Proc. Estonian Acad. Sci. Phys. Math., 2002, **51**, 4, 221–237
https://doi.org/10.3176/phys.math.2002.4.02

Quadratic spline collocation method for weakly singular integral equations on graded grids

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Received 7 May 2002, in revised form 20 June 2002

Abstract. The spline collocation method for weakly singular Fredholm integral equations of the second kind is studied. Using smooth quadratic splines, we derive the rate of uniform convergence of this method on graded grids.

Key words: weakly singular integral equation, collocation method, quadratic splines, graded grids.

1. INTRODUCTION

We consider the integral equation

$$y(t) = \int_a^b g(t,s)\kappa(t-s)y(s)ds + f(t), \qquad t \in [a,b], \tag{1}$$

where $a, b \in \mathbf{R}$, $a < b$, the given functions $g : [a,b] \times [a,b] \to \mathbf{R}$, $\kappa : [a-b, b-a] \setminus \{0\} \to \mathbf{R}$, and $f : [a,b] \to \mathbf{R}$ are at least continuous and the function κ may have at most a weak singularity at 0: $|\kappa(\tau)| \le \mathrm{const}|\tau|^{-\alpha}$, $0 \le \alpha < 1$. The main difficulty with equations of type (1) (which arise in the potential theory, atmospheric physics, and many other fields) is the nonsmoothness of the solution y: even if the functions g and f are smooth, the derivatives of the function y (starting from a certain order) are, in general, unbounded at the boundary points $t = a$ and $t = b$ (see, for example, [1]). Therefore it is quite complicated to construct high-order approximation methods for the numerical solution of equations of type (1) (see, for example, [1–4]).

In [5] the numerical solution of a wide class of weakly singular integral equations is studied using the collocation method with smooth quadratic splines on quasi-uniform grids. In this paper we consider the same class of integral equations with weakly singular kernels. We establish the conditions which guarantee the convergence of the approximate solutions obtained by the collocation method with continuously differentiable quadratic splines on graded grids. We also derive uniform convergence estimates for that method and present some numerical examples. The main results of the paper are stated in Theorems 1–3.

2. INTEGRAL EQUATION

Throughout this paper we denote by $C[a,b]$ the Banach space of continuous functions $x:[a,b]\to\mathbf{R}$ with the norm $||x||_{C[a,b]}=\max_{a\le t\le b}|x(t)|$. We denote by $C^m(\Omega),\ \Omega\subset\mathbf{R}^n,\ m,n\in\mathbf{N}$, the set of m times continuously differentiable functions $x:\Omega\to\mathbf{R}$. For a Banach space E we denote by $\mathcal{L}(E)$ the Banach space of linear bounded operators $A:E\to E$ with the norm $||A||_{\mathcal{L}(E)}=\sup\{||Ax||_E : x\in E,\ ||x||_E=1\}$.

Let the following assumptions about the functions g, κ, and f appearing in Eq. (1) be fulfilled:

(A1) $g\in C^3([a,b]\times[a,b]),\ \kappa\in C^2([a-b,b-a]\setminus\{0\})$, and

$$|\kappa''(\tau)|\le \mathrm{const}|\tau|^{-\beta}\qquad(0<\beta<3)$$

for any $\tau\in[a-b,b-a]\setminus\{0\}$;

(A2) $f\in C^{3,\beta}[a,b]$, where

$$C^{3,\beta}[a,b]=\left\{y\in C[a,b]\cap C^3(a,b):\ \sup_{a<t<b}\frac{|y'''(t)|}{(t-a)^{-\beta}+(b-t)^{-\beta}}<\infty\right\}.$$

Notice that $C^{3,\beta}[a,b]$ is a Banach space with respect to the norm

$$||y||_{C^{3,\beta}[a,b]}=\max_{a\le t\le b}|y(t)|+\sup_{a<t<b}\frac{|y'''(t)|}{(t-a)^{-\beta}+(b-t)^{-\beta}},\quad y\in C^{3,\beta}[a,b].$$

It follows from the definition of the space $C^{3,\beta}[a,b]$ that if $y\in C^{3,\beta}[a,b]$, then

$$|y'''(t)|\le d_3\left[(t-a)^{-\beta}+(b-t)^{-\beta}\right],\qquad t\in(a,b),\tag{2}$$

where d_3 is a positive constant.

