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Riemannian manifolds of conullity two admitting semiparallel isometric immersions

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Abstract. A Riemannian manifold (M, g) is semisymmetric if $R(X, Y) \circ R = 0$. An isometric immersion of (M, g) into a Euclidean space is semiparallel if $\bar{R}(X, Y) \circ h = 0$ holds for the second fundamental tensor h. Due to Gauss and Ricci equations the second condition leads to the first one. Especially $R(X, Y) \circ R = 0$ holds if (M, g) is foliated by codimension two locally Euclidean leaves (equivalently, is of conullity two). Here the planar, hyperbolic, parabolic, and elliptic types can be specified. For many cases of these manifolds of conullity two it has been shown already that their isometric semiparallel immersions into a Euclidean space are possible only if the manifold is of planar type. Now the same is established for the rather general manifolds of conullity two; it is claimed that this holds perhaps for all of them.

Key words: semisymmetric Riemannian manifolds, manifolds of conullity two, planar type, semiparallel immersions.

1. INTRODUCTION

The geometry of a Riemannian manifold (M, g) depends essentially on its Levi–Civita connection ∇ and the curvature tensor R. If R is parallel with respect to ∇, i.e. if $\nabla R = 0$, then M is said to be *locally symmetric*. É. Cartan has developed the famous theory of such manifolds, both local and global (see, e.g., [1]).

The geometry of an isometric immersion of (M^m, g) into a Euclidean space E^n or a space form $N^n(c)$ depends essentially on its van der Waerden–Bortolotti connection $\bar{\nabla}$ (which is actually a pair of ∇ and of the normal connection $\nabla^\perp$) and on the second fundamental (mixed) tensor h. The famous Gauss, Peterson–Codazzi, and Ricci equations establish the well-known relationships between h, R, $\bar{\nabla}$, and $R^\perp$ (here the last one is the curvature (mixed) tensor of $\nabla^\perp$).

Such an immersion is said to be *parallel* if $\bar{\nabla}h = 0$. A conclusion from the Gauss equation is that a parallel immersion admits only the locally symmetric (M^m, g).

The differential systems $\nabla R = 0$ and $\bar{\nabla}h = 0$ have their integrability conditions $\Omega \circ R = 0$ and $\bar{\Omega} \circ h = 0$, respectively, where the first ingredients are the curvature 2-form operators; the same integrability conditions can be written also as $R(X,Y) \circ R = 0$ and $\bar{R}(X,Y) \circ h = 0$, respectively. The manifold and immersion satisfying these conditions are called a *semisymmetric manifold* and a *semiparallel immersion*, respectively. From the Gauss and Ricci equations it follows that a semiparallel immersion admits only the semisymmetric manifold.

The local classification of semisymmetric Riemannian manifolds (M, g) is given by Szabó [2]. The most interesting is the class of so-called foliated semisymmetric manifolds (M^m, g), every one of which is foliated by locally Euclidean leaves of codimension two; subsequently they will be called the *manifolds of conullity two*. Kowalski has given for the dimension $m = 3$ a more detailed partition in this class, first in a 1991 preprint and then in [3]. Afterwards it was extended by Boeckx [4] for the arbitrary dimension m. So the *planar*, *hyperbolic*, *parabolic*, and *elliptic* manifolds of conullity two have been distinguished (see [5], Ch. 7).

The concept of semiparallel isometric immersion and the first results on it were given in [6] and then summarized, together with the further results, in [7]. Recent results published in [8–10] make plausible the following conjecture: *if a semiparallel isometric immersion into a Euclidean space E^n realizes a Riemannian manifold (M^m, g) of conullity two, then the latter can be only of planar type.* In [10] it is shown that this is true for arbitrary n if m is 3. In [8] and [9] it is established that this conjecture is valid if such an immersion gives a submanifold with plane generators of codimension two or a normally flat submanifold, respectively. The problem arises: can this conjecture be verified in general?

In the present paper the validity of the above conjecture will be established in a rather general situation using the following known facts.

In [11] it is shown that a submanifold M^m in a Euclidean space E^n is semiparallel ($\equiv$ semisymmetric, extrinsically) if and only if M^m is a second order envelope of the symmetric submanifolds. The last ones are described by Ferus [12–14] as the extrinsic products of two submanifolds; the first of them is the extrinsic product of standard embeddings of symmetric R-spaces, and the second is the extrinsic product of some circles and a plane, i.e. $S^1(c_1) \times ... \times S^1(c_q) \times E^{m_0}$.

The last product is obviously locally Euclidean and on the above second order envelope M^m the tangent subspaces of these products form a foliation with locally Euclidean leaves. There are two possibilities for this envelope to be intrinsically of conullity two.

The first, simpler possibility realizes when these locally Euclidean leaves are of codimension two in this envelope M^m, in other words, when the other component of the symmetric extrinsic product is two-dimensional, and consequently these components envelope the semiparallel surfaces. But the semiparallel surfaces in a Euclidean space are classified completely (see [6,7]): such a surface is either (i) a surface with flat $\bar{\nabla}$, or (ii) a sphere $S^2(c)$, or (iii) a second order envelope of the Veronese surfaces $V^2(c)$. This enables us to solve the problem for this possibility; see Theorems 1 and 2 below.

There is another possibility: the extrinsic product of standard embeddings of symmetric R-spaces carries a foliation, whose leaves have flat ∇ and, together with the leaves enveloped by $S^1(c_1) \times ... \times S^1(c_q) \times E^{m_0}$, generate the locally Euclidean submanifolds of codimension two in M^m. Here the problem is solved for a principal case, where in the role of the standard embedding of a symmetric R-space is the three-dimensional Segre submanifold $S_{(1,2)}(k)$ (see Theorem 3 below).

