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The complete enumeration of binary perfect forms over the algebraic number field $\mathbb{Q}(\sqrt{6})$

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Abstract. A complete list of binary perfect forms with coefficients in $\mathbb{Q}(\sqrt{6})$ is presented and the theoretical background of Voronoï's algorithm of perfect quadratic forms with coefficients in a totally real normal algebraic number field is discussed.

Key words: perfect quadratic forms, Voronoï's algorithm, totally real normal algebraic number fields.

1. INTRODUCTION

In this paper we present the complete enumeration of binary perfect forms over real quadratic extension $\mathbb{Q}(\sqrt{6})$. It is a continuation of the work started by Ong [1], where complete lists of perfect forms over $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{5})$ are given. Ong studied the cases where the real quadratic field is the maximal totally real subfield of a cyclotomic field. The present paper considers a more general case, which appears to be unknown in the literature.

The perfect quadratic forms are also related to the description of a reduction domain of quadratic forms (see [2]). Koecher showed in [2] that a reduction domain is a union of perfect polyhedral cones, which are associated to perfect forms of the same rank. Until now, neither the explicit description of the reduction domain nor reduction algorithm for positive definite quadratic forms with algebraic coefficients have been published (except for unary quadratic forms over the real quadratic number field or totally real cyclic cubic fields [3]). Explicit results in the reduction theory of quadratic forms are interesting for people working in the field of the geometry of numbers and its applications, and this motivates the enumeration of perfect forms as well.

The perfect quadratic forms with rational coefficients are closely related to the extreme forms by the well-known Voronoï's theorem, thus those forms are also interesting for applications of the geometry of numbers. For instance, the significance of extreme quadratic forms relies on the interesting properties of the associated lattices in the real n-dimensional space $\mathbb{R}^n$. The lattices associated to critical forms represent the densest lattice packing of equal spheres in $\mathbb{R}^n$.

Unfortunately, the theory of perfect quadratic forms with algebraic coefficients is much less studied than the theory of rational perfect quadratic forms, therefore the results are important mainly in connection with the corresponding rational quadratic forms. For example, there are several constructions known for perfect forms with algebraic coefficients, which correspond to perfect forms with rational coefficients (see [4,5] and Chapter 7 in [6]).

Although the main result of the paper (Theorem 2) includes the complete enumeration of perfect binary quadratic forms with coefficient in $\mathbb{Q}(\sqrt{6})$, we also present some theoretical results such as

1. the construction of a unary perfect form with a coefficient in $\mathbb{Q}(\sqrt{D})$ for arbitrary square-free $D > 1$ (Theorem 1);
2. the use of field automorphisms in the application of Voronoï's algorithm (Propositions 2, 3 and Corollary 1).

The latter results are necessary for applying Voronoï's algorithm to perfect quadratic forms with coefficients in a normal extension of $\mathbb{Q}$.

The main result agrees with Prestel's [7] result of counting inequivalent elliptic fixed points under the action of the Hilbert modular group $\mathrm{SL}(2, \mathcal{O}_{\mathbb{Q}(\sqrt{6})})$ (see Section 4).

The rational quadratic forms $\mathrm{Tr}_{\mathbb{Q}(\sqrt{6})/\mathbb{Q}}(\phi_8)$ and $\mathrm{Tr}_{\mathbb{Q}(\sqrt{6})/\mathbb{Q}}(\phi_9)$ are also perfect. Moreover, they are critical quaternary quadratic forms with rational coefficients. But the rational quadratic form $\mathrm{Tr}_{\mathbb{Q}(\sqrt{6})/\mathbb{Q}}(f)$ is not perfect for any $f \in \{\phi_0, \ldots, \phi_7, \phi_{10}, \phi_{11}, \phi_{12}\}$ since the set of minimum vectors of f is not big enough (see Subsections 5.1–5.13).

2. DEFINITIONS AND VORONOÏ'S ALGORITHM

Let $\mathbb{K}$ be a totally real algebraic number field and let $\mathcal{O}_{\mathbb{K}}$ be the ring of algebraic integers in $\mathbb{K}$. Let $r = [\mathbb{K} \colon \mathbb{Q}]$ and $\sigma_1, \ldots, \sigma_r$ be the embeddings of $\mathbb{K}$ into $\mathbb{R}$.

Let f be a quadratic form over $\mathbb{K}$, i.e. $f(x) = f(x_1, \ldots, x_n) = \sum_{i,j} f_{ij} x_i x_j$ ($f_{ij} = f_{ji}$) with $f_{ij} \in \mathbb{K}$ for all i and j.

Definition 1. *A quadratic form $f(x) = \sum_{i,j} f_{ij} x_i x_j$ is called positive definite over $\mathbb{K}$ if $\sigma_k(f(x)) = \sum_{i,j} \sigma_k(f_{ij}) x_i x_j$ is positive definite for all $k = 1, \ldots, r$.*

Let $\omega_1, \ldots, \omega_r$ be a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$. If $f(x)$ is a positive definite quadratic form over $\mathbb{K}$ of rank n, then

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(x)) &= \sum_{k=1}^{r}\sum_{i,j=1}^{n}\sigma_k(f_{ij})(x_{i,1}\sigma_k(\omega_1)+\ldots+x_{i,r}\sigma_k(\omega_r)) \\
&\quad \times(x_{j,1}\sigma_k(\omega_1)+\ldots+x_{j,r}\sigma_k(\omega_r)) \\
&= \sum_{i,j=1}^{n}\sum_{l,m=1}^{r} x_{i,l}x_{j,m}\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f_{ij}\omega_l\omega_m) \qquad (f_{ij}=f_{ji}) \quad (1)
\end{aligned}$$

is a positive definite quadratic form over $\mathbb{Q}$ of rank rn.

We write $\mu(f)$ for the smallest positive value of $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(x))$ on $\mathcal{O}_{\mathbb{K}}^{\mathrm{rank}(f)}$, that is,

$$\mu(f)=\min\left\{\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(X))|X\in\mathcal{O}_{\mathbb{K}}^{\mathrm{rank}(f)}\setminus\{0\}\right\}.$$

If f is a positive definite quadratic form over $\mathbb{K}$, then we set

$$\mathcal{M}(f)=\left\{X\in\mathcal{O}_{\mathbb{K}}^{\mathrm{rank}(f)}|\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(X))=\mu(f)\right\}.$$

The set $\mathcal{M}(f)$ is called the set of minimum vectors of f. Throughout this paper we do not distinguish between m and $-m$ for each $m\in\mathcal{M}(f)$.

Definition 2. *The positive definite quadratic form f over $\mathbb{K}$ is called perfect (in the sense of Voronoï) if it is uniquely determined by the set $\mathcal{M}(f)$ and the first minimum $\mu(f)$ of $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(x))$.*

Proposition 1. *Let f be a positive definite quadratic form of rank n. Then f is perfect (in the sense of Koecher [2]) if and only if there exist $r\frac{n(n+1)}{2}$ block matrices*

$$\mathrm{diag}\{\sigma_1(mm^t),\ldots,\sigma_r(mm^t)\},\quad m\in\mathcal{M}(f),$$

that are linearly independent over $\mathbb{R}$, or equivalently, there exist $r\frac{n(n+1)}{2}$ matrices mm^t, $m\in\mathcal{M}(f)$, that are linearly independent over $\mathbb{Q}$.

Proof. Obvious. □

Proposition 1 was the definition for the perfection given by Koecher [2] (see also [1]).

If $f(x)$ is a perfect form of rank n, then obviously $\#\mathcal{M}(f)\geq r\frac{n(n+1)}{2}$.

Let $\mathcal{P}_{n,\mathbb{K}}$ denote the set of all positive definite quadratic forms of rank n over $\mathbb{K}$. Write $\det(f)=\det(f_{ij})$.

Definition 3. *The positive definite quadratic form $f\in\mathcal{P}_{n,\mathbb{K}}$ is called critical if the function*

$$\gamma_{\mathbb{K}}:\mathcal{P}_{n,\mathbb{K}}\to\mathbb{R}_{>0},\quad \gamma_{\mathbb{K}}(g)=\frac{\mu(g)}{(\mathrm{Nm}\det(g))^{1/nr}}$$

attains global maximum at f.

Definition 4. *A matrix* $S \in \mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ *is called an automorph of* $f \in \mathcal{P}_{n,\mathbb{K}}$, $f(x) = x^t(f_{ij})x$, *if* $(f_{ij}) = S^t(f_{ij})S$.

For each $S \in \mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ we write $\mathcal{T}_S$ for the mapping $\mathcal{P}_{n,\mathbb{K}} \to \mathcal{P}_{n,\mathbb{K}}$ defined by $\mathcal{T}_S(f) = S^t(f_{ij})S$, where $f(x) = x^t(f_{ij})x$. We call quadratic forms $f, g \in \mathcal{P}_{n,\mathbb{K}}$ equivalent and write $f \sim g$ if there exist $\mathcal{T}_S$ such that $g = \mathcal{T}_S(f)$.