Remark 1. For the estimates of $y,\ y'$, and y'' of the function $y\in C^{3,\beta}[a,b]$ see [5].

Remark 2. If under the assumptions (A1) and (A2) Eq. (1) has an integrable solution y, then $y\in C^{3,\beta}[a,b]$ (see, for example, [1], p. 7; [6]).

3. INTERPOLATION BY QUADRATIC SPLINES ON GRADED GRIDS

Let $n = 2k,\ k \in \mathbf{N}$, and let

$$a = t_{n,0} < t_{n,1} < ... < t_{n,n} = b \tag{3}$$

be a partition of the interval $[a, b]$ such that

$$\begin{cases} t_{n,i} = a + \left(\dfrac{b-a}{2}\right)\left(\dfrac{2i}{n}\right)^r, & i = 0, ..., \dfrac{n}{2}; \\ t_{n,n/2+i} = a + b - t_{n,n/2-i}, & i = 1, ..., \dfrac{n}{2}; \end{cases} \tag{4}$$

where $r \in [1, \infty)$ is a fixed real number not depending on n. The partition {(3),(4)} is called a graded grid, the parameter r characterizes the nonuniformity of the partition: if $r = 1$, then we obtain a uniform grid, if $r > 1$, then the knots $t_{n,i},\ i = 0, ..., n$, are more densely located in the neighbourhood of the points $t = a$ and $t = b$.

Denote

$$\Delta_n^r = \{t_{n,i} :\ i = 0, ..., n\};$$
$$h_i = t_{n,i+1} - t_{n,i}, \quad i = 0, ..., n-1.$$

Let $S_{2,1}(\Delta_n^r)$ be the linear space of smooth quadratic splines defined as

$$S_{2,1}(\Delta_n^r) = \left\{z \in C^1[a, b] :\ z|_{[t_{n,i},\ t_{n,i+1}]} \in \pi_2,\ i = 0, ..., n-1,\ t_{n,i} \in \Delta_n^r,\ i = 0, ..., n\right\},$$

where π_2 is the set of polynomials of the second order. We introduce the interpolation operator $P_n : C[a, b] \to C[a, b]$ which assigns to any function $y \in C[a, b]$ the function $P_n y \in S_{2,1}(\Delta_n^r) \subset C[a, b]$ satisfying

$$(P_n y)(x_i) = y(x_i), \quad i = 0, ..., n+1. \tag{5}$$

Here $x_i,\ i = 0, ..., n+1$, are the interpolation points defined by the formulas

$$x_0 = a, \qquad x_i = t_{n,i-1} + \eta h_{i-1}, \quad i = 1, ..., n, \qquad x_{n+1} = b, \tag{6}$$

where $\eta \in (0, 1)$ is a fixed real number not depending on n. It is easy to show that the operator P_n is well defined, i.e. the function $P_n y \in S_{2,1}(\Delta_n^r)$ satisfying the conditions (5) is uniquely determined for any function $y \in C[a, b]$ (in [5] it is proved for a quasi-uniform grid, but the proof is same for the graded grid {(3),(4)}). It is also obvious (due to the linearity of the interpolation conditions (5)) that the operator P_n is linear.

Lemma 1. *Let $P_n : C[a, b] \to C[a, b]$ be the interpolation operator given by the conditions* (5). *Then for any $y \in C[a, b]$*

$$||P_n y - y||_{C[a,b]} \to 0, \qquad n \to \infty. \tag{7}$$

Proof. As in [5] we present the function $P_n y$ on every interval $[t_{n,i}, t_{n,i+1}]$, $i = 0, ..., n-1$, in the form

$$(P_n y)(t) = y_{i+1} + \left[\frac{(1-\eta)^2 h_i}{2} - \frac{(t_{n,i+1} - t)^2}{2h_i}\right] m_i + \left[\frac{(t - t_{n,i})^2}{2h_i} - \frac{\eta^2 h_i}{2}\right] m_{i+1},$$

