^n$ is generated by time-like line l with unit direction time-like vector $e(s)$ along a space-like curve α. For this ruled surface

$$\psi(s,v) = \alpha(s) + ve(s)$$

is a parameterization. Throughout this paper, α is supposed to be an orthogonal trajectory of the generators.

Let $\{e, e_1\}$ be an orthonormal base field of $\chi(M)$, so that

$$\langle e, e\rangle = -1, \quad \langle e_1, e_1\rangle = 1, \quad \langle e, e_1\rangle = 0. \tag{3}$$

Let us denote the Levi-Civita connection of the Minkowski space $\mathbb{R}_1^n$ by $\bar{D}$. Because the lines in $\mathbb{R}_1^n$ are geodesics, we have

$$\bar{D}_e e = 0. \tag{4}$$

If we substitute this equation into Eq. (1), we get

$$V(e,e) = 0.$$

Considering Eq. (3), we can easily see that $\bar{D}_e e_1 \perp e$ and $\bar{D}_e e_1 \perp e_1$. This implies $\bar{D}_e e_1 \in \chi^{\perp}(M)$. Therefore,

$$\bar{D}_e e_1 = V(e, e_1). \tag{5}$$

Let $\{\xi_1, \xi_2, \ldots, \xi_{n-2}\}$ be vector fields, which constitute an orthonormal base $T_M{}^{\perp}(p)$. Then $\{e_1, e_2, \xi_1, \xi_2, \ldots, \xi_{n-2}\}$ is a base of $T_{\mathbb{R}_1^n}(p)$ at $p \in \mathbb{R}_1^n$. Together with (2) we can write

$$\begin{aligned} \bar{D}_e \xi_j &= a_{11}^j e + a_{12}^j e_1 + \sum_{i=1}^{n-2} b_{1i}^j \xi_i, \quad 1 \le j \le n-2, \\ \bar{D}_{e_1} \xi_j &= a_{21}^j e + a_{22}^j e_1 + \sum_{i=1}^{n-2} b_{2i}^j \xi_i, \quad 1 \le j \le n-2. \end{aligned} \tag{6}$$

Comparing Eqs. (6) with Eq. (4) leads us to

$$a_{21}^j = -a_{12}^j, \quad a_{11}^j = 0, \quad 1 \le j \le n-2.$$

Moreover, we find

$$A_{\xi_j} = \begin{bmatrix} 0 & a_{12}^j \\ -a_{12}^j & a_{22}^j \end{bmatrix}.$$

The matrix A_{ξ_j} corresponds to the shape operator of M and A_{ξ_j} is a symmetric matrix in the sense of Lorentz.

The Lipschitz–Killing curvature at $p \in M$ in the direction of ξ_j is given by

$$G(p, \xi_j) = -(a_{12}^j)^2. \tag{7}$$

If we use Eqs. (6), we see

$$a_{12}^j = \left\langle \bar{D}_e \xi_j, e_1 \right\rangle = -\left\langle \xi_j, \bar{D}_e e_1 \right\rangle \tag{8}$$

and from (5) and with (8) we get

$$\bar{D}_e e_1 = V(e, e_1) = \sum_{j=1}^{n-2} \left\langle \xi_j, \bar{D}_e e_1 \right\rangle \xi_j = -\sum_{j=1}^{n-2} a_{12}^j \xi_j. \tag{9}$$

In addition, the Gaussian curvature of M denoted by G is expressed by (see [6])

$$G = -\left\langle \bar{D}_e e_1, \bar{D}_e e_1 \right\rangle.$$

With the elements of A_{ξ_j}, the Gaussian curvature of M is

$$G = -\sum_{j=1}^{n-2} (a_{12}^j)^2. \tag{10}$$

Hence, from Eqs. (7) and (10) we obtain

$$G(p) = \sum_{j=1}^{n-2} G(p, \xi_j), \quad p \in M. \tag{11}$$

Moreover, if the Lipschitz–Killing curvature $G(p, \xi_j)$ is equal to zero at $p \in M$ for each j, $1 \le j \le n-2$, then Gaussian curvature $G(p)$ will be zero. This shows that M is an intrinsically developable surface, i.e., locally isometric to open sets of Minkowski plane. Conversely, if M is intrinsically developable, then $G(p, \xi_j)$ is equal to zero at $p \in M$ for each j, $1 \le j \le n-2$. Therefore, one may say that M is intrinsically developable if and only if the Lipschitz–Killing curvature is zero at each point [6].