For a quadratic form f and $S \in \mathrm{GL}(\mathrm{rank} f, \mathcal{O}_{\mathbb{K}})$ we write $f[S]$ for $S^t(f_{ij})S$.

Definition 5. *A mapping* $\mathcal{T}_S$ *is called an automorphism of* f *if* S *is an automorph of* f.

Throughout this paper we work with values of $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}} f(x)$ only, therefore we are interested in automorphisms of $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}} f(x)$ to exploit the "symmetry" of Voronoï's algorithm. This will be done by considering the actions of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ if $\mathbb{K}$ is a normal extension, and the automorphism group of $f \in \mathcal{P}_{n,\mathbb{K}}$. This motivates the following definition.

Definition 6. *By the automorphism of* $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}} f(x)$ ($f \in \mathcal{P}_{n,\mathbb{K}}$) *we mean a mapping of the form* $\tau \circ \mathcal{T}_S$, *where* $S \in \mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ *and* $\tau \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ *such that*

$$(\tau \circ \mathcal{T}_S)(f) = f.$$

If $\mathbb{K}$ *is not a normal extension, then we set* $\tau = \mathrm{Id}$.

We emphasize that Definition 6 does not involve arbitrary automorphism of the rational quadratic form defined by (1).

Clearly, every automorphism of f is also an automorphism of $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}} f(x)$. We denote the automorphism group of f and $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}} f(x)$ by $\mathrm{Aut}(f)$ and $\mathrm{Aut}_{\mathrm{Tr}}(f)$, respectively. By $\mathrm{Stab}(f)$ we mean the subgroup $\{\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q}) | \sigma(f) \sim f\}$.

Voronoï's algorithm is discussed in detail in [1,8,9]. Theoretical background of perfect polyhedral cones with respect to bilinear product can be found in [2,9].

Here we give a short outline of the algorithm and in Section 3 we discuss some theoretical results concerning the method.

Let $\mathcal{M}(f) = \{m_1, \ldots, m_t\}$. To each trace minimum vector m_k we associate a tuple of linear forms

$$\widehat{\lambda}_k = (\lambda_{k;1}, \ldots, \lambda_{k;r})$$

such that

$$\lambda_{k;i}(x) = \sigma_i(m_k) \cdot x = \sigma_i(m_{k,1})x_1 + \ldots + \sigma_i(m_{k,n})x_n.$$

Write $\widehat{\lambda}_k^2 = (\lambda_{k;1}^2, \ldots, \lambda_{k;r}^2)$. The perfect polyhedral cone Π_f associated to perfect form f with trace minimums $m_1, \ldots, m_t$ is defined as a polyhedral cone generated by tuples of quadratic forms $\lambda_1^2, \ldots, \lambda_t^2$, that is

$$\Pi_f = \left\{ \sum_{k=1}^{t} \rho_i \widehat{\lambda}_k^2 \,|\, \rho_1 \geqslant 0, \ldots, \rho_t \geqslant 0 \right\}.$$

The perfect polyhedral cones partition the set $\overline{\mathcal{P}_{n,\mathbb{K}}}$, thus two perfect polyhedral cones $\Pi_f \neq \Pi_g$ have no common inner points by Proposition 2 in [2].

We write $\mathrm{Sym}_n(\mathbb{K})$ for the set of all symmetric $n \times n$-matrices with entries in $\mathbb{K}$. The set of all symmetric positive semidefinite $n \times n$-matrices with entries in $\mathbb{K}$ will be denoted by $\mathrm{Sym}_{n,\geqslant 0}(\mathbb{K})$. Let

$$\langle A, B\rangle = \sum_{i=1}^{r} \mathrm{TR}(A^{(i)}B^{(i)}),$$

where $A = (A^{(1)}, \ldots, A^{(r)}), B = (B^{(1)}, \ldots, B^{(r)}) \in \mathbb{R} \otimes \mathrm{Sym}_n(\mathbb{K})$.

The perfect polyhedral cone can be described also in terms of symmetric matrices, which satisfy a certain number of homogeneous linear inequalities

$$\Pi_f = \{B | B \in \mathbb{R} \otimes \mathrm{Sym}_n(\mathbb{K}), \psi_\ell(B) = \langle A_\ell, B\rangle \geqslant 0, \ell = 1, \ldots, u\},$$

where $A_\ell \in \mathbb{R} \otimes \mathrm{Sym}_{n,\geqslant 0}(\mathbb{K})$. The dimension of Π_f is $N = r\frac{n(n+1)}{2}$. The cone Π_f is determined by u hyperplanes $H_\ell = \{B \in \mathbb{R} \otimes \mathrm{Sym}_n(\mathbb{K}) | \langle \Psi_\ell, B\rangle = 0\}$, $\Psi_\ell \in \mathbb{R} \otimes \mathrm{Sym}_n(\mathbb{K})$, and bounded by u faces $W_\ell = \Pi_f \cap H_\ell$ of dimension $N-1$. Write $A_k = \big(\sigma_1(m_k m_k^t), \ldots, \sigma_r(m_k m_k^t)\big)$. Every edge

$$\Lambda_k = \{\rho A_k | \rho \in \mathbb{R}_{\geqslant 0}\}, \quad k = 1, \ldots, s,$$

is the solution of a system of $N-1$ linearly independent equations $\langle \Psi_\ell, B\rangle = 0$, $\ell = 1, \ldots, N-1$. Moreover, Ψ_ℓ is determined by $N-1$ linearly independent edges

$$\langle \Psi_\ell, A_k\rangle = 0, \quad k = 1, \ldots, N-1,$$

with an unknown Ψ_ℓ.

To each $(N-1)$-dimensional face W of Π_f there corresponds a uniquely determined neighbouring perfect polyhedral cone Π_g with $\Pi_f \cap \Pi_g = W$, such that the perfect form g is not multiple of f (see [2]; Koecher also proved that if Π_f and Π_g are arbitrary perfect polyhedral cones, then the intersection $\Pi_f \cap \Pi_g$ is an r-dimensional face of Π_f and Π_g, where $0 \leqslant r < N$).

We call the forms f and g neighbouring forms along the face W, or simply, neighbouring forms. As Barnes [8] pointed out, the practical efficiency of Voronoï's method lies in the simplicity of the relation between neighbouring forms. Let f and g be neighbouring forms along the face $W = \{A \in \mathbb{R} \otimes \mathrm{Sym}_{n,\geqslant 0}(\mathbb{K}) | \langle \Psi, A\rangle = 0\}$ and $\Psi \in \mathbb{R} \otimes \mathrm{Sym}_n(\mathbb{K})$. Denote by $\psi(x)$ the indefinite quadratic form

$$\psi(x) = \sum_{i,j=1}^{n} b_{ij} x_i x_j \quad (b_{ij} = b_{ji})$$

corresponding to the face W, that is, if $B = (b_{ij})$, then $\Psi = (\sigma_1(B), \ldots, \sigma_r(B))$. Koecher [2] proved (if $\mathbb{K}$ is a real quadratic extension, see also Theorem 3.1.6

in [1]) that if W is a face of Π_f determined by the $N-1$ independent edge forms $\lambda_1, \ldots, \lambda_u$, then the neighbouring form g corresponding to the neighbouring cone Π_g along the face W, i.e. $W = \Pi_f \cap \Pi_g$, is

$$g = f + \lambda\psi$$

and $\lambda > 0$ is uniquely determined by

$$\lambda = \inf\left\{ \frac{\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(x)) - \mu(f)}{-\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi(x))} \middle| \, x \in \mathcal{O}_{\mathbb{K}}^n \wedge \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi(x)) < 0 \right\}.$$

Moreover, λ is a rational number. The edge forms $\lambda_1, \ldots, \lambda_u$ ($u \geq N-1$) determine the face W which is defined by the quadratic form ψ iff

$$\begin{aligned} \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi(m_i)) = 0 \quad (i = 1, \ldots, u), \\ \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi(m_i)) > 0 \quad (i > u) \end{aligned} \tag{2}$$

and the system (2) has rank $N-1$.