where $y_{i+1} = y(x_{i+1}),\ i = 0, ..., n-1$, are given and $m_i = (P_n y)'(t_{n,i})$, $i = 0, ..., n$, are unknowns to be determined. Considering the continuity conditions of the function $P_n y$ on the interval $[a, b]$ (i.e. $(P_n y)(t_{n,i} - 0) = (P_n y)(t_{n,i} + 0)$, $i = 1, ..., n-1$) and the conditions $(P_n y)(x_0) = y(x_0)$ and $(P_n y)(x_{n+1}) = y(x_{n+1})$, we come to the following linear system with respect to the parameters m_i:

$$\begin{cases} \mu_0 m_0 + \nu_0 m_1 = g_0; \\ \lambda_i m_{i-1} + \mu_i m_i + \nu_i m_{i+1} = g_i, \quad i = 1, ..., n-1; \\ \lambda_n m_{n-1} + \mu_n m_n = g_n; \end{cases} \tag{8}$$

where

$$\mu_0 = 2\eta - \eta^2, \quad \nu_0 = \eta^2, \quad g_0 = \frac{2(y_1 - y_0)}{h_0},$$

$$\lambda_i = \frac{(1-\eta)^2 h_{i-1}}{(1-\eta^2)h_{i-1} + (2\eta - \eta^2)h_i}, \quad \mu_i = 1, \quad \nu_i = \frac{\eta^2 h_i}{(1-\eta^2)h_{i-1} + (2\eta - \eta^2)h_i},$$

$$g_i = \frac{2(y_{i+1} - y_i)}{(1-\eta^2)h_{i-1} + (2\eta - \eta^2)h_i}, \quad i = 1, ..., n-1,$$

$$\lambda_n = (1-\eta)^2, \quad \mu_n = 1 - \eta^2, \quad g_n = \frac{2(y_{n+1} - y_n)}{h_{n-1}}.$$

Now we define additional knots $t_{n,-i} = t_{n,0} - ih_0,\ t_{n,n+i} = t_{n,n} + ih_{n-1}$, $i = 1, ..., l$, where $l \in \mathbf{N}$ is independent of n and will be specified later (so we have $h_{-i} = h_0,\ i = 1, ..., l$ and $h_{n+i} = h_{n-1},\ i = 0, ..., l-1$). Making in the system (8) the change of variable $z_i = m_i(h_{i-l} \ldots h_{i+l-1})^{1/(2l)},\ i = 0, ..., n$, we obtain a new linear system with respect to new unknowns z_i:

$$\begin{cases} \mu_0 z_0 + \nu_0 \left(\dfrac{h_{-l}}{h_l}\right)^{1/(2l)} z_1 = g_0 \left(h_{-l} \ldots h_{l-1}\right)^{1/(2l)}, \\ \lambda_i \left(\dfrac{h_{i+l-1}}{h_{i-l-1}}\right)^{1/(2l)} z_{i-1} + \mu_i z_i + \nu_i \left(\dfrac{h_{i-l}}{h_{i+l}}\right)^{1/(2l)} z_{i+1} \\ \qquad = g_i \left(h_{i-l} \ldots h_{i+l-1}\right)^{1/(2l)}, \ i = 1, ..., n-1, \\ \lambda_n \left(\dfrac{h_{n+l-1}}{h_{n-l-1}}\right)^{1/(2l)} z_{n-1} + \mu_n z_n = g_n \left(h_{n-l} \ldots h_{n+l-1}\right)^{1/(2l)}. \end{cases} \tag{9}$$

As

$$d_0 = \mu_0 - \nu_0 \left(\frac{h_{-l}}{h_l}\right)^{1/(2l)} = 2\eta - \eta^2 - \eta^2 \left(\frac{h_{-l}}{h_l}\right)^{1/(2l)} \geq 2\eta(1-\eta) > 0 \quad (10)$$

and

$$d_n = \mu_n - \lambda_n \left(\frac{h_{n+l-1}}{h_{n-l-1}}\right)^{1/(2l)} = 1 - \eta^2 - (1-\eta)^2 \left(\frac{h_{n+l-1}}{h_{n-l-1}}\right)^{1/(2l)} \geq 2\eta(1-\eta) > 0, \quad (11)$$

the diagonal dominance of the matrix of the system (9) (and thus the solvability of the system (9)) depends on the quantities $d_i,\ i = 1, ..., n-1$,

$$d_i = \mu_i - \lambda_i \left(\frac{h_{i+l-1}}{h_{i-l-1}}\right)^{1/(2l)} - \nu_i \left(\frac{h_{i-l}}{h_{i+l}}\right)^{1/(2l)}.$$