It is claimed that these results solve perhaps the whole problem above.

2. CLASSIFICATION OF SEMISYMMETRIC RIEMANNIAN MANIFOLDS

A general classification of the semisymmetric Riemannian manifolds (M, g) is provided by Szabó, locally in [2]. First he proves by means of the infinitesimal and the local holonomy groups that for every semisymmetric Riemannian manifold (M, g) there exists a dense open subset U such that around the points of U the manifold M is locally isometric to a direct product of semisymmetric manifolds $M_0 \times M_1 \times ... \times M_r$, where M_0 is the open part of a Euclidean space and the manifolds M_i, $i > 0$, are infinitesimally irreducible simple semisymmetric leaves. Here a semisymmetric M is called a *simple leaf* if at its every point x the primitive holonomy group determines a simple decomposition $T_xM = V_x^{(0)} + V_x^{(1)}$, where this group acts trivially on $V_x^{(0)}$ and there is only one other subspace $V_x^{(1)}$ which is invariant to this group. A simple leaf is said to be infinitesimally irreducible if at least at one point the infinitesimal holonomy group acts irreducibly on $V_x^{(1)}$.

The dimension $\nu(x) = \mathrm{dim} V_x^{(0)}$ is called the *index of nullity* at x and $u(x) = \mathrm{dim} M - \nu(x)$ the *index of conullity* at x.

The classification theorem by Szabó [2] asserts the following (according to the formulation given in [5]).

Theorem A. *Let (M, g) be an infinitesimally irreducible simple semisymmetric leaf and x a point of M. Then one of the following cases occurs*:

(a) $\nu(x) = 0$ *and* $u(x) > 2$: (M, g) *is locally symmetric and hence locally isometric to a symmetric space*;

(b) $\nu(x) = 1$ *and* $u(x) > 2$: (M, g) *is locally isometric to an elliptic, a hyperbolic or a Euclidean cone*;

(c) $\nu(x) = 2$ *and* $u(x) > 2$: (M, g) *is locally isometric to a Kählerian cone*;

(d) $\nu(x) = \dim M - 2$ *and* $u(x) = 2$: (M, g) *is locally isometric to a space foliated by Euclidean leaves of codimension two or to a two-dimensional manifold* (*the last for the case where* $\dim M = 2$).

Note that for (M, g) of the case (d) with $\nu(x) > 2$ the term *manifold of conullity two* is used in [5].

Kowalski, considering the three-dimensional M, introduced for the manifold of conullity two the geometric concept of *asymptotic foliation* in a preprint of 1991. Afterwards this concept was published in [3] and generalized by Boeckx [4] to the arbitrary dimension of M (see also [5]).

Namely, a codimension one submanifold of a Riemannian manifold (M, g) of conullity two is called the *asymptotic leaf* if it is generated by the codimension two Euclidean leaves of this M and if its tangent spaces are parallel along each of the latter leaves (with respect to the Levi–Civita connection ∇ of (M, g)).

A codimension one foliation on such an M is called the *asymptotic foliation* if its integral manifolds are asymptotic leaves.

In what follows a treatment of the asymptotic foliations is given according to Kowalski [3] (and also [4,5]).

Let $O(M)$ be the bundle of orthonormal frames $(e_1, ..., e_m)$ on M, $m = \dim M$. For the bundle $O^*(M)$ of the dual coframes $(\omega^1, ..., \omega^m)$ the following structure equations hold:

$$\mathrm{d}\omega^i = \omega^j \wedge \omega^i_j, \;\; \mathrm{d}\omega^i_j = \omega^k_j \wedge \omega^i_k + \Omega^i_j \tag{2.1}$$

(see [15], Chs. III and IV), where ω^i_j and Ω^i_j are, respectively, the connection 1-forms and the curvature 2-forms of ∇. Here orthonormality yields $\omega^i_j + \omega^j_i = 0$, $\Omega^i_j + \Omega^j_i = 0$.

Let M be of conullity two. Then $O(M)$ and $O^*(M)$ can be adapted to this M so that $(e_3, ..., e_m)$ are tangent to one of the Euclidean leaves and thus the latter are determined by $\omega^1 = \omega^2 = 0$. Since this last differential system is totally integrable, dω^1 and dω^2 must vanish as the algebraic consequences of $\omega^1 = \omega^2 = 0$ (due to the Frobenius theorem, second version; see [16]). This, together with the fact that Euclidean leaves are totally geodesic because M is a simple leaf, yields

$$\omega^1_u = A_u\omega^1 + B_u\omega^2, \;\; \omega^2_u = C_u\omega^1 + F_u\omega^2; \tag{2.2}$$

here (and also further) $u \in \{3, ..., m\}$.