For the rest of this section we assume that $\mathbb{K}$ is a normal extension. We can simplify our computations by making the following observations:

1. If f is perfect, then $\sigma(f)$ is also perfect for any $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$.
2. If f and g are neighbouring forms, then $\sigma(f)$ and $\sigma(g)$ are neighbouring forms too for any $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. Once we have the neighbouring forms of f, we have also the neighbouring forms of $\sigma(f)$ for any field automorphism $\sigma \neq 1$ without applying Voronoï's algorithm.
3. The set of neighbours of f is a union of orbits by $\mathrm{Stab}(f)$ (see Proposition 4).
4. The equivalent faces W, W' of Π_f under action by $\mathrm{Aut}(f)$ yield equivalent neighbouring forms g, g' (see Proposition 5). Let $\varphi \in \mathrm{Aut}_{\mathrm{Tr}}(f)$ and let the face W be determined by minimum vectors $m_1, \ldots, m_u \in \mathcal{M}(f)$ by (2). Write $\varphi = \tau \circ \mathcal{T}_S$, where $\tau \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ and $S \in \mathrm{GL}(\mathrm{rank} f, \mathcal{O}_{\mathbb{K}})$. If the positive semidefinite quadratic form ψ corresponds to the face W, then by permuting minimum vectors by $x \rightsquigarrow S\tau^{-1}(x)$ we obtain a new face, say W', determined by $S\tau^{-1}(m_1), \ldots, S\tau^{-1}(m_u)$

$$\begin{aligned} \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi'(S\tau^{-1}(m_i))) &= \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\varphi(\psi')(m_i)) = 0 \quad (i = 1, \ldots, u), \\ \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi'(S\tau^{-1}(m_i))) &= \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\varphi(\psi')(m_i)) > 0 \quad (i > u). \end{aligned}$$

   Thus we may take $\varphi(\psi') = \psi$. Since S is an automorphism of f, the faces W, W' are equivalent under $\mathrm{Aut}_{\mathrm{Tr}}(f)$ (see Proposition 5 and Corollary 1). Also, we have

$$f + \lambda\psi' = \varphi^{-1}(f) + \lambda\varphi^{-1}(\psi) = \varphi^{-1}(f + \lambda\psi).$$

   The group $\mathrm{Aut}_{\mathrm{Tr}}(f)$ partitions the set $\{M \subseteq \mathcal{M}(f) | \#M = N-1\}$ into orbits $\mathcal{K}_1, \ldots, \mathcal{K}_U$. According to Corollary 1 we apply Voronoï's algorithm to the representatives of $\mathcal{K}_1, \ldots, \mathcal{K}_U$ only.

Starting with any perfect form f, we find all its inequivalent neighbours, discarding any neighbour equivalent to f. We now find all inequivalent neighbours of these forms, discarding any neighbour equivalent to perfect forms already found, and so on. This process stops after a finite number of steps because there exist only finitely many inequivalent (up to homothety) perfect forms (see Proposition 8 in [2]). At each step (i.e. finding all inequivalent neighbours of a perfect form f), if the group $\mathrm{Aut}_{\mathrm{Tr}}(f)$ is nontrivial, we partition the set $\mathcal{M}(f)$ into orbits of $\mathrm{Aut}_{\mathrm{Tr}}(f)$ and study the representatives of each orbit. Then we apply Corollary 1 to the result.

The total number of systems of $N-1$ linear equations is $\binom{\#\mathcal{M}(f)}{N-1}$. It can be very large even if we have used the partition by $\mathrm{Aut}_{\mathrm{Tr}}(f)$, thus computer algorithm was used to study those systems.

However, in a computational number theory software which handles systems of linear equations with algebraic coefficients (such as KASH), it is possible to use rational numbers only. Let $\omega_1, \ldots, \omega_r$ be a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$ and let I_n be the identity matrix with n rows. Set

$$\mathcal{B} = \begin{pmatrix} \sigma_1(\omega_1)I_n & \sigma_1(\omega_2)I_n & \ldots & \sigma_1(\omega_r)I_n \\ \vdots & \vdots & & \vdots \\ \sigma_r(\omega_1)I_n & \sigma_r(\omega_2)I_n & \ldots & \sigma_r(\omega_r)I_n \end{pmatrix}.$$

Let A be the symmetric matrix associated to the quadratic form f of rank n. Then

$$\mathcal{B}^t \mathrm{diag}\{\sigma_1(A), \ldots, \sigma_r(A)\}\mathcal{B}$$

is a symmetric matrix with rational entries and with rn rows.

The computer algorithm was implemented twice:

1. using computational algebraic number theory software KASH[1]. In this case the algebraic coefficients were treated directly;
2. using the C programming language with CARAT[2] library and only the matrices with rational entries.

The results obtained using both implementations coincide.

Voronoï [10] proved the perfection of the quadratic form

$$\phi_0 = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j,$$

which can be used as an initial perfect form for Voronoï's algorithm in the rational case. In our case, we do not have such a nice result. But, we have:

1. if ax^2 is a perfect unary form over $\mathbb{K}$, then $a\phi_0$ is also perfect over $\mathbb{K}$;
2. if $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ with square-free $D > 1$, then the unary form given in Theorem 1 provides us with a perfect unary form over $\mathbb{K}$.

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1 http://www.math.tu-berlin.de/~kant/kash.html

2 http://wwwb.math.rwth-aachen.de/carat/

3. THEOREMS

Proposition 2. *If f is a perfect quadratic form over $\mathbb{K}$, then $\sigma(f)$ is also perfect for all $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$.*

Proof. Obvious.

Proposition 3. *If f and g are neighbouring forms, then $\sigma(f)$ and $\sigma(g)$ are neighbouring forms for each $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$.*

Proof. Since λ is defined as a minimum of

$$\frac{\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(f(x)) - \mu(f)}{-\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\psi(x))}$$

over $\mathcal{O}_{\mathbb{K}}^{\mathrm{rank} f}$ with $\mathrm{Tr}(\psi(x)) < 0$, we have that λ is a rational number. Let f and g be neighbouring forms along the face determined by the quadratic form ψ. Since $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, we have

$$g = f + \lambda\psi \Longrightarrow \sigma(g) = \sigma(f + \lambda\psi) = \sigma(f) + \lambda\sigma(\psi). \qquad \square$$

Proposition 4. *The set of neighbouring forms of f is a union of the orbits of* $\mathrm{Stab}(f)$.

Since equivalent perfect quadratic forms have equivalent neighbours, this proves the proposition.

Proposition 5. *Let W, W' be faces of Π_f and W, W' be equivalent under $\mathrm{Aut}(f)$. Suppose that the quadratic forms ψ, ψ' correspond to W, W', respectively. Then the perfect neighbouring forms $f + \lambda\psi$ and $f + \lambda\psi'$are equivalent.*

Proof. Let $\mathcal{T}_S$ be the automorphism of f such that $\psi'(x) = \psi(Sx)$. Since $f(x) = f(Sx)$, we have

$$(f + \lambda\psi')(x) = f(x) + \lambda\psi'(x) = f(Sx) + \lambda\psi(Sx) = (f + \lambda\psi)(Sx).$$

Hence $f + \lambda\psi'$ and $f + \lambda\psi$ are equivalent. $\square$

Combining Propositions 4 and 5, we obtain the following corollary.

Corollary 1. *Let f be a perfect quadratic form over $\mathbb{K}$. Then $\mathrm{Aut}(f)$ decomposes the set of neighbours of f into orbits by $\mathrm{Stab}(f)$.*

Theorem 1. *Let $D > 1$ be a square-free integer.*

1. *Suppose that $|k^2 - D|$ attains a minimum at integer $k > 0$. If $D \equiv 2 \pmod 4$ or $D \equiv 3 \pmod 4$, then the unary form $ax^2 = (a_1 + a_2\sqrt{D})x^2$, with*

$$a_1 = 2kD, \qquad a_2 = k^2 + D - 1,$$

is perfect and $\{1, k - \sqrt{D}\} \subseteq \mathcal{M}(ax^2)$.

2. *Let $k > 0$ be the smallest integer such that $|(2k-1)^2 - D|$ is minimal. If $D \equiv 1 \pmod 4$, then the unary form $ax^2 = (a_1 + a_2\frac{1+\sqrt{D}}{2})x^2$, with*

$$a_1 = 1 - k^2 + (1+D)k - \frac{1+3D}{4}, \qquad a_2 = 2k^2 - 2k + \frac{1+D}{2} - 2$$

is perfect and $\{1, -k + \frac{1+\sqrt{D}}{2}\} \subseteq \mathcal{M}(ax^2)$.

Proof. Clearly, 1 and $(k - \sqrt{D})^2$ (respectively 1 and $(-k + \frac{1+\sqrt{D}}{2})^2$) are linearly independent over $\mathbb{Q}$. This proves the perfectness.

To show that 1 and $k - \sqrt{D}$ (respectively 1 and $-k + \frac{1+\sqrt{D}}{2}$) are the trace minimum vectors of ax^2, we use rational binary quadratic forms. The trace $\mathrm{Tr}_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(ax^2)$ of the unary quadratic form can be considered as a binary quadratic form $f(x_1, x_2)$ over $\mathbb{Q}$ and x_1, x_2 are the coefficients of an algebraic integer on $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$.

Let $D \equiv 2 \pmod 4$ or $D \equiv 3 \pmod 4$. We have

$$\frac{1}{2}\mathrm{Tr}((a_1 + a_2\sqrt{D})(x_1 + x_2\sqrt{D})^2) = 2kDx_1^2 + 2D(k^2 + D - 1)x_1x_2 + 2kD^2x_2^2.$$

Replacing x_1 by $x_1 - kx_2$, we obtain the equivalent binary quadratic form

$$2kDx_1^2 + 2D(D - k^2 - 1)x_1x_2 + 2kDx_2^2,$$

which is reduced in the sense that

$$0 \leqslant |2D(D - k^2 - 1)| \leqslant 2kD.$$

This proves the claim.