Let us study the quantities $h_{i+l-1}/h_{i-l-1},\ \ i = 1, ..., n-1,\ \ l \geq 1$. For $0 \leq i-l-1 < n/2$ we have

$$\frac{h_{i+l-1}}{h_{i-l-1}} \leq \frac{h'_{i+l-1}}{h_{i-l-1}} = \frac{(i+l)^r - (i+l-1)^r}{(i-l)^r - (i-l-1)^r} = \frac{(\xi+l)^{r-1}}{(\xi-l)^{r-1}}, \quad i-1 < \xi < i,$$

where

$$h'_j = \left(\frac{b-a}{2}\right)\left(\frac{2}{n}\right)^r \left[(j+1)^r - j^r\right], \quad j = 0, 1, ...$$

The expression

$$\frac{(\xi+l)^{r-1}}{(\xi-l)^{r-1}} = \left(1 + \frac{2l}{\xi - l}\right)^{r-1}$$

obtains its maximum value if $i = l+1$, so we get

$$\frac{h_{i+l-1}}{h_{i-l-1}} \leq (2l+1)^r - (2l)^r, \quad 0 \leq i-l-1 < \frac{n}{2}.$$

If $i-l-1 < 0$, then

$$\frac{h_{i+l-1}}{h_{i-l-1}} \leq \frac{h'_{i+l-1}}{h_0} = (i+l)^r - (i+l-1)^r = r(\xi+l)^{r-1}, \quad i-1 < \xi < i.$$

The expression $r(\xi+l)^{r-1}$ obtains its maximum value if $i = l$, so

$$\frac{h_{i+l-1}}{h_{i-l-1}} \leq (2l)^r - (2l-1)^r \leq (2l+1)^r - (2l)^r, \quad i-l-1 < 0.$$

Finally, if $i-l-1 \geq n/2$, then

$$\frac{h_{i+l-1}}{h_{i-l-1}} < 1 \leq (2l+1)^r - (2l)^r.$$

Hence

$$\left(\frac{h_{i+l-1}}{h_{i-l-1}}\right)^{1/(2l)} \leq \left[(2l+1)^r - (2l)^r\right]^{1/(2l)},$$

and, as $h_{i-l}/h_{i+l} = h_{i'+l}/h_{i'-l},\ i' = n-i-1,\ i = 1, ..., n-1$, also

$$\left(\frac{h_{i-l}}{h_{i+l}}\right)^{1/(2l)} \leq \left[(2l+1)^r - (2l)^r\right]^{1/(2l)}.$$

In the process $l \to \infty$ we have $\left[(2l+1)^r - (2l)^r\right]^{1/(2l)} \to 1$, so we can choose $l = l_0 \in \mathbf{N}$ such that $\left[(2l_0+1)^r - (2l_0)^r\right]^{1/(2l_0)} < 1+\varepsilon$, where $\varepsilon \in \left(0, \frac{3\eta(1-\eta)}{2(\eta^2-\eta+1)}\right)$ does not depend on n. Thus

$$\left(\frac{h_{i+l_0-1}}{h_{i-l_0-1}}\right)^{1/(2l_0)} < 1+\varepsilon, \quad \left(\frac{h_{i-l_0}}{h_{i+l_0}}\right)^{1/(2l_0)} < 1+\varepsilon,$$

and we get for $i = 1, ..., n-1$:

$$\begin{aligned}
d_i &= \mu_i - \lambda_i \left(\frac{h_{i+l_0-1}}{h_{i-l_0-1}}\right)^{1/(2l_0)} - \nu_i \left(\frac{h_{i-l_0}}{h_{i+l_0}}\right)^{1/(2l_0)} \\
&> 1 - \frac{(1-\eta)^2 h_{i-1}}{(1-\eta^2)h_{i-1} + (2\eta-\eta^2)h_i}(1+\varepsilon) \\
&\quad - \frac{\eta^2 h_i}{(1-\eta^2)h_{i-1} + (2\eta-\eta^2)h_i}(1+\varepsilon) \\
&> \left(1 - \frac{1-\eta}{1+\eta} - \frac{\eta}{2-\eta}\right) - \left(\frac{1-\eta}{1+\eta} + \frac{\eta}{2-\eta}\right)\varepsilon \\
&= \frac{3\eta(1-\eta) - 2(\eta^2-\eta+1)\varepsilon}{(1+\eta)(2-\eta)} > 0.
\end{aligned}$$

The last expression, together with the inequalities (10) and (11), tells us that the matrix of the system (9) is diagonally dominant and therefore the system (9) is uniquely solvable (see, for example, [[7]], p. 333).