In [6] it is shown that the mean curvature vector H of the time-like ruled surface M is

$$H = \frac{1}{2} V(e_1, e_1).$$

Theorem 1. *Let M be a 2-dimensional time-like ruled surface in $\mathbb{R}_1^n$. Then the generators of M are asymptotic and geodesic of M.*

Proof. Since the generators are the geodesics of $\mathbb{R}_1^n$, we write

$$\bar{D}_e e = 0.$$

If we set this into the Gauss equation, we find

$$D_e e + V(e,e) = 0 \text{ or } D_e e = -V(e,e).$$

Since $D_e e \in \chi(M)$ and $V(e,e) \in \chi^{\perp}(M)$ we reach $D_e e = 0$ and $V(e,e) = 0.$ This completes the proof of the theorem.

Definition 1. *Let M be a 2-dimensional time-like ruled surface in $\mathbb{R}_1^n$. If the tangent planes of M are constant along the generators of M, then M is called developable* [7].

Theorem 2. *Let M be a 2-dimensional time-like ruled surface in $\mathbb{R}_1^n$. Then M is developable and minimal if and only if M is totally geodesic.*

Proof. Assume that M is developable and minimal. If we have $X = ae + be_1$ and $Y = ce + de_1$ in $\chi(M)$, then

$$V(X,Y) = acV(e,e) + (ad+bc)V(e,e_1) + bdV(e_1,e_1).$$

Since the lines in $\mathbb{R}_1^n$ are geodesic and M is minimal, we find that $V(e,e) = V(e_1,e_1) = 0$. Moreover, $\bar{D}_e e_1$ is equal to zero since M is developable. From Eq. (5) we get

$$V(e,e_1) = 0.$$

Hence, we have $V(X,Y) = 0$ for all $X, Y \in \chi(M)$. This means that M is totally geodesic.

Conversely, assume that $V(X,Y) = 0$ for all $X, Y \in \chi(M)$. Therefore we have the relations

$$V(e,e) = 0, \quad V(e_1,e_1) = 0, \quad V(e,e_1) = 0.$$

By using these equations and Eq. (9) we find $\bar{D}_e e_1 = 0.$ This shows that M is totally developable. Moreover, $V(e_1,e_1) = 0$ implies that $H = 0.$ This means that M is minimal.

3. SOME CHARACTERIZATIONS FOR 2-DIMENSIONAL TIME-LIKE RULED DEVELOPABLE SURFACES IN THE MINKOWSKI SPACE $\mathbb{R}_1^n$

Let $\{e, e_1\}$ be an orthonormal basis of $\chi(M)$, as above, and $\{\xi_1, \xi_2, \ldots, \xi_{n-2}\}$ be an orthonormal basis of $\chi^{\perp}(M)$. We give covariant derivative equations of the orthonormal basis $\{e_1, e_2, \xi_1, \xi_2, \ldots, \xi_{n-2}\}$ of $\chi(\mathbb{R}_1^n)$ as follows:

$$\begin{aligned}
&\bar{D}_{e_1} e = c_{11}e + c_{12}e_1 + c_{13}\xi_1 + \ldots + c_{1n}\xi_{n-2},\\
&\bar{D}_{e_1} e_1 = c_{21}e + c_{22}e_1 + c_{23}\xi_1 + \ldots + c_{2n}\xi_{n-2},\\
&\bar{D}_{e_1} \xi_1 = c_{31}e + c_{32}e_1 + c_{33}\xi_1 + \ldots + c_{3n}\xi_{n-2},\\
&\vdots\\
&\bar{D}_{e_1} \xi_{n-2} = c_{n1}e + c_{n2}e_1 + c_{n3}\xi_1 + \ldots + c_{nn}\xi_{n-2}.
\end{aligned} \tag{12}$$

If we calculate the coefficient c_{st}, $1 \le s,t \le n$, and write Eqs. (12) in the matrix form, we obtain

$$\begin{bmatrix} \bar{D}_{e_1} e \\ \bar{D}_{e_1} e_1 \\ \bar{D}_{e_1} \xi_1 \\ \vdots \\ \bar{D}_{e_1} \xi_{n-2} \end{bmatrix} = \begin{bmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{12} & 0 & c_{23} & \cdots & c_{2n} \\ c_{13} & -c_{23} & 0 & \cdots & c_{3n} \\ \vdots & & & & \\ c_{1n} & -c_{2n} & -c_{3n} & \cdots & 0 \end{bmatrix} \begin{bmatrix} e \\ e_1 \\ \xi_1 \\ \vdots \\ \xi_{n-2} \end{bmatrix}. \tag{13}$$

By using Eq. (13) we can prove the following theorem.