The proof is similar for the case $D \equiv 1 \pmod 4$. □

Corollary 2. *The unary form $(8 + 3\sqrt{6})x^2$ is perfect over $\mathbb{Q}(\sqrt{6})$.*

Up to equivalence and scaling, there are only two unary perfect forms: $(8 + 3\sqrt{6})x^2$ and its field conjugate $(8 - 3\sqrt{6})x^2$.

Theorem 2 (Main theorem). *There exist exactly* 22 *classes of binary perfect forms with coefficients in $\mathbb{Q}(\sqrt{6})$.* (*The list of representatives is given in Subsection* 5.14).

The proof is based on applying Voronoï's algorithm to inequivalent perfect binary quadratic forms, starting at the initial form

$$\phi_0 = (8+3\sqrt{6})(x^2+xy+y^2),$$

which is perfect by Corollary 2 and by Remark 1 in [11]. The results of applying Voronoï's algorithm to the initial form and perfect forms already found are summarized in Subsections 5.1–5.13.

4. BINARY PERFECT FORMS AND ELLIPTIC FIXED POINTS OF THE HILBERT MODULAR GROUP

In this section we recall some basic facts about binary perfect forms and elliptic fixed points of the Hilbert modular group. We refer to [12] for more facts about Hilbert modular groups. We attach to each matrix $M \in \mathrm{GL}(2,\mathbb{K})$ the tuple

$$(\sigma_1(M),\dots,\sigma_r(M)) \in \mathrm{GL}(2,\mathbb{R})^r$$

and obtain the imbedding

$$\mathrm{GL}(2,\mathbb{K}) \hookrightarrow \mathrm{GL}(2,\mathbb{R})^r. \tag{3}$$

Write $\mathrm{GL}^+(2,\mathbb{K}) = \{M \in \mathrm{GL}(2,\mathbb{K}) | \det(M) \text{ is totally positive}\}$. Let $\mathfrak{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$. Using the imbedding (3), one defines the action of $\mathrm{GL}^+(2,\mathbb{K})$ on $\mathfrak{H}^r$ as a componentwise fractional-linear transformation, that is, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}^+(2,\mathbb{K})$ and $(z_1,\dots,z_r) \in \mathfrak{H}^r$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1,\dots,z_r) = \left(\frac{\sigma_i(a)z_i+\sigma_i(b)}{\sigma_i(c)z_i+\sigma_i(d)}\right)_{i=1}^r .$$

We shall identify the matrix $M \in \mathrm{GL}^+(2,\mathbb{K})$ and its image under the embedding into $\mathrm{GL}(2,\mathbb{R})^r$.

Let $f(x,y) = ax^2+bxy+cy^2$ be a positive definite binary quadratic form over $\mathbb{K}$ and assume that $\sigma_i(f)$ can be factored as follows:

$$\sigma_i(a)(x-\tau_i y)(x-\bar{\tau}_i y), \quad (\tau_1,\dots,\tau_r) \in \mathfrak{H}^r.$$

Here $\bar{\tau}_i$ denotes the complex conjugate of τ_i. Therefore we obtain the map from the set of all positive definite binary quadratic forms over $\mathbb{K}$ into $\mathfrak{H}^r$

$$f \rightsquigarrow (\tau_1,\dots,\tau_r).$$

Write $\Gamma = \mathrm{SL}(2,\mathcal{O}_{\mathbb{K}})$ for the Hilbert modular group of $\mathbb{K}$.

It is known that the Hilbert modular group acts discontinuously on $\mathfrak{H}^r$ (see Remark 3.2 in [12]). The points z, $z' \in \mathfrak{H}^r$ are called equivalent and will be denoted by $z \sim z'$ if there exists an $M \in \Gamma$ such that $Mz = z'$. One can show that if positive definite quadratic forms f and f' satisfy $\alpha f' = f[M]$ for some $M \in \Gamma$ ($M \in \mathrm{GL}^+(2, \mathcal{O}_{\mathbb{K}})$) and totally positive $\alpha \in \mathbb{K}$, then the corresponding points z, $z' \in \mathfrak{H}^r$ are equivalent under Γ (under $\mathrm{GL}^+(2, \mathcal{O}_{\mathbb{K}})$, respectively).

By the elliptic fixed point of Γ we mean the point $z \in \mathfrak{H}^r$ if its stabilizer $\Gamma_z \leqslant \Gamma$ is not equal to $\{\pm E\}$. Moreover, Γ_z is finite and $\Gamma_z/\{\pm E\}$ is cyclic by Remark 2.14 in [12].

For a real quadratic number field $\mathbb{K}$ with totally positive fundamental unit ε we write $\Gamma_{(\varepsilon)} = \mathrm{GL}^+(2, \mathcal{O}_{\mathbb{K}})/\{M \in \mathrm{GL}^+(2, \mathcal{O}_{\mathbb{K}}) | \det(M) = \varepsilon^{2l}, l \in \mathbb{Z}\}$ for the extended Hilbert modular group. Since $\mathbb{K} = \mathbb{Q}(\sqrt{6})$ has a totally positive fundamental unit $\varepsilon = 5 + 2\sqrt{6}$, we consider elliptic fixed points with respect to the extended Hilbert modular group $\Gamma_{(\varepsilon)}$ too.

Some binary perfect forms have a large automorphism group, hence the points in $\mathfrak{H}^r$ associated to those forms are elliptic fixed points. The number of inequivalent elliptic fixed points of order e ($e \in \{2, \ldots, 6\}$) for real quadratic extension $\mathbb{Q}(\sqrt{D})$ with $D \leq 97$ was computed by Prestel [7]. As a result of our computation for the enumeration of the inequivalent binary perfect forms over $\mathbb{Q}(\sqrt{6})$ we found the following classes of forms:

1. One class of perfect forms corresponding to the elliptic fixed point of order 6 under the action by $\Gamma_{(\varepsilon)}$. The representative of this class is

$$\phi_8 = \frac{24 - 8\sqrt{6}}{3}x^2 + 2 \cdot 4xy + \frac{24 + 8\sqrt{6}}{3}y^2$$

and the corresponding elliptic point is

$$(z_1, z_2) = \left(\frac{-3 + \sqrt{-3}}{2(3 - \sqrt{6})}, \frac{-3 + \sqrt{-3}}{2(3 + \sqrt{6})} \right). \tag{4}$$

Under the action by Γ the points (z_1, z_2), (z_2, z_1) are inequivalent and have order 3 (see also Remark 2).

2. One class of perfect forms corresponding to the elliptic fixed point of order 4 with respect to the group $\Gamma_{(\varepsilon)}$. The quadratic form is

$$\phi_9 = \frac{24 + 6\sqrt{6}}{3}x^2 + 2\frac{12 + 8\sqrt{6}}{3}xy + \frac{24 + 4\sqrt{6}}{3}y^2$$

with the corresponding elliptic fixed point

$$(w_1, w_2) = \left(\frac{-(3 + 2\sqrt{6}) + \sqrt{-3\varepsilon}}{2(3 + \sqrt{6})}, \frac{-(3 - 2\sqrt{6}) + \sqrt{-3\varepsilon^{-1}}}{2(3 - \sqrt{6})} \right). \tag{5}$$

Under the action by Γ the points $(w_1, w_2) \not\sim (w_2, w_1)$ are of order 2 (cf. Remark 3).

3. Two classes of perfect binary forms corresponding to the elliptic fixed point of order 3. The representatives of these classes are the initial perfect form and its field conjugate

$$\begin{aligned} \phi_0 &= (8+3\sqrt{6})(x^2+xy+y^2), \\ \overline{\phi_0} &= (8-3\sqrt{6})(x^2+xy+y^2). \end{aligned}$$

The corresponding elliptic fixed point of order 3 with respect to the groups $\Gamma_{(\varepsilon)}$ and Γ is (τ, τ), where $\tau = \frac{-1+\sqrt{-3}}{2} \in \mathfrak{H}$ is the elliptic fixed point of order 3 with respect to the group $\mathrm{SL}(2, \mathbb{Z})$. Since ϕ_0 and $\overline{\phi_0}$ differ only by totally positive scalar multiple, they correspond to the same point $z \in \mathfrak{H}^2$.

This result coincides with the number of elliptic fixed points of orders 6 and 3 found by Prestel (see p. 208 in [7]). Under the action by Γ the inequivalent elliptic fixed points of order 3 are

$$(\tau, \tau), \quad (z_1, z_2), \quad \text{and} \quad (z_2, z_1).$$

With respect to the group $\Gamma_{(\varepsilon)}$, there is an elliptic fixed point (z_1, z_2) of order 6 and an elliptic fixed point (τ, τ) of order 3.