Denote $\sigma = 2^r - 1$. As $\max_{|i-j|=1} h_i/h_j \leq \sigma$, we have

$$\frac{h_{i-l_0}}{h_i} = \frac{h_{i-l_0}}{h_{i-l_0+1}} \frac{h_{i-l_0+1}}{h_{i-l_0+2}} \cdots \frac{h_{i-1}}{h_i} \leq \sigma^{l_0}$$

and also $\frac{h_{i+l_0}}{h_i} \leq \sigma^{l_0}$. Consequently

$$\frac{(h_{i-l_0}h_{i-l_0+1}\ ...\ h_{i+l_0-2}h_{i+l_0-1})^{1/(2l_0)}}{h_i} \leq \left(\sigma^{l_0}\sigma^{l_0-1}\ ...\ \sigma 1\sigma\ ...\ \sigma^{l_0-2}\sigma^{l_0-1}\right)^{1/(2l_0)}$$
$$= \left(\sigma^{l_0^2}\right)^{1/(2l_0)} = \sigma^{l_0/2}$$

and

$$\frac{(h_{i-l_0}h_{i-l_0+1}\ ...\ h_{i+l_0-2}h_{i+l_0-1})^{1/(2l_0)}}{h_{i-1}} \leq \sigma^{l_0/2}.$$

Using the last two inequalities, we can now estimate the right-hand sides of the system (9):

$$|g_0\,(h_{-l_0}\ ...\ h_{l_0-1})^{1/(2l_0)}\,| = 2\frac{(h_{-l_0}\ ...\ h_{l_0-1})^{1/(2l_0)}}{h_0}|y_1-y_0| \leq \frac{2}{1-\eta^2}\sigma^{l_0/2}\omega_n(y),$$

$$|g_i\,(h_{i-l_0}\ ...\ h_{i+l_0-1})^{1/(2l_0)}\,| = 2\frac{(h_{i-l_0}\ ...\ h_{i+l_0-1})^{1/(2l_0)}}{(1-\eta^2)h_{i-1}+(2\eta-\eta^2)h_i}|y_{i+1}-y_i|$$
$$\leq \frac{2}{1-\eta^2}\frac{(h_{i-l_0}\ ...\ h_{i+l_0-1})^{1/(2l_0)}}{h_{i-1}}|y_{i+1}-y_i|$$
$$\leq \frac{2}{1-\eta^2}\sigma^{l_0/2}\omega_n(y), \quad i=1,...,n-1,$$

and

$$|g_n\,(h_{n-l_0}\ ...\ h_{n+l_0-1})^{1/(2l_0)}\,| = 2\frac{(h_{n-l_0}\ ...\ h_{n+l_0-1})^{1/(2l_0)}}{h_{n-1}}|y_{n+1}-y_n|$$
$$\leq \frac{2}{1-\eta^2}\sigma^{l_0/2}\omega_n(y),$$

where

$$\omega_n(y) = \max_{0\leq i\leq n-1}\ \max_{t_{n,i}\leq t',\ t''\leq t_{n,i+1}} |y(t')-y(t'')|.$$

So, denoting

$$q = \max\left\{2\eta(1-\eta), \frac{3\eta(1-\eta)-2(\eta^2-\eta+1)\varepsilon}{(1+\eta)(2-\eta)}\right\}$$

and considering that the matrix of the system (9) is diagonally dominant, we get

$$|z_i| \leq \frac{2\sigma^{l_0/2}}{q(1-\eta^2)}\omega_n(y), \quad i=0,...,n,$$

or

$$|m_i| = \frac{h_i}{h_i} \frac{|z_i|}{(h_{i-l_0} \, ... \, h_{i+l_0-1})^{1/(2l_0)}} \leq \frac{\sigma^{l_0/2}|z_i|}{h_i} \leq \frac{2\sigma^{l_0}}{q(1-\eta^2)h_i}\omega_n(y),$$

$$i = 0, ..., n-1,$$

and also

$$|m_{i+1}| \leq \frac{2\sigma^{l_0}}{q(1-\eta^2)h_i}\omega_n(y), \quad i = 0, ..., n-1.$$