Let the unit vector $X = e_1 \cos\varphi + e_2 \sin\varphi$ be taken so that span$\{X, e_3, ..., e_m\}$ is the tangent plane of an asymptotic leaf. Then $\nabla_{e_u} X = \nabla_X e_u + [e_u, X]$ must belong to the tangent plane of this asymptotic leaf for every value of u. Since the tangent distribution of these leaves is a foliation, this tangent plane contains $[e_u, X]$. Thus this plane must contain also

$$\nabla_X e_u = \nabla_{e_1} e_u \cos\varphi + \nabla_{e_2} e_u \sin\varphi = (\omega^k_u(e_1)e_k)\cos\varphi + (\omega^k_u(e_2)e_k)\sin\varphi.$$

Hence

$$(A_u e_1 + C_u e_2)\cos\varphi + (B_u e_1 + E_u e_2)\sin\varphi$$

must belong to span$\{X, e_3, ..., e_m\}$ and therefore must be a multiple of $X = e_1\cos\varphi + e_2\sin\varphi$. The last condition is equivalent to

$$B_u \sin^2\varphi + (A_u - E_u)\cos\varphi\sin\varphi - C_u\cos^2\varphi = 0.$$

But along this asymptotic leaf $\omega^1\sin\varphi = \omega^2\cos\varphi$, so that this condition reduces to

$$C_u(\omega^1)^2 + (E_u - A_u)\omega^1\omega^2 - B_u(\omega^2)^2 = 0. \tag{2.3}$$

According to [3,5] a foliated M is said to be *planar* if it admits infinitely many asymptotic foliations. If it admits just two (or one, or none, respectively) asymptotic foliations, it is said to be *hyperbolic* (or *parabolic*, or *elliptic*, respectively).

From (2.3) it is seen that the planar foliated M is characterized by $A_u - E_u = B_u = C_u = 0$, i.e. by the fact that (2.2) reduces to

$$\omega_u^1 = A_u\omega^1, \quad \omega_u^2 = A_u\omega^2. \tag{2.4}$$

Then $A = \sum_u A_u e_u$ determines a vector field on such an M. The relations (2.4) can be written as $\omega_u^a = A_u\omega^a$, where $a, b, ...$ run over $\{1, 2\}$. By exterior differentiation, using the structure equations (2.1), from here

$$(\mathrm{d}A_u - A_v\omega_u^v + A_u A_v\omega^v)\wedge\omega^a - \Omega_u^a = 0, \tag{2.5}$$

$\Omega_j^i = \frac{1}{2}R^i_{j,kl}\omega^k\wedge\omega^l$ being the curvature 2-forms of the Riemannian M. Due to Cartan's lemma, from this exterior equation it follows that $\mathrm{d}A_u - A_v\omega_u^v$ is a linear combination of all ω^i. Since the latter turn to zero at an arbitrary fixed point $x \in M$, $\mathrm{d}A_u = A_v\omega_u^v$ at x. Moreover, $\mathrm{d}e_u = e_v\omega_u^v$ at x. Hence $\mathrm{d}A = \sum_u(A_v\omega_u^v e_u + A_u e_v\omega_u^v) = 0$ at x, due to the orthonormality. This shows that A is invariant at x, indeed. Actually the vector field A consists of vector fields on the locally Euclidean leaves of codimension two.

3. THE FIRST POSSIBILITY OF SEMIPARALLEL IMMERSED MANIFOLDS OF CONULLITY TWO

In Introduction it is noted, using the results of [11−14], that there are two possibilities of a semiparallel immersed manifold of conullity two.

Let us start now with the first possibility where the considered submanifold M^m is the second order envelope of $M^2 \times S^1(c_1) \times ... \times S^1(c_q) \times E^{m_0}$ in E^n, $m = 2 + q + m_0$, $n > m$, here M^2 being a parallel surface. The orthonormal frame bundle can be adapted to such a submanifold M^m, following [17], so that at an arbitrary point $x \in M^m$ the basic vectors e_{i_1} are tangent to M^2, e_f are tangent to $S^1(c_{f-1})$, and e_{i_0} belong to E^{m_0}; here $i_1, j_1, ...$ run over $\{1, 2\}$, $f, g, ...$ run over

$\{3, ..., q+2\}$, and $i_0, j_0, ...$ over $\{q+3, ..., m\}$. Moreover, let e_{m+1}, normal to M^m, be directed to the centre of the sphere S^{n_1-1} in which M^2 lies fully and minimally, let the next e_{α_1} be normal to M^m and tangent to this S^{n_1-1} (here $\alpha_1, \beta_1, ...$ run over $\{m+2, ..., m+n_1\}$), and let e_{n^*+f} be directed to the centre of the circle $S^1(c_{f-1})$ (here $n^* = m + n_1 - 2$); if there are more frame vectors normal to M^m at x, they will be denoted by e_ξ.

Then M^m is determined in E^n by the following Pfaff system as one of its integral submanifolds (see [17]):

$$\omega^{m+1} = \omega^{\alpha_1} = \omega^{n^*+f} = \omega^\xi = 0,$$

$$\omega^{m+1}_{i_1} - k\omega^{i_1} = \omega^{\alpha_1}_{i_1} - h^{\alpha_1}_{i_1 j_1}\omega^{j_1} = \omega^{n^*+f}_{i_1} = \omega^\xi_{i_1} = 0, \tag{3.1}$$

$$\omega^{m+1}_f = \omega^{\alpha_1}_f = \omega^{n^*+g}_f - \delta^g_f k_f \omega^f = \omega^\xi_f = 0, \tag{3.2}$$

$$\omega^{m+1}_{i_0} = \omega^{\alpha_1}_{i_0} = \omega^{n^*+f}_{i_0} = \omega^\xi_{i_0} = 0. \tag{3.3}$$