Remark 1. Due to the result by Prestel [7] there are two inequivalent elliptic fixed points of order 4 with respect to the group $\Gamma_{(\varepsilon)}$. Thus there exists a class of imperfect binary quadratic forms corresponding to an elliptic fixed point of order 4.

The same issue arises for binary quadratic forms over $\mathbb{Q}(\sqrt{3})$. Due to the result by Prestel there is an elliptic fixed point of order 4 with respect to the group $\Gamma_{(\varepsilon)}$. Ong [1] found a complete list of binary perfect forms over $\mathbb{Q}(\sqrt{3})$, but none of them corresponds to the elliptic fixed point of order 4. The binary imperfect quadratic form associated to the fixed point (see Proposition 4 in [13])

$$\frac{1}{2}(1+\sqrt{3}) + \frac{i}{2}|1-\sqrt{3}|$$

is

$$x^2 - (1+\sqrt{3})xy + 2y^2. \tag{6}$$

The imperfection of the quadratic form (6) follows from the fact that the number of minimal vectors is strictly less than 6.

5. LIST OF BINARY PERFECT FORMS OVER $\mathbb{Q}(\sqrt{6})$

Since neither explicit description of the reduction domain for binary quadratic forms over $\mathbb{Q}(\sqrt{6})$ nor the reduction algorithm has been published until now, we shall compare invariants such as the determinant of the quadratic form and norm of the determinant in order to detect the inequivalence of quadratic forms. If $\mathrm{Nm}\det(f) = \mathrm{Nm}\det(g)$ for binary perfect forms f, g, then we study further those forms to determine whether they are equivalent or not. For example, from $f \sim g$ it follows that $\#\mathcal{M}(f) = \#\mathcal{M}(g)$ and $\mathrm{Aut}(f) \cong \mathrm{Aut}(g)$. Throughout this section we write ε for the totally positive fundamental unit $5+2\sqrt{6}$ of $\mathcal{O}_{\mathbb{Q}(\sqrt{6})}$. If quadratic forms f and g are equivalent, then $\det(f) = \varepsilon^{2l}\det(g)$ for some integer l.

We denote by $\overline{*}$ the image of $*$ by nontrivial Galois' automorphism of $\mathbb{Q}(\sqrt{6})$. If $*$ is a quadratic form, then the automorphism is applied to each coefficient of $*$.

For any perfect form f already found, we skip the investigation of neighbours of the quadratic form $\bar{f}$ due to Propositions 2 and 3 if f is not equivalent to $\bar{f}$. This means that for each pair of perfect forms $f, \bar{f}$ we pick up only one member for detailed investigation.

For a binary quadratic form ϕ we write

$$M_5(\phi) = \{M \subset \mathcal{M}(\phi) | \#M = 5\}.$$

5.1. Initial perfect form ϕ_0 and its neighbours

We start with the perfect form

$$\phi_0 = (8+3\sqrt{6})(x^2+xy+y^2).$$

The invariants are $\det(\phi_0) = \frac{177+72\sqrt{6}}{2}$ and $\mathrm{Nm}\det(\phi_0) = \frac{225}{4}$. Since

$$\frac{\det(\phi_0)}{\det(\overline{\phi_0})} = \frac{6937+2832\sqrt{6}}{25} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{6})},$$

it implies that $\phi_0 \not\sim \overline{\phi_0}$.

The minimum vectors of ϕ_0 are

$$\mathcal{M}(\phi_0) = \left\{ \begin{pmatrix} 0 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 2-\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

The set $\mathcal{M}(\phi_0)$ is decomposed into two orbits by the action of the group

$$\mathrm{Aut}(\phi_0) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \cong S_3.$$

Hence, there are at most two inequivalent neighbours.

The forms

$$\begin{aligned} \phi_1 &= (8+3\sqrt{6})x^2 - 2(2+\sqrt{6})xy + (8+3\sqrt{6})y^2, \\ \phi_2 &= (8+3\sqrt{6})x^2 + 2\frac{20+9\sqrt{6}}{5}xy + (8+3\sqrt{6})y^2 \end{aligned}$$

are new perfect forms because

$$\operatorname{Nm}\det(\phi_1) = 48 \neq \operatorname{Nm}\det(\phi_0) \neq \operatorname{Nm}\det(\phi_2) = \frac{26496}{625}.$$

5.2. Perfect form ϕ_1 and its neighbours

Let us consider the perfect form

$$\phi_1 = (8+3\sqrt{6})x^2 - 2(2+\sqrt{6})xy + (8+3\sqrt{6})y^2,$$

with invariants $\det(\phi_1) = 108 + 44\sqrt{6}$ and $\operatorname{Nm}\det(\phi_1) = 48$. But $\det(\phi_1) = 108 + 44\sqrt{6} = (108 - 44\sqrt{6})\varepsilon^3 = \det(\overline{\phi_1})\varepsilon^3$, hence $\phi_1 \not\sim \overline{\phi_1}$.

The minimum vectors of ϕ_1 are

$$\begin{aligned}\mathcal{M}(\phi_1) \quad = \quad & \left\{ \begin{pmatrix} 0 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} 3-\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 2-\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 2-\sqrt{6} \\ -3+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.\end{aligned}$$

The group $\operatorname{Aut}(\phi_1) = \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ decomposes the set $M_5(\phi_1)$ into 12 orbits. Investigating these orbits by computer, we found the following five classes of forms:

$$\phi_7 = (8+3\sqrt{6})x^2 - 2(2+\sqrt{6})xy + (20+8\sqrt{6})y^2,$$

$$\phi_3 = (8+3\sqrt{6})x^2 + 2xy + \frac{16+5\sqrt{6}}{2}y^2,$$

and ϕ_0, $\overline{\phi_1}$, ϕ_2.

Clearly, ϕ_3 represents a new class of perfect forms, because applying any matrix $M \in \mathrm{GL}(2, \mathcal{O}_{\mathbb{Q}(\sqrt{6})})$ to an integral quadratic form, we get an integral form too. Since ϕ_3 is not integral, but ϕ_0, ϕ_1, ϕ_7, and ϕ_2 are, the result follows. Investigating the form ϕ_7 (see Subsection 5.8), we find that the number of minimum vectors of ϕ_7 is 9, which exceeds the number of minimum vectors of the forms ϕ_0, ϕ_1, and ϕ_2. Therefore ϕ_7 represents a new class of perfect forms.

5.3. Perfect form ϕ_2 and its neighbours

The perfect form

$$\phi_2 = (8+3\sqrt{6})x^2 + 2\frac{20+9\sqrt{6}}{5}xy + (8+3\sqrt{6})y^2$$

has the invariants $\det(\phi_2) = \frac{2064+840\sqrt{6}}{25}$ and $\operatorname{Nm}\det(\phi_2) = \frac{26496}{625}$. Since $\det(\phi_2)/\det(\overline{\phi_2}) = \frac{7373+3010\sqrt{6}}{23}$ is not a unit, $\phi_2 \not\sim \overline{\phi_2}$.

The minimum vectors of ϕ_2 are

$$\mathcal{M}(\phi_2) = \left\{ \begin{pmatrix} -3+\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ -3+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 0 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

The automorphism group $\mathrm{Aut}(\phi_2) = \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ decomposes the set $M_5(\phi_2)$ into 12 orbits.

Again, investigating these orbits by computer, we found the following five classes of perfect forms: ϕ_0, ϕ_1, $\overline{\phi_2}$, ϕ_3, and

$$\phi_4 = (8+3\sqrt{6})x^2 + 2 \cdot 2xy + \frac{16+\sqrt{6}}{2}y^2.$$

From the comparison of $\mathrm{Nm}\det(\phi_4) = 57$ to norm of determinant of other perfect forms already found it follows that ϕ_4 represents a new class of perfect forms.

5.4. Perfect form ϕ_3 and its neighbours

Consider the perfect form

$$\phi_3 = (8+3\sqrt{6})x^2 + 2xy + \frac{16+5\sqrt{6}}{2}y^2.$$

Since the invariants $\det(\phi_3) = 108 + 44\sqrt{6}$, $\mathrm{Nm}\det(\phi_3) = 48$ coincide with the invariants of ϕ_1 (see Subsection 5.2), we can apply the same argument to prove that $\phi_3 \not\sim \overline{\phi_3}$.

The minimum vectors of ϕ_3 are

$$\mathcal{M}(\phi_3) = \left\{ \begin{pmatrix} 3-\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 2-\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 5-2\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 5-2\sqrt{6} \\ -3+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Investigating the neighbours of ϕ_3, we found seven classes of perfect forms with the following representative elements: ϕ_1, ϕ_2, ϕ_4, ϕ_7, and

$$\phi_5 = (8+3\sqrt{6})x^2 - 2(3+\sqrt{6})xy + \frac{24-7\sqrt{6}}{3}y^2,$$

$$\phi_6 = (8+\sqrt{6})x^2 + 2\frac{3-\sqrt{6}}{3} + (8+3\sqrt{6})y^2,$$

$$\phi_9 = \frac{24+8\sqrt{6}}{3}x^2 + 2\frac{12+8\sqrt{6}}{3}xy + \frac{24+4\sqrt{6}}{3}y^2.$$

Since $\operatorname{Nm}\det(\phi_9) = \frac{256}{9}$, it follows that ϕ_9 represents a new class of perfect forms. The same argument applies to ϕ_5 ($\operatorname{Nm}\det(\phi_5) = \frac{139}{3}$) and to ϕ_6 ($\operatorname{Nm}\det(\phi_6) = \frac{457}{9}$).