Let now $t \in [t_{n,i}, t_{n,i+1}], \ i = 0, ..., n-1$. Then

$$|(P_n y)(t) - y(t)|$$

$$= \left| y_{i+1} + \left[\frac{(1-\eta)^2 h_i}{2} - \frac{(t_{n,i+1}-t)^2}{2h_i} \right] m_i + \left[\frac{(t-t_{n,i})^2}{2h_i} - \frac{\eta^2 h_i}{2} \right] m_{i+1} - y(t) \right|$$

$$\leq |y_{i+1} - y(t)| + \left| \frac{(1-\eta)^2 h_i}{2} - \frac{(t_{n,i+1}-t)^2}{2h_i} \right| |m_i| + \left| \frac{(t-t_{n,i})^2}{2h_i} - \frac{\eta^2 h_i}{2} \right| |m_{i+1}|$$

$$\leq \omega_n(y) + h_i|m_i| + h_i|m_{i+1}| \leq \left[1 + \frac{4\sigma^{l_0}}{q(1-\eta^2)} \right] \omega_n(y).$$

Since the interval $[t_{n,i}, t_{n,i+1}]$ was arbitrary, we have

$$||P_n y - y||_{C[a,b]} \leq \left[1 + \frac{4\sigma^{l_0}}{q(1-\eta^2)} \right] \omega_n(y).$$

The convergence (7) is now evident, considering that $\omega_n(y) \to 0$ as $n \to \infty$.

Remark 3. An analogous result in case of cubic splines is obtained, for example, in [1], p. 55; [8].

From Lemma 1 we can make one important corollary, the proof of which is an elementary application of the Banach–Steinhaus theorem.

Lemma 2. *Let $P_n : C[a,b] \to C[a,b]$ be the interpolation operator given by the conditions* (5). *Then there exists $M > 0$ such that for any $n = 2k, \ k \in \mathbf{N}$,*

$$||P_n||_{\mathcal{L}(C[a,b])} \leq M. \tag{12}$$

For the following result we introduce the B-splines of the second degree $B_{2,i} : \mathbf{R} \to \mathbf{R}, \ i = 0, ..., n+1$ (cf. [5,7]):

$$B_{2,i}(t) = \begin{cases} \dfrac{(t - t_{n,i-2})^2}{(t_{n,i} - t_{n,i-2})(t_{n,i-1} - t_{n,i-2})}, \quad t \in [t_{n,i-2}, t_{n,i-1}); \\ \dfrac{(t - t_{n,i-2})(t_{n,i} - t)}{(t_{n,i} - t_{n,i-2})(t_{n,i} - t_{n,i-1})} + \dfrac{(t_{n,i+1} - t)(t - t_{n,i-1})}{(t_{n,i+1} - t_{n,i-1})(t_{n,i} - t_{n,i-1})}, \\ \qquad t \in [t_{n,i-1}, t_{n,i}); \\ \dfrac{(t_{n,i+1} - t)^2}{(t_{n,i+1} - t_{n,i-1})(t_{n,i+1} - t_{n,i})}, \quad t \in [t_{n,i}, t_{n,i+1}); \\ 0, \quad \text{otherwise}, \end{cases} \tag{13}$$

if $i = 0, ..., n$, and

$$B_{2,n+1}(t) = \begin{cases} \dfrac{(t - t_{n,n-1})^2}{(t_{n,n} - t_{n,n-1})^2}, \quad t \in [t_{n,n-1}, t_{n,n}]; \\ 0, \quad \text{otherwise}. \end{cases} \tag{14}$$

Here $t_{n,-2} = t_{n,-1} = t_{n,0},\ t_{n,n+2} = t_{n,n+1} = t_{n,n},\ t_{n,i} \in \Delta_n^r,\ i = 0, ..., n$. Notice that $B_{2,i} \in S_{2,1}(\Delta_n^r),\ i = 0, ..., n+1$.