By exterior differentiation the equations $\omega^{m+1}_{i_0} = 0$ in (3.3) and $\omega^{m+1}_f = 0$ in (3.2) give, respectively, $\sum_{j_1} \omega^{j_1}_{i_0} \wedge k\omega^{j_1} = 0$ and $\sum_{j_1} \omega^{j_1}_f \wedge k\omega^{j_1} + k_f\omega^f \wedge \omega^{m+1}_{n^*+f} = 0$, thus due to Cartan's lemma

$$\omega^{i_0}_{j_1} = \lambda^{i_0}_{j_1 l_1}\omega^{l_1}, \quad \omega^f_{i_1} = \lambda^f_{i_1 j_1}\omega^{j_1} + \mu^f_{i_1}\omega^f, \tag{3.4}$$

$$k^{-1}k_f\omega^{n^*+f}_{m+1} = \mu^f_{i_1}\omega^{i_1} + \nu^f\omega^f.$$

In the same manner the equations $\omega^{\alpha_1}_{i_0} = 0$ in (3.3) and $\omega^{\alpha_1}_f = 0$ in (3.2) give $\omega^{j_1}_{i_0} \wedge h^{\alpha_1}_{j_1 k_1}\omega^{k_1} = 0$ and $\omega^{i_1}_f \wedge h^{\alpha_1}_{i_1 j_1}\omega^{j_1} + k_f\omega^f \wedge \omega^{\alpha_1}_{n^*+f} = 0$, thus

$$\sum_{j_1}(h^{\alpha_1}_{j_1 k_1}\lambda^{i_0}_{j_1 l_1} - h^{\alpha_1}_{j_1 l_1}\lambda^{i_0}_{j_1 k_1}) = 0, \quad \sum_{i_1}(h^{\alpha_1}_{i_1 k_1}\lambda^f_{i_1 j_1} - h^{\alpha_1}_{i_1 j_1}\lambda^f_{i_1 k_1}) = 0, \tag{3.5}$$

$$\omega^{\alpha_1}_{n^*+f} + k_f^{-1}\sum_{i_1}\mu^f_{i_1}h^{\alpha_1}_{i_1 k_1}\omega^{k_1} = \lambda^{\alpha_1}_{fg}\omega^g; \tag{3.6}$$

here the coefficients with two subindices are symmetric with respect to these subindices.

The equations $\omega^{n^*+f}_{i_1} = 0$ in (3.1) lead to

$$\omega^f_{i_1} \wedge k_f\omega^f + k\omega^{i_1} \wedge \omega^{n^*+f}_{m+1} + h^{\alpha_1}_{i_1 j_1}\omega^{j_1} \wedge \omega^{n^*+f}_{\alpha_1} = 0.$$

After substitutions from (3.4) and (3.6) this implies

$$\sum_{l_1}\Big\{\sum_{\alpha_1} h^{\alpha_1}_{i_1[j_1}h^{\alpha_1}_{k_1]l_1} + k^2\delta_{i_1[j_1}\delta_{k_1]l_1}\Big\}\mu^f_{l_1} = 0,$$

where $_{[...]}$ means the alternation. Here the expressions between braces $\{$ and $\}$ are the components of the Levi–Civita curvature tensor of M^2. Since M^2 is supposed to be not locally Euclidean, $\mu^f_{l_1} = 0$, and so due to (3.4) and (3.5)

$$\omega^u_{i_1} = \lambda^u_{i_1 j_1}\omega^{j_1}, \quad \sum_{i_1}(h^{\alpha_1}_{i_1 k_1}\lambda^u_{i_1 j_1} - h^{\alpha_1}_{i_1 j_1}\lambda^u_{i_1 k_1}) = 0, \tag{3.7}$$

where the index u runs over the scopes of both f and i_0.

As is noted in Introduction, a parallel, not locally Euclidean surface M^2 in a Euclidean space E^n is either a sphere $S^2(c)$, or a Veronese surface $V^2(k)$. These two cases will further be considered separately.

3.1. Case of the sphere $S^2(c)$

Here in (3.1) $h^{\alpha_1}_{i_1 j_1} = 0$, and so the scope of α_1 is empty, $n_1 = 3$, $n^* = m+1 = 3 + q + m_0$.

The equations $\omega^{m+1}_{i_1} - k\omega^{i_1} = 0$ in (3.1) give after exterior differentiation and using Cartan's lemma that

$$\delta_{i_1 j_1}\mathrm{d}\ln k - \sum_u \lambda^u_{i_1 j_1}\omega^u = \Lambda_{i_1 j_1 k_1}\omega^{k_1},$$

where $\Lambda_{i_1 j_1 k_1}$ is symmetric with respect to all three indices. For $i_1 \neq j_1$ this implies that only λ^u_{11} and λ^u_{22} can be nonzero, but for $i_1 = j_1 = 1$ and $i_1 = j_1 = 2$ it implies that $\lambda^u_{11} = \lambda^u_{22} = \lambda^u$. Hence the first relations in (3.7) reduce to

$$\omega^u_1 = \lambda^u\omega^1, \quad \omega^u_2 = \lambda^u\omega^2, \tag{3.8}$$

which in comparison with (2.4) show that the considered M^m of conullity two is of planar type and now $A_u = -\lambda^u$.

This result can be formulated as

Theorem 1. *A semiparallel submanifold M^m in E^n, which is the second order envelope of product-submanifolds $S^2(c) \times S^1(c_2) \times ... \times S^1(c_{1+q}) \times E^{m_0}$, is intrinsically of conullity two of planar type.*

Note that the relation above leads to

$$\mathrm{d}\ln k = \sum_u \lambda^u\omega^u, \tag{3.9}$$

because now all $\Lambda_{i_1 j_1 k_1}$ turn to zero.