5.5. Perfect form ϕ_4 and its neighbours

The perfect form

$$\phi_4 = (8+3\sqrt{6})x^2 + 2\cdot 2xy + \frac{16+\sqrt{6}}{2}y^2,$$

with $\det(\phi_4) = 69 + 28\sqrt{6}$, $\operatorname{Nm}\det(\phi_4) = 57$, and

$$\mathcal{M}(\phi_4) = \left\{\begin{pmatrix}7-3\sqrt{6}\\-2+\sqrt{6}\end{pmatrix}, \begin{pmatrix}-2+\sqrt{6}\\1\end{pmatrix}, \begin{pmatrix}-3+\sqrt{6}\\1\end{pmatrix}, \begin{pmatrix}-2+\sqrt{6}\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$$

is inequivalent to $\overline{\phi_4}$ since $\det(\phi_4)/\det(\overline{\phi_4}) = \frac{3155+1288\sqrt{6}}{19} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{6})}$.

In order to find all perfect neighbours of ϕ_4, we must investigate all elements of $M_5(\phi_4)$ since $\operatorname{Aut}(\phi_4)$ is a trivial group. The investigation by computer yielded the following perfect neighbours: ϕ_2, ϕ_3, ϕ_6, ϕ_7, ϕ_9, and ϕ_{10}.

5.6. Perfect form ϕ_5 and its neighbours

Consider the perfect form

$$\phi_5 = (8+3\sqrt{6})x^2 - 2(3+\sqrt{6})xy + \frac{24-7\sqrt{6}}{3}y^2$$

with minimum vectors

$$\mathcal{M}(\phi_5) = \left\{\begin{pmatrix}-1+\sqrt{6}\\2+\sqrt{6}\end{pmatrix}, \begin{pmatrix}-2+\sqrt{6}\\0\end{pmatrix}, \begin{pmatrix}2\\3+\sqrt{6}\end{pmatrix}, \begin{pmatrix}2\\2+\sqrt{6}\end{pmatrix}, \begin{pmatrix}1\\2+\sqrt{6}\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

Since $\det(\phi_5) = \frac{21-2\sqrt{6}}{3}$ and $\operatorname{Nm}\det(\phi_5) = \frac{139}{3}$, we have $\det(\phi_5)/\det(\overline{\phi_5}) = \frac{155-28\sqrt{6}}{139} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{6})}$. This gives $\phi_5 \not\sim \overline{\phi_5}$.

The group $\operatorname{Aut}(\phi_5)$ is trivial. After examining the set $M_5(\phi_5)$, we found the following neighbours: ϕ_3, $\overline{\phi_5}$, ϕ_6, ϕ_7, and

$$\phi_8 = \frac{24-8\sqrt{6}}{3}x^2 + 2\cdot 4xy + \frac{24+8\sqrt{6}}{3}y^2,$$

$$\phi_{11} = (8+3\sqrt{6})x^2 + 2\frac{2\sqrt{6}}{3}xy + (8-3\sqrt{6})y^2,$$

$$\phi_{12} = (8+3\sqrt{6})x^2 + 2\frac{6+\sqrt{6}}{2}xy + \frac{96-31\sqrt{6}}{12}y^2.$$

An easy computation shows that $\#\mathcal{M}(\phi_8) = \#\mathcal{M}(\phi_9) = 12$, $\det(\phi_8) = \frac{16}{3}$ and $\det(\phi_9) = \frac{80+32\sqrt{6}}{3}$ (see Subsections 5.9 and 5.10). Hence, $\phi_8 \not\sim \phi_9$ and ϕ_8 represents a new class of perfect forms. By comparing the norms of determinants, one immediately verifies that the quadratic forms ϕ_{11} and ϕ_{12} represent new classes of perfect forms.

5.7. Perfect form ϕ_6 and its neighbours

The perfect form

$$\phi_6 = (8+\sqrt{6})x^2 + 2\frac{6+\sqrt{6}}{2}xy + (8+3\sqrt{6})y^2,$$

has invariants $\det(\phi_6) = \frac{241+98\sqrt{6}}{3}$ and $\operatorname{Nm}\det(\phi_6) = \frac{457}{9}$. We have $\phi_6 \not\sim \overline{\phi_6}$, because $\det(\phi_6)/\det(\overline{\phi_6}) = \frac{115705+47236\sqrt{6}}{457}$ is not a unit. As

$$\mathcal{M}(\phi_6) = \left\{ \begin{pmatrix} -2+\sqrt{6} \\ 5-2\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 7-3\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 0 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

and $\operatorname{Aut}(\phi_6)$ is a trivial group, we must investigate each element of $M_5(\phi_6)$. As a result we have the following neighbouring forms: ϕ_3, ϕ_4, ϕ_5, ϕ_6, ϕ_9, and ϕ_{10}.

5.8. Perfect form ϕ_7 and its neighbours

Consider the perfect form

$$\phi_7 = (8+3\sqrt{6})x^2 - 2(2+\sqrt{6})xy + (20+8\sqrt{8})y^2.$$

The invariants are $\det(\phi_7) = 294 + 120\sqrt{6}$ and $\operatorname{Nm}\det(\phi_7) = 36$. Put

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -\varepsilon^{-1} \\ 0 & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

then

$$\phi_7(x,y) = (8+3\sqrt{6})x'^2 + 2\cdot 2x'y' + (8-3\sqrt{6})y'^2.$$

For the sake of simplicity, we set

$$\phi_7 = \phi_7(x,y) = (8+3\sqrt{6})x^2 + 2\cdot 2xy + (8-3\sqrt{6})y^2.$$

It is easy to see that $\phi_7 \sim \overline{\phi_7}$, and $\chi = \bar{\ } \circ \mathcal{T}_P$, where $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, is the only nontrivial automorphism of $\operatorname{Tr}\phi_7$. The automorphism χ is of order 2. By Proposition 4 the neighbours of ϕ_7 occur in pairs $f, \bar{f}$. Since

$$\mathcal{M}(\phi_7) \;=\; \left\{\begin{pmatrix}-1\\3+\sqrt{6}\end{pmatrix},\begin{pmatrix}0\\2+\sqrt{6}\end{pmatrix},\begin{pmatrix}-1\\2+\sqrt{6}\end{pmatrix},\begin{pmatrix}2-\sqrt{6}\\2+\sqrt{6}\end{pmatrix},\begin{pmatrix}-2+\sqrt{6}\\1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}-3+\sqrt{6}\\1\end{pmatrix},\begin{pmatrix}-2+\sqrt{6}\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

and $\mathrm{Aut}_{\mathrm{Tr}}(\phi_7) \cong C_2$, we have to investigate 66 elements of $M_5(\phi_7)$. Hence, the inequivalent neighbours of ϕ_7 are $\phi_1, \overline{\phi_1}$, ϕ_3, $\overline{\phi_3}$, ϕ_4, $\overline{\phi_4}$, ϕ_5, $\overline{\phi_5}$, ϕ_7, ϕ_8, ϕ_{11}, ϕ_{12}, and $\overline{\phi_{12}}$.

5.9. Critical perfect form ϕ_8 and its neighbours

The perfect form

$$\phi_8 = \frac{24-8\sqrt{6}}{3}x^2 + 2\cdot 4xy + \frac{24+8\sqrt{6}}{3}y^2,$$

with $\det(\phi_8) = \frac{16}{3}$ and $\mathrm{Nm}\det(\phi_8) = \frac{256}{9}$, is a critical binary form. Since $\phi_8 \sim \overline{\phi_8}$, we have that ϕ_8 is fixed by $\chi = {}^{-} \circ \mathcal{T}_R$, where

$$R = \begin{pmatrix} 3+\sqrt{6} & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, the automorphism groups are

$$\mathrm{Aut}(\phi_8) = \langle \mathcal{T}_S, \mathcal{T}_T\rangle \cong S_3, \quad \mathrm{Aut}_{\mathrm{Tr}}(\phi_8) = \langle \chi, \mathcal{T}_S, \mathcal{T}_T\rangle \cong D_{12},$$

where

$$S = \begin{pmatrix} 2 & 3+\sqrt{6} \\ -3+\sqrt{6} & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 3+\sqrt{6} \\ 0 & -1 \end{pmatrix}.$$

Remark 2. Let z (z') be the elliptic fixed point (4) associated with the form ϕ_8 (respectively $\overline{\phi_8}$). The stabilizer $\Gamma_z \leqslant \Gamma$ ($\Gamma_z \leqslant \Gamma_{(\varepsilon)}$) is generated by S (respectively S and $S' = \left(\begin{smallmatrix}1&0\\0&\varepsilon\end{smallmatrix}\right)$). The equality $S'z = z'$ follows from immediate computations. Thus, $z \not\sim z'$ under Γ, but $z \sim z'$ under $\Gamma_{(\varepsilon)}$. Due to immediate computations, we have that Γ_z has order 3 in Γ (6 in $\Gamma_{(\varepsilon)}$). See also the table on page 208 in [7].