Lemma 3. [9] *Let $Q_n : C[a,b] \to C[a,b]$ be the operator defined by*

$$Q_n y = \sum_{i=0}^{n+1} \left[-\frac{1}{2} y(t_{n,i-1}) + 2y\left(\frac{t_{n,i-1} + t_{n,i}}{2}\right) - \frac{1}{2} y(t_{n,i}) \right] B_{2,i},$$

where $t_{n,-1} = t_{n,0},\ t_{n,n+1} = t_{n,n},\ t_{n,i} \in \Delta_n^r,\ i = 0, ..., n$, and $B_{2,i}$ is the ith *B-spline of the second degree defined by the expressions* {(13),(14)}*. Then*

$$||y - Q_n y||_{C[t_{n,i},\, t_{n,i+1}]} \le 4\text{dist}_{[t_{n,i-1},\, t_{n,i+2}]}(y, \pi_2), \quad i = 0, ..., n-1, \tag{15}$$

where $\text{dist}_{[u,v]}(y, \pi_2) = \inf_{p \in \pi_2} ||y - p||_{C[u,v]}$ and π_2 is the set of polynomials of the second order.

In the next theorem we state the main result of this section.

Theorem 1. *Let $P_n : C[a,b] \to C[a,b]$ $(n = 4, 6, 8, ...)$ be the interpolation operator given by the conditions* (5)*. Then for any $y \in C^{3,\beta}[a,b]$ $(0 < \beta < 3)$*

$$||y - P_n y||_{C[a,b]} \le c \begin{cases} n^{-3}, & r \ge \dfrac{3}{3-\beta}; \\ n^{-r(3-\beta)}, & 1 \le r < \dfrac{3}{3-\beta}. \end{cases} \tag{16}$$

Here

$$c = (1+M)d_3(b-a)^{3-\beta}\max\left\{9r^3 2^{r(\beta+3)+\beta-1}, \frac{2^{\beta-2}6^{r(3-\beta)}}{3-\beta} + \frac{6^{3r}2^{\beta(r+1)}}{12}\right\},$$

where M and d_3 are the positive constants from the inequalities (12) *and* (2), *respectively.*

Proof. Let $y \in C^{3,\beta}[a,b]$ be given. Using the operator Q_n defined in Lemma 3 and the estimate (12), we get

$$\begin{aligned} ||y - P_n y||_{C[a,b]} &\le ||y - Q_n y||_{C[a,b]} + ||Q_n y - P_n y||_{C[a,b]} \\ &= ||y - Q_n y||_{C[a,b]} + ||P_n(Q_n y - y)||_{C[a,b]} \\ &\le ||y - Q_n y||_{C[a,b]} + ||P_n||_{\mathcal{L}(C[a,b])}||y - Q_n y||_{C[a,b]} \\ &\le (1+M)||y - Q_n y||_{C[a,b]}. \qquad (17) \end{aligned}$$

Now we divide the proof into two parts. In part I we obtain the estimate for the norm $||y - Q_n y||_{C[t_{n,0},\, t_{n,2}]}$ in case $n \ge 4$, in part II we consider the norms $||y - Q_n y||_{C[t_{n,i},\, t_{n,i+1}]},\ i = 2, ..., n/2 - 1$, in case $n \ge 6$.

I. Let $n \ge 4$ and $t \in [t_{n,0}, t_{n,3}]$ $(t_{n,0}, t_{n,3} \in \Delta_n^r)$. Consider the Taylor expansion of the function y at the point $t = t_{n,3}$:

$$y(t) = T_{2,1}(t) + \frac{1}{2}\int_{t_{n,3}}^{t}(t-s)^2 y'''(s)ds,$$

where

$$T_{2,1}(t) = y(t_{n,3}) + y'(t_{n,3})(t - t_{n,3}) + \frac{y''(t_{n,3})}{2}(t - t_{n,3})^2.$$