3.2. Case of the Veronese orbit $V^2(k)$

Due to [7] here the matrices $h^{\alpha_1} = \|h^{\alpha_1}_{i_1 j_1}\|$ are as follows:

$$h^{m+2} = \begin{pmatrix} k\sqrt{3} & 0 \\ 0 & k\sqrt{3} \end{pmatrix}, \quad h^{m+3} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \quad h^{m+4} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \quad h^{\xi} = 0.$$

The second relation in (3.7) can be considered as the condition that the product of two symmetric matrices h^{α_1} and $\lambda^u = \|\lambda^u_{i_1 j_1}\|$ is a symmetric matrix. For $\alpha_1 = m+2$ this condition is satisfied trivially. Since

$$h^{m+3} \cdot \lambda^u = \begin{pmatrix} k\lambda^u_{11} & k\lambda^u_{12} \\ -k\lambda^u_{21} & -k\lambda^u_{22} \end{pmatrix}, \qquad h^{m+4} \cdot \lambda^u = \begin{pmatrix} k\lambda^u_{21} & k\lambda^u_{22} \\ k\lambda^u_{11} & k\lambda^u_{12} \end{pmatrix},$$

the same condition for $\alpha_1 = m+3$ and $\alpha_1 = m+4$ implies $\lambda^u_{21} = -\lambda^u_{12}$ and $\lambda^u_{11} = \lambda^u_{22}$. This, together with the symmetricity condition $\lambda^u_{21} = \lambda^u_{12}$, leads to $\lambda^u_{12} = \lambda^u_{21} = 0$, so that

$$\omega^u_1 = \lambda^u \omega^1, \qquad \omega^u_2 = \lambda^u \omega^2, \tag{3.10}$$

where λ^u is the common value of λ^u_{11} and λ^u_{22}. The comparison with (2.4) shows that the considered M^m is intrinsically of conullity two of planar type and now $A_u = -\lambda^u$.

This result can be formulated as

Theorem 2. *A semiparallel submanifold M^m in E^n, which is the second order envelope of product-submanifolds $V^2(k) \times S^1(c_2) \times ... \times S^1(c_{1+q}) \times E^{m_0}$, is intrinsically of conullity two of planar type.*

4. GEOMETRICAL DESCRIPTIONS

The geometry of submanifolds M^m in E^n, considered in Theorems 1 and 2, is described by the property that they are intrinsically of conullity two of planar type. A further description is possible due to the fact that for both of these cases there hold similar formulae (3.8) and (3.10). They allow us to introduce at an arbitrary point $x \in M^m$ the vector $l = e_u \lambda^u$, which is due to $\lambda^u = -A_u$ opposite to the vector A, introduced above at the end of Section 2. So a vector field l is determined on every locally Euclidean leaf of codimension two in M^m.

For submanifolds M^m of Theorems 1 and 2 the field l has a special quality. Recall that for every submanifold in E^n the formula $\Omega^j_i = \omega^\alpha_i \wedge \omega^j_\alpha$ holds (see [7], formula (2.6)). For M^m considered here thus $\Omega^a_u = \omega^{m+1}_u \wedge \omega^a_{m+1} + \omega^{n^*+g}_u \wedge \omega^a_{n^*+g}$, where a is in the role of i_1 in Eqs. (3.1)–(3.3), and u is either f or i_0. Due to the same equations $\omega^{m+1}_u = 0$ and $\omega^a_{n^*+g} = -\omega^{n^*+g}_a = 0$, hence here $\Omega^a_u = 0$. Since a runs over $\{1, 2\}$, from (2.5) it follows that now

$$\mathrm{d}\lambda^u = -\sum_v \lambda^v(\omega_v^u - \lambda^u\omega^v). \tag{4.1}$$

This leads to

$$\begin{aligned}
\mathrm{d}l &= \mathrm{d}e_u\lambda^u + e_u\mathrm{d}\lambda^u \\
&= (\omega_u^a e_a + \omega_u^v e_v + \omega_u^{n^*+f} e_{n^*+f})\lambda^u + e_u\Big[-\sum_v \lambda^v(\omega_v^u - \lambda^u\omega^v)\Big] \\
&= l\sum_v \lambda^v\omega^v + \theta,
\end{aligned}$$

where θ is a vector valued 1-form, normal to the locally Euclidean leaf, but this shows that the integral lines of the vector field l are the geodesic lines of this leaf. Since the leaf is locally Euclidean, the geodesic lines can be taken for the coordinate lines of some system of Descartes' coordinates $x^3, ..., x^m$ with orthogonal net on this leaf. Then $e_u = \frac{\partial}{\partial x^u}$ and $\omega^u = \mathrm{d}x^u$, $\omega_u^v = 0$.

These considerations have been used already in [9] for the case of spheres $S^2(c)$, when the second order envelope M^m has flat normal connection. Now they are extended also to the case of Veronese orbits $V^2(k)$. But there is an essential difference, which concerns the 2-dimensional submanifolds (surfaces) intersecting orthogonally the locally Euclidean leaves. These surfaces are the second order envelopes of the spheres or Veronese orbits, respectively.

Since the spheres are totally umbilical, these envelopes, as the orthogonal surfaces, are here the spheres themselves. In [9] it is shown that the radius of this orthogonal sphere along the geodesic line tangent to l is a linear function of the arc length of this geodesic. Indeed, these orthogonal surfaces are tangent to e_a and for them

$$\mathrm{d}e_a = e_b\omega_a^b + e_u\omega_a^u + e_{m+1}\omega_a^{m+1} = e_b\omega_a^b + (l + ke_{m+1})\omega^a. \tag{4.2}$$

Thus these surfaces are the spheres with the radius $r = (l^2 + k^2)^{-1/2}$. From (4.1) and (3.9) it follows that $\mathrm{d}r = r_u\omega^u$, where $r_u = -r\lambda^u$ and thus $\mathrm{d}r_u = r_v\omega_u^v$. If we consider this in the Descartes' coordinates above, then due to $\omega_u^v = 0$ here $r_u = c_u = \mathrm{const}$, and due to $\omega^u = \mathrm{d}x^u$ thus $r = c_u x^u + c$.