The minimum vectors of ϕ_8 are

$$\mathcal{M}(\phi_8) \;=\; \left\{\begin{pmatrix}1\\-2+\sqrt{6}\end{pmatrix},\begin{pmatrix}2\\-3+\sqrt{6}\end{pmatrix},\begin{pmatrix}1\\-3+\sqrt{6}\end{pmatrix},\begin{pmatrix}-1-\sqrt{6}\\-1+\sqrt{6}\end{pmatrix},\right.$$
$$\begin{pmatrix}-2-\sqrt{6}\\-1+\sqrt{6}\end{pmatrix},\begin{pmatrix}-1-\sqrt{6}\\-2+\sqrt{6}\end{pmatrix},\begin{pmatrix}3+\sqrt{6}\\-1\end{pmatrix},\begin{pmatrix}2+\sqrt{6}\\-1\end{pmatrix},$$
$$\left.\begin{pmatrix}3+\sqrt{6}\\-2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

The group $\mathrm{Aut}(\phi_8)$ partitions the set $M_5(\phi_8)$ into 76 orbits. Each orbit, if any element of it determines a face of Π_{ϕ_8}, gives us the pair of neighbours f and $\overline{f}$ of ϕ_8 by Corollary 1. Investigating these orbits by computer, we obtain a complete list of the neighbours of ϕ_8: ϕ_5, $\overline{\phi_5}$, ϕ_7, ϕ_9, ϕ_{12}, and $\overline{\phi_{12}}$.

5.10. Critical perfect form ϕ_9 and its neighbours

Let us consider the perfect form

$$\phi_9 = \frac{24+8\sqrt{6}}{3}x^2 + 2\frac{12+8\sqrt{6}}{3}xy + \frac{24+4\sqrt{6}}{3}y^2$$

with $\det(\phi_9) = \frac{80+32\sqrt{6}}{3}$ and $\mathrm{Nm}\det(\phi_9) = \frac{256}{9}$.

It follows from immediate computations that ϕ_9 represents another class of critical forms.

Let

$$R = \begin{pmatrix} \varepsilon^{-1} & \varepsilon^{-1} \\ 2-\sqrt{6} & 3-\sqrt{6} \end{pmatrix}.$$

A trivial verification shows that $\chi = {}^{-} \circ \mathcal{T}_R$ fixes ϕ_9. Therefore $\phi_9 \sim \overline{\phi_9}$. Writing

$$S = \begin{pmatrix} -1+\sqrt{6} & 3-\sqrt{6} \\ -2 & 1-\sqrt{6} \end{pmatrix}, \quad T = \begin{pmatrix} -1+\sqrt{6} & 4-\sqrt{6} \\ -2 & 1-\sqrt{6} \end{pmatrix},$$

we have $\mathrm{Aut}(\phi_9) = \langle \mathcal{T}_S, \mathcal{T}_T \rangle \cong V_4$ and $\mathrm{Aut}_{\mathrm{Tr}}(\phi_9) = \langle \mathcal{T}_S, \mathcal{T}_T, \chi \rangle \cong D_8$.

Remark 3. Let w (w') be the elliptic fixed point (5) associated with the form ϕ_9 ($\overline{\phi_9}$, respectively). The stabilizer Γ_w in Γ ($in \Gamma_{(\varepsilon)}$) is generated by T (respectively T and $S' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)$). Since $S'w = w'$, we have $w \not\sim w'$ under Γ, but $w \sim w'$ under $\Gamma_{(\varepsilon)}$ (cf. table on page 208 in [7]).

The minimum vectors of ϕ_9 are

$$\mathcal{M}(\phi_9) = \left\{ \begin{pmatrix} -\varepsilon^{-1} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -4+2\sqrt{6} \\ -3+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -\varepsilon^{-1} \\ -3+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -3+\sqrt{6} \\ -1+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -4+\sqrt{6} \\ -1+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -3+\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -1+\sqrt{6} \\ -1 \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ -1 \end{pmatrix}, \begin{pmatrix} -1+\sqrt{6} \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

The group $\mathrm{Aut}_{\mathrm{Tr}}(\phi_9)$ decomposes the set $M_5(\phi_9)$ into 104 orbits. Let O be an orbit. If an element of O determines a face of Π_{ϕ_9} such that f and ϕ_9 are neighbouring forms along this face, then $\bar{f}$ is also a neighbouring form of ϕ_9 by Corollary 1. Examining these orbits by computer gave the following neighbours of ϕ_9: ϕ_3, $\overline{\phi_3}$, ϕ_4, $\overline{\phi_4}$, ϕ_6, $\overline{\phi_6}$, ϕ_8, ϕ_9, ϕ_{10}, and $\overline{\phi_{10}}$.

5.11. Perfect form ϕ_{10} and its neighbours

Let us consider the perfect form

$$\phi_{10} = (8+3\sqrt{6})x^2 + 2\frac{6+\sqrt{6}}{2}xy + \frac{96+13\sqrt{6}}{12}y^2$$

with $\det(\phi_{10}) = \frac{219+89\sqrt{6}}{3}$ and $\operatorname{Nm}\det(\phi_{10}) = \frac{145}{3}$. The inequivalence $\phi_{10} \not\sim \overline{\phi_{10}}$ follows from comparing the determinants $\det(\phi_{10})/\det(\overline{\phi_{10}}) = \frac{31829+12994\sqrt{6}}{145} \notin \mathcal{O}_{\mathbb{K}}$. The group $\operatorname{Aut}_{\mathrm{Tr}}(\phi_{10})$ is a trivial group. Since

$$\mathcal{M}(\phi_{10}) = \left\{ \begin{pmatrix} \varepsilon^{-1} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 7-3\sqrt{6} \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -3+\sqrt{6} \\ 1 \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

there are at most 6 neighbours (up to equivalence). Applying the generalization of Voronoï's algorithm to ϕ_{10}, we obtain the following list of perfect neighbours: ϕ_4, ϕ_6, ϕ_9, and $\overline{\phi_{12}}$.

5.12. Perfect form ϕ_{11} and its neighbours

The perfect form

$$\phi_{11} = (8+3\sqrt{6})x^2 + 2\frac{2\sqrt{6}}{3}xy + (8-3\sqrt{6})y^2$$

is equivalent to $\overline{\phi_{11}}$. The values of interesting invariants are

$$\det(\phi_{11}) = \frac{22}{3}, \qquad \operatorname{Nm}\det(\phi_{11}) = \frac{484}{9}.$$

The group of proper automorphisms is a trivial one, but the group $\operatorname{Aut}(\phi_{11})$ is generated by the element $^{-} \circ \mathcal{T}_R$, where $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\operatorname{Aut}(\phi_{11}) \cong C_2$.

The perfect form ϕ_{11} has six minimums:

$$\mathcal{M}(\phi_{11}) = \left\{ \begin{pmatrix} 0 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -1 \\ -2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

thus $\operatorname{Aut}(\phi_{11})$ decomposes the set $\mathcal{M}(\phi_{11})$ into three orbits, each of it giving the pair $f, \overline{f}$ of neighbours. By means of computer we found the following list of the neighbours of ϕ_{11}: ϕ_7, ϕ_5, $\overline{\phi_5}$, and ϕ_{11}.

5.13. Perfect form ϕ_{12} and its neighbours

Let us consider the perfect form

$$\phi_{12} = (8+3\sqrt{6})x^2 + 2\frac{6+\sqrt{6}}{2}xy + \frac{96-31\sqrt{6}}{12}y^2$$

with $\det(\phi_{12}) = \frac{21+\sqrt{6}}{3}$ and $\mathrm{Nm}\det(\phi_{12}) = \frac{145}{3}$. Since $\det(\phi_{12})/\det(\overline{\phi_{12}}) = \frac{149+14\sqrt{6}}{145}$ is not a square of a unit, $\phi_{12} \not\sim \overline{\phi_{12}}$.

The minimum vectors of ϕ_{12} are

$$\mathcal{M}(\phi_{12}) = \left\{ \begin{pmatrix} -1 \\ 2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1-\sqrt{6} \\ 2+\sqrt{6} \end{pmatrix}, \begin{pmatrix} -3+\sqrt{6} \\ 1 \end{pmatrix}, \begin{pmatrix} -2+\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

The automorphism groups are trivial ones, thus all elements of $M_5(\phi_{12})$ should be investigated. By means of computer we found a complete list of the neighbours of ϕ_{12}: ϕ_5, ϕ_7, ϕ_8, and $\overline{\phi_{10}}$.