Theorem 3. *Let* M *be a* 2*-dimensional time-like ruled surface in* $\mathbb{R}_1^n$, *and* $\{e, e_1\}$ *be an orthonormal base field of the tangential bundle* $\chi(M)$, *as above. In this case, the following propositions are equivalent*:

(i) M *is developable,*

(ii) *the Lipschitz–Killing curvature* $G(p,\xi_j)$, $1 \le j \le n-2$, *is equal to zero,*

(iii) *the Gaussian curvature* G *is equal to zero,*

(iv) *in Eq.* (13), c_{2k}, $3 \le k \le n$, *is equal to zero,*

(v) $A_{\xi_j}(e)$ *is equal to zero,*

(vi) $\bar{D}_e e_1$ *is an element of* $\chi(M)$.

Proof. (i) $\Rightarrow$ (ii): Assume that M is developable, i.e., $\bar{D}_e e_1 = 0$. Equation (7) says that the Lipschitz–Killing curvature at the point p in the direction of ξ_j is given by

$$G(p,\xi_j) = -(a_{12}^j(p))^2, \quad 1 \le j \le n-2. \tag{14}$$

Due to $\bar{D}_e e_1 = 0$ and from Eq. (9)

$$\bar{D}_e e_1 = -\sum_{j=1}^{n-2} (a_{12}^j)\xi_j = 0. \tag{15}$$

Considering Eqs. (14) and (15) yields

$$G(p,\xi_j) = 0, \quad 1 \le j \le n-2.$$

(ii) $\Rightarrow$ (iii): This follows directly from Eq. (11), as shown above.

(iii) $\Rightarrow$ (iv): Assume that $G = 0$, $\forall p \in M$. From Eq. (10) we have $a_{12}^j = 0$, $1 \le j \le n-2$. Since $a_{21}^j = -a_{12}^j$ in (6), also $a_{12}^j = 0$. This means that $\bar{D}_{e_1}\xi_j$ has no component in the direction e. Hence, we see that $c_{2k} = 0$, $3 \le k \le n$, in Eqs. (12), due to (13).

(iv) $\Rightarrow$ (v): Suppose that $c_{2k} = 0$, $3 \le k \le n$, in Eqs. (12). This shows that $\bar{D}_e\xi_j$ has no component in the direction e. Thus we have $a_{12}^j = 0$, $1 \le j \le n-2$, in Eqs. (6).

Moreover, using Weingarten equation (2), we write

$$A_{\xi_j}(e) = 0, \quad 1 \le j \le n-2,$$

since $a_{11}^j = -\left\langle \bar{D}_e\xi_j, e\right\rangle = \left\langle \xi_j, \bar{D}_e e\right\rangle = 0.$

(v) $\Rightarrow$ (vi): Let $A_{\xi_j}(e)$ be equal to zero. Then, from Weingarten equation (2) we have $a_{11}^j = 0$, $a_{12}^j = 0$, $1 \le j \le n-2$. Since $\left\langle e, \xi_j\right\rangle = 0$ implies $\left\langle \bar{D}_e e_1, \xi_j\right\rangle = \left\langle e, \bar{D}_{e_1}\xi_j\right\rangle = -a_{12}^j$, we find

$$\left\langle \bar{D}_e e_1, \xi_j\right\rangle = 0.$$

From this equation we get

(vi) $\Rightarrow$ (i): Let $\bar{D}_e e_1$ be an element of $\chi(M)$. Then $\left\langle \bar{D}_e e_1, \xi_j\right\rangle$ will be equal to $-a_{12}^j$, $1 \le j \le n-2$, which is again equal to zero. On the other hand, $\left\langle e_1, e_1\right\rangle = 1$ implies that $\left\langle \bar{D}_e e_1, e_1\right\rangle = 0$ and $\left\langle e_1, e\right\rangle = 0$ implies that $\left\langle \bar{D}_e e_1, e\right\rangle = 0$. Thus $\bar{D}_e e_1 \in \chi^{\perp}(M)$.

Using Eq. (9), we get that $\bar{D}_e e_1 = 0$, since a_{12}^j, $1 \le j \le n-2$, is equal to zero. This means that the tangent planes of M are constant along the generator e of M, i.e., M is developable. This finishes the proof.

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Ajasarnaste kahemõõtmeliste joonpindade omadusi Minkowski ruumis $\mathbb{R}_1^n$

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Mõned tulemused, mis on hästi tuntud joonpindade puhul eukleidilises ruumis $\mathbb{R}^n$, on üldistatud siin ruumi $\mathbb{R}_1^n$ juhule. Nii on tõestatud, et ajasarnasel joonpinnal ruumis $\mathbb{R}_1^n$ on puutujatasand piki iga moodustajat konstantne siis ja ainult siis, kui Gaussi kõverus on null; lisaks sellele on taoline joonpind minimaalne siis ja ainult siis, kui ta on täielikult geodeetiline.