Hence here M^m is intrinsically a Riemannian product of an elliptic cone and a locally Euclidean manifold.

The situation is different in the case of Veronese orbits. In [18] it is established that in the Euclidean space of dimension > 5 the second order envelope of Veronese orbits $V^2(k)$ needs not be such an orbit itself (or its open part), but in dimension ≥ 7 this envelope can be an arbitrary surface of positive Gaussian curvature.

In [19] the subcase is investigated, when a semiparallel M^m in E^n is the second order envelope of $V^2(k) \times E^{m_0}$, i.e. there are no circular factors in the product of Theorem 2; here $m = m_0 + 2$, of course. The formulae (3.9) have been obtained for this subcase already in [19] (see the formulae (3.4) there), but their interpretation as showing the planarity is not given yet in [19], because this concept was introduced much later. Also the vector field l works in [19], and the absence of circular factors makes the situation simpler. It is shown that if $l \neq 0$, then M^{m_0+2} is a product-submanifold $M^3 \times E^{m_0-1}$, where M^3 is the second order envelope of $V^2(k) \times E^1$, which is a cone with a point-vertex and 1-dimensional generators, whose directrix on the sphere S^{n-m_0} around the vertex is the second order envelope of Veronese surfaces $V^2(k)$ in S^{n-m_0} (see [19], Theorem 2 for $m = 2$).

5. ANOTHER POSSIBILITY

Another possibility of a semiparallel isometric immersion of the manifold M^m of conullity two into E^n is noted in Introduction and is the case where there is only one standard embedding of a symmetric *R*-space, which is a Segre submanifold $S_{(1,2)}(k)$. The latter is a 3-dimensional complete parallel submanifold in a sphere $S^5(k^2) \subset E^6$, generated by 2-dimensional great spheres $S^2(k)$ of $S^5(k^2)$, intersected orthogonally by great circles of $S^5(k^2)$, and is immersed into E^6 symmetric space $O(6, \mathrm{R})/O(2, \mathrm{R}) \times O(3, \mathrm{R})$ (see, e.g., [7], Sec. 21).

So let M^m be the second order envelope of product-submanifolds $S_{(1,2)}(k) \times S^1(c_1) \times ...S^1(c_q) \times E^{m_0}$ with variable k and $c_1, ..., c_q$, $m = 3 + q + m_0$. The orthonormal frame bundle will be adapted to this M^m so that at an arbitrary point $x \in M^m$ the vectors e_a, where $a, b, ...$ run over $\{1, 2\}$, are tangent to the generator sphere $S^2(k)$ and e_3 is tangent to the generator circle $S^1(k)$ of $S_{(1,2)}(k)$, going through x. Moreover, let e_f, where $f, g, ...$ run over $\{4, ..., 3 + q\}$, be tangent to the circle $S^1(c_{f-3})$ and e_{i_0}, where $i_0, j_0, ...$ run over $\{3 + q + 1, ..., 3 + q + m_0\}$, belong to E^{m_0}. Among the normal to M^m basic vectors of the orthonormal frame at x, let e_{m+1} be directed to the centre of the sphere $S^5(k^2)$ containing $S_{(1,2)}(k)$, and e_{m+1+a} be normal to $S_{(1,2)}(k)$ and tangent to $S^5(k^2)$ (recall, here $a \in \{1, 2\}$). Finally, let e_{m+f} be directed to the centre of the circle $S^1(c_{f-3})$ at x, and e_ξ be the remaining normal to M^m basic vectors in E^n.

Then M^m is determined by the following Pfaff system, as one of its integral submanifolds:

$$\omega^{m+1} = \omega^{m+1+a} = \omega^{m+f} = \omega^\xi = 0,$$

$$\omega_a^{m+1} = k\omega^a, \ \ \omega_3^{m+1} = k\omega^3, \ \ \omega_b^{m+1+a} = \delta_b^a k\omega^3, \ \ \omega_3^{m+1+a} = k\omega^a, \tag{5.1}$$

$$\omega_a^{m+f} = \omega_3^{m+f} = \omega_a^\xi = \omega_3^\xi = 0, \tag{5.2}$$

$$\omega_f^{m+1} = \omega_f^{m+1+a} = \omega_f^{m+f} - \gamma_f\omega^f = \omega_f^\xi = 0, \tag{5.3}$$

$$\omega_{i_0}^{m+1} = \omega_{i_0}^{m+1+a} = \omega_{i_0}^{m+f} = \omega_{i_0}^\xi = 0. \tag{5.4}$$

Note that Eqs. (5.1) here turn to Eqs. (21.5) with $c = 0$ in [7] for a Segre orbit $S_{(1,2)}(k)$ if we take $m = 3$ and use the renumeration $1 \to 2$, $2 \to 3$, $3 \to 1$.