5.14. Complete list of binary perfect forms over $\mathbb{Q}(\sqrt{6})$

Perfect form f	Field conjugate of $\overline{f}$
$\phi_0:\ (8+3\sqrt{6})(x^2+xy+y^2)$	$\overline{\phi_0}:\ (8-3\sqrt{6})(x^2+xy+y^2)$
$\phi_1:\ (8+3\sqrt{6})x^2-(4+2\sqrt{6})xy+(8+3\sqrt{6})y^2$	$\overline{\phi_1}:\ (8-3\sqrt{6})x^2-(4-2\sqrt{6})xy+(8-3\sqrt{6})y^2$
$\phi_2:\ (8+3\sqrt{6})x^2+2\frac{20+9\sqrt{6}}{5}xy+(8+3\sqrt{6})y^2$	$\overline{\phi_2}:\ (8-3\sqrt{6})x^2+2\frac{20-9\sqrt{6}}{5}xy+(8-3\sqrt{6})y^2$
$\phi_3:\ (8+3\sqrt{6})x^2+2xy+\frac{16+5\sqrt{6}}{2}y^2$	$\overline{\phi_3}:\ (8-3\sqrt{6})x^2+2xy+\frac{16-5\sqrt{6}}{2}y^2$
$\phi_4:\ (8+3\sqrt{6})x^2+4xy+\frac{16+\sqrt{6}}{2}y^2$	$\overline{\phi_4}:\ (8-3\sqrt{6})x^2+4xy+\frac{16-\sqrt{6}}{2}y^2$
$\phi_5:\ (8+3\sqrt{6})x^2-(6+2\sqrt{6})xy+\frac{24-7\sqrt{6}}{3}y^2$	$\overline{\phi_5}:\ (8-3\sqrt{6})x^2-(6-2\sqrt{6})xy+\frac{24+7\sqrt{6}}{3}y^2$
$\phi_6:\ (8+\sqrt{6})x^2+(6+\sqrt{6})xy+(8+3\sqrt{6})y^2$	$\overline{\phi_6}:\ (8-\sqrt{6})x^2+(6-\sqrt{6})xy+(8-3\sqrt{6})y^2$
$\phi_7:\ (8+3\sqrt{6})x^2+4xy+(8-3\sqrt{6})y^2$	ϕ_7
$\phi_8:\ \frac{24-8\sqrt{6}}{3}x^2+8xy+\frac{24+8\sqrt{6}}{3}y^2$	ϕ_8
$\phi_9:\ \frac{24+8\sqrt{6}}{3}x^2+2\frac{12+8\sqrt{6}}{3}xy+\frac{24+4\sqrt{6}}{3}y^2$	ϕ_9
$\phi_{10}:\ (8+3\sqrt{6})x^2+(6+\sqrt{6})xy+\frac{96+13\sqrt{6}}{12}y^2$	$\overline{\phi_{10}}:\ (8-3\sqrt{6})x^2+(6-\sqrt{6})xy+\frac{96-13\sqrt{6}}{12}y^2$
$\phi_{11}:\ (8+3\sqrt{6})x^2+2\frac{2\sqrt{6}}{3}xy+(8-3\sqrt{6})y^2$	ϕ_{11}
$\phi_{12}:\ (8+3\sqrt{6})x^2+(6+\sqrt{6})xy+\frac{96-31\sqrt{6}}{12}y^2$	$\overline{\phi_{12}}:\ (8-3\sqrt{6})x^2+(6-\sqrt{6})xy+\frac{96+31\sqrt{6}}{12}y^2$

5.15. List of perfect forms and their neighbours

Perfect form f	Neighbours of f
ϕ_0	ϕ_1, ϕ_2
$\overline{\phi_0}$	$\overline{\phi_1}, \overline{\phi_2}$
ϕ_1	$\phi_7, \phi_3, \phi_2, \overline{\phi_1}, \phi_0$
$\overline{\phi_1}$	$\phi_7, \overline{\phi_3}, \overline{\phi_2}, \phi_1, \overline{\phi_0}$
ϕ_2	$\phi_0, \phi_1, \phi_3, \overline{\phi_2}, \phi_4$
$\overline{\phi_2}$	$\overline{\phi_0}, \overline{\phi_1}, \overline{\phi_3}, \phi_2, \overline{\phi_4}$
ϕ_3	$\phi_1, \phi_2, \phi_7, \phi_4, \phi_5, \phi_9, \phi_6$
$\overline{\phi_3}$	$\overline{\phi_1}, \overline{\phi_2}, \phi_7, \overline{\phi_4}, \overline{\phi_5}, \phi_9, \overline{\phi_6}$
ϕ_4	$\phi_2, \phi_3, \phi_6, \phi_7, \phi_9, \phi_{10}$
$\overline{\phi_4}$	$\overline{\phi_2}, \overline{\phi_3}, \overline{\phi_6}, \phi_7, \phi_9, \overline{\phi_{10}}$
ϕ_5	$\phi_3, \overline{\phi_5}, \phi_6, \phi_7, \phi_8, \phi_{11}, \phi_{12}$
$\overline{\phi_5}$	$\overline{\phi_3}, \phi_5, \overline{\phi_6}, \phi_7, \phi_8, \phi_{11}, \overline{\phi_{12}}$
ϕ_6	$\phi_3, \phi_4, \phi_5, \phi_6, \phi_9, \phi_{10}$
$\overline{\phi_6}$	$\overline{\phi_3}, \overline{\phi_4}, \overline{\phi_5}, \overline{\phi_6}, \phi_9, \overline{\phi_{10}}$
ϕ_7	$\phi_1, \overline{\phi_1}, \phi_3, \overline{\phi_3}, \phi_4, \overline{\phi_4}, \phi_5, \overline{\phi_5}, \phi_7, \phi_{11}, \phi_{12}, \overline{\phi_{12}}, \phi_8$
ϕ_8	$\phi_5, \overline{\phi_5}, \phi_7, \phi_9, \phi_{12}, \overline{\phi_{12}}$
ϕ_9	$\phi_3, \overline{\phi_3}, \phi_4, \overline{\phi_4}, \phi_6, \overline{\phi_6}, \phi_8, \phi_9, \phi_{10}, \overline{\phi_{10}}$
ϕ_{10}	$\phi_4, \phi_6, \phi_9, \overline{\phi_{12}}$
$\overline{\phi_{10}}$	$\overline{\phi_4}, \overline{\phi_6}, \phi_9, \phi_{12}$
ϕ_{11}	$\phi_5, \overline{\phi_5}, \phi_7, \phi_{11}$
ϕ_{12}	$\phi_5, \phi_7, \phi_8, \overline{\phi_{10}}$
$\overline{\phi_{12}}$	$\overline{\phi_5}, \phi_7, \phi_8, \phi_{10}$

6. DISCUSSION

There are 22 classes of binary perfect forms with coefficients in $\mathbb{Q}(\sqrt{6})$ (by Theorem 2). From [1] it follows that the numbers of equivalence classes of binary perfect forms over $\mathbb{Q}(\sqrt{D})$ with $D = 2, 3, 5$ are $2, 3, 2$, respectively. One reason for the rapid increase in the number of inequivalent binary perfect forms is clearly the fact that the quadratic number fields $\mathbb{Q}(\sqrt{D})$, $D = 2, 3, 5$, are the only real quadratic number fields that are maximal totally real subfields of cyclotomic fields. If the ground field is the maximal totally real subfield of a cyclotomic field, there exist binary perfect forms having large automorphism groups. Consequently, the number of perfect neighbours is small. If square-free $D > 5$ (i.e. $\mathbb{Q}(\sqrt{D})$ is not the maximal totally real subfield of a cyclotomic field), then it can be shown that for any binary positive quadratic form f over $\mathbb{Q}(\sqrt{D})$ we have the upper bound $\#\mathrm{Aut}_{\mathrm{Tr}}(f) \leq 12$.

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Täielik loetelu täiuslikest binaarsetest ruutvormidest kordajatega arvukorpusest $\mathbb{Q}(\sqrt{6})$

Alar Leibak

On esitatud täielik loetelu täiuslikest binaarsetest ruutvormidest (ekvivalentsi täpsusega rühma $\mathrm{GL}(2, \mathcal{O}_{\mathbb{Q}(\sqrt{6})})$ toime suhtes), kusjuures ruutvormide kordajad on arvukorpusest $\mathbb{Q}(\sqrt{6})$. On tõestatud täiuslike ruutvormide omadusi üle täielikult reaalse normaallaiendi $\mathbb{K}$, mis on seotud Voronoï algoritmiga: täiusliku algruutvormi ehitamine üle reaalse ruutlaiendi ja ruutvormi automorfismirühma toime kasutamine.