The first equations in (5.1) give by exterior differentiation

$$-\mathrm{d}\ln k\wedge\omega^a+\omega^{m+1+a}_{m+1}\wedge\omega^3+\omega^a_f\wedge\omega^f+\omega^a_{i_0}\wedge\omega^{i_0}=0,$$
$$-\mathrm{d}\ln k\wedge\omega^3+\sum_a\omega^{m+1+a}_{m+1}\wedge\omega^a+\omega^3_f\wedge\omega^f+\omega_{i_0}\wedge\omega^{i_0}=0,$$

and from here by Cartan's lemma

$$-\mathrm{d}\ln k=\kappa\omega^3+A_f\omega^f+A_{i_0}\omega^{i_0}, \tag{5.5}$$
$$\omega^{m+1+a}_{m+1}=\kappa\omega^a+B^a_f\omega^f+B^a_{i_0}\omega^{i_0}, \tag{5.6}$$
$$\omega^a_f=A_f\omega^a+B^a_f\omega^3+C^a_{fg}\omega^g+C^a_{fi_0}\omega^{i_0}, \tag{5.7}$$
$$\omega^a_{i_0}=A_{i_0}\omega^a+B^a_{i_0}\omega^3+C^a_{fi_0}\omega^f+C^a_{i_0j_0}\omega^{j_0}, \tag{5.8}$$
$$\omega^3_f=\sum_a B^a_f\omega^a+A_f\omega^3+D_{fg}\omega^g+E_{fi_0}\omega^{i_0}, \tag{5.9}$$
$$\omega^3_{i_0}=\sum_a B^a_{i_0}\omega^a+A_{i_0}\omega^3+E_{fi_0}\omega^f+F_{i_0j_0}\omega^{j_0}; \tag{5.10}$$

here is symmetry with respect to subscripts fg and i_0j_0.

The remaining equations of (5.1) give by exterior differentiation

$$\begin{aligned}(\delta^a_b\omega^3_c+\delta^a_c\omega^3_b-\delta^b_c\omega^{m+1+a}_{m+1})\wedge\omega^c+[-\delta^a_b\mathrm{d}\ln k+(\omega^a_b-\omega^{m+1+a}_{m+1+b})]\wedge\omega^3\\+\delta^a_b(\omega^3_f\wedge\omega^f+\omega^3_{i_0}\wedge\omega^{i_0})=0,\end{aligned}$$

$$[-\delta^a_b\mathrm{d}\ln k+(\omega^a_b-\omega^{m+1+a}_{m+1+b})]\wedge\omega^b+(2\omega^a_3-\omega^{m+1+a}_{m+1})\wedge\omega^3+\omega^a_f\wedge\omega^f+\omega^a_{i_0}\wedge\omega^{i_0}=0.$$

Here, in the first exterior equation, the first terms reduce by $a=b=1$, $a=b=2$, respectively, to

$$(2\omega^3_1-\omega^{m+2}_{m+1})\wedge\omega^1+\omega^3_2\wedge\omega^2,\quad \omega^3_1\wedge\omega^1+(2\omega^3_2-\omega^{m+3}_{m+1})\wedge\omega^2.$$

Thus these equations, together with (5.5)–(5.10), give

$$\omega^3_a=P_{(a)}\omega^a+B^a_f\omega^f+B^a_{i_0}\omega^{i_0}.$$

Now the same equations by $a=b$ lead to $B^a_f=B^a_{i_0}=0$, $P_{(a)}=-\kappa$, and $\omega^2_1=\omega^{m+3}_{m+2}$, so that

$$\omega^a_3=\kappa\omega^a, \tag{5.11}$$
$$\omega^a_f=A_f\omega^a+C^a_{fg}\omega^g+C^a_{fi_0}\omega^{i_0},\;\; \omega^a_{i_0}=A_{i_0}\omega^a+C^a_{fi_0}\omega^f+C^a_{i_0j_0}\omega^{j_0}. \tag{5.12}$$

The equations $\omega_a^{m+f} = 0,\ \omega_3^{m+f} = 0$ in (5.2) give

$$k^{-1}\omega_a^f \wedge \gamma_f\omega^f + \omega^a \wedge \omega_{m+1}^{m+f} + \omega^3 \wedge \omega_{m+1+a}^{m+f} = 0,$$
$$k^{-1}\omega_3^f \wedge \gamma_f\omega^f + \omega^3 \wedge \omega_{m+1}^{m+f} + \omega^a \wedge \omega_{m+1+a}^{m+f} = 0,$$

and from here $C_{fg}^a = C_{fi_0}^a = D_{fd} = E_{fi_0} = 0$, $\omega_{m+1}^{m+f} - k^{-1}A_f\gamma_f\omega^f = 0$, $\omega_{m+1+a}^{m+f} = 0$.

The equations $\omega_{i_0}^{m+1} = 0$ in (5.4) lead to $\sum_a \omega_{i_0}^a \wedge \omega^a + \omega_{i_0}^a \wedge \omega^3 = 0$, and thus $C_{i_0 j_0}^a = F_{i_0 j_0} = 0$. Hence (5.12), (5.9), (5.10) reduce to

$$\omega_f^a = A_f\omega^a,\ \ \omega_{i_0}^a = A_{i_0}\omega^a,\ \ \omega_f^3 = A_f\omega^3,\ \ \omega_{i_0}^3 = A_{i_0}\omega^3. \qquad (5.13)$$

The distribution, which is determined by the system $\omega^a = 0$ (recall, a runs over $\{1, 2\}$), is a foliation, because due to (5.11) and (5.13)

$$\mathrm{d}\omega^a = \omega^b \wedge \omega_b^a + \omega^3 \wedge \kappa\omega^a + \omega^f \wedge A_f\omega^a + \omega^{i_0} \wedge A_{i_0}\omega^a.$$

The leaves of this foliation are generated by the second order envelopes of the products of circular generators of $S_{(1,2)}(k)$ and of $S^1(c_2) \times ... \times S^1(c_{1+q}) \times E^{m_0}$.

For them the indices $u, v, ...$ can be introduced, which run over the scope containing 3 and the scopes of f and i_0. These leaves are intrinsically locally Euclidean, because e_a, e_{m+1}, e_{m+1+a}, e_{m+f}, e_ξ are normal to them and due to (5.1)–(5.4)

$$\Omega_u^v = \omega_u^a \wedge \omega_a^v + \omega_u^{m+1} \wedge \omega_{m+1}^v + \omega_u^{m+1+a} \wedge \omega_{m+1+a}^v + \omega_u^{m+f} \wedge \omega_{m+f}^v + \omega_u^\xi \wedge \omega_\xi^v = 0.$$

Hence the considered M^m in E^n is intrinsically of conullity two. Since (5.11) and the first equations in (5.13) can be joined into $\omega_u^a = A_u\omega^a$ if we denote $\kappa = A_3$, it is of planar type, as shows comparison with (2.4).

The result can be formulated as follows.

Theorem 3. *A semiparallel submanifold M^m in E^n, which is the second order envelope of product-submanifolds $S_{(1,2)}(k) \times S^1(c_1) \times ... \times S^1(c_q) \times E^{m_0}$, is intrinsically of conullity two of planar type.*

To characterize the geometry of such an M^m, the analysis of the first part of Section 4 can be repeated. The only difference is that now the scope of the index u contains also the value 3, so that among Ω_u^a in (2.5) there is also Ω_3^a, which is zero too, because due to (5.1) and (5.2)

$$\Omega_3^a = \omega_3^{m+1} \wedge \omega_{m+1}^a + \omega_3^{m+1+a} \wedge \omega_{m+1+a}^a + \sum_f \omega_3^{m+f} \wedge \omega_{m+f}^a$$
$$= k\omega^3 \wedge (-k\omega^a) + k\omega^a \wedge (-k\omega^3) = 0.$$

Denoting, as above, $A_u = -\lambda^u$, from (2.5) the same $\mathrm{d}\lambda^u = -\sum_v \lambda^v(\omega_v^u - \lambda^u\omega^v)$ can be deduced, only now the scope of u, v contains one more value 3. For the vector $l = e_u\lambda^u$ there now holds

$$\mathrm{d}l = -\lambda^u e_a\omega^a + \delta_{u3}k\Big(e_{m+1}\omega^3 + \sum_a e_{m+1+a}\omega^a\Big) + \delta_{uf}\gamma_f e_{m+f}\omega^f + l\sum_v \lambda^v\omega^v,$$

where all terms, except the last one, are normal to the locally Euclidean leaf. Thus the integral lines of the vector field l on this leaf are geodesics of this leaf. The Descartes' coordinates $x^3, ..., x^m$ can be taken on this leaf as before in Section 4, so that these geodesics are some coordinate lines. Then $\omega^u = \mathrm{d}x^u$, $\omega_u^v = 0$.

The surfaces, orthogonal to these leaves, are tangent to e_a and for them (4.2) hold, as before. Thus these surfaces are two-dimensional spheres whose radius is a linear function of the coordinate along the geodesics. Hence the submanifold M^m is intrinsically a Riemannian product of an elliptic cone and a locally Euclidean manifold.

The second order envelopes of the Segre orbits are investigated in [20,21]. It is shown, in particular, that such an envelope of $S_{(1,2)}(k)$ with variable k is a "logarithmic spiral tube", generated by a family of concentric two-dimensional spheres whose orthogonal trajectories are the congruent logarithmic spirals with the common pole in the centre of the family spheres.

The spheres above, whose radius is of linear dependence on the coordinate along the geodesics, are the generating spheres of these "logarithmic spiral tubes".

6. CONCLUSIONS

In connection with the conjecture, formulated in Introduction, the following question arises: is this conjecture verified completely by Theorems 1–3 above?

The answer concerning the first possibility is positive because the spheres and Veronese surfaces are the only irreducible two-dimensional parallel submanifolds, which are the main symmetric orbits.

For the other possibility the answer is also positive if one restricts oneself to the case of $(m_0 + q + 3)$-dimensional submanifolds. The three-dimensional parallel submanifolds in a Euclidean space are classified in [22,23] (see also [7], Secs. 20 and 21). Among them the only ones carrying a foliation whose leaves have flat ∇ are Segre orbits $S_{(1,2)}(k)$; see Theorem 3.

The case of dimension $> m_0 + q + 3$ remains open. The list of symmetric R-spaces is given in [24] (see also [25]), but it is not yet clear which of them have the needed now property.

Nevertheless, it is very plausible that the submanifolds of Theorems 1–3 are the only semiparallel submanifolds in E^n, which are intrinsically of conullity two.

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Semiparalleelset isomeetrilist sisestust lubavad Riemanni muutkonnad konullisusega kaks

Ülo Lumiste

Riemanni muutkond konullisusega kaks võib olla kas planaarne, hüperboolne, paraboolne või elliptiline. Mitme erijuhu puhul on varem näidatud, et semiparalleelset isomeetrilist sisestust eukleidilisse ruumi lubavad ainult planaarset tüüpi sellised muutkonnad. Nüüd on see tõestatud üsna üldiste muutkondade jaoks konullisusega kaks. On oletatud, et see võib kehtida ka kõigi viimast tüüpi muutkondade puhul.