As $T_{2,1} \in \pi_2$,

$$\mathrm{dist}_{[t_{n,0},\, t_{n,3}]}(y, \pi_2) \le ||y - T_{2,1}||_{C[t_{n,0},\, t_{n,3}]},$$

so by the inequalities (15) (taking $i = 1$) we have

$$||y - Q_n y||_{C[t_{n,1},\, t_{n,2}]} \le 4||y - T_{2,1}||_{C[t_{n,0},\, t_{n,3}]} \qquad (18)$$

and, as $\mathrm{dist}_{[t_{n,0},\, t_{n,2}]}(y, \pi_2) \le \mathrm{dist}_{[t_{n,0},\, t_{n,3}]}(y, \pi_2)$, also (taking $i = 0$)

$$||y - Q_n y||_{C[t_{n,0},\, t_{n,1}]} \le 4||y - T_{2,1}||_{C[t_{n,0},\, t_{n,3}]}. \qquad (19)$$

In order to estimate the norm $||y - T_{2,1}||_{C[t_{n,0},\, t_{n,3}]}$ we use the inequality (2):

$$|y(t) - T_{2,1}(t)| = \frac{1}{2}\Bigg|\int\limits_{t_{n,3}}^{t} (t-s)^2 y'''(s)ds\Bigg| \le \frac{1}{2}\int\limits_{t}^{t_{n,3}} (s-t)^2 |y'''(s)|ds$$

$$\le \frac{1}{2}d_3 \int\limits_{t}^{t_{n,3}} (s-t)^2 \left[(s-a)^{-\beta} + (b-s)^{-\beta}\right] ds$$

$$\le \frac{1}{2}d_3 \int\limits_{t}^{t_{n,3}} (s-t)^{2-\beta} ds + \frac{1}{2}d_3 \int\limits_{t}^{t_{n,3}} (s-t)^2 (b-s)^{-\beta} ds$$

$$\le \frac{1}{2(3-\beta)} d_3 (t_{n,3}-t)^{3-\beta} + \frac{1}{6} d_3 (b-t_{4,3})^{-\beta}(t_{n,3}-t)^3$$

$$\le \frac{1}{2(3-\beta)} d_3 (t_{n,3}-t_{n,0})^{3-\beta} + \frac{1}{6} d_3 (b-t_{4,3})^{-\beta}(t_{n,3}-t_{n,0})^3.$$

Considering that $t \in [t_{n,0},\ t_{n,3}]$ was arbitrary,

$$t_{n,3} - t_{n,0} \le \left(\frac{b-a}{2}\right) 6^r n^{-r}, \quad n \ge 4,$$

and

$$b - t_{4,3} = t_{4,1} - a = \frac{b-a}{2^{r+1}},$$

we have

$$||y - T_{2,1}||_{C[t_{n,0},\, t_{n,3}]}$$
$$\le d_3 \left[\frac{(b-a)^{3-\beta} 6^{r(3-\beta)}}{2^{4-\beta}(3-\beta)} + \frac{(b-a)^{3-\beta} 6^{3r} 2^{\beta(r+1)}}{48}\right] n^{-r(3-\beta)}.$$

Therefore, due to the inequalities (18) and (19),

$$||y - Q_n y||_{C[t_{n,0},\, t_{n,2}]}$$
$$\le d_3 \left[\frac{2^{\beta-2}(b-a)^{3-\beta} 6^{r(3-\beta)}}{3-\beta} + \frac{(b-a)^{3-\beta} 6^{3r} 2^{\beta(r+1)}}{12}\right] n^{-r(3-\beta)}.$$

II. Let $n \ge 6$, $t \in [t_{n,i-1}, t_{n,i+2}]$, $i = 2, ..., n/2-1$, $t_{n,i} \in \Delta_n^r$, $i = 1, ..., n/2+1$. In this case consider the Taylor expansion of the function y at the point $t = t_{n,i+2}$:

$$y(t) = T_{2,i}(t) + \frac{1}{2}\int\limits_{t_{n,i+2}}^{t} (t-s)^2 y'''(s)ds,$$

where

$$T_{2,i}(t) = y(t_{n,i+2}) + y'(t_{n,i+2})(t - t_{n,i+2}) + \frac{y''(t_{n,i})}{2}(t - t_{n,i+2})^2.$$

The function $T_{2,i}$ is a polynomial of the second order, so we have

$$\operatorname{dist}_{[t_{n,i-1},\, t_{n,i+2}]}(y, \pi_2) \le ||y - T_{2,i}||_{C[t_{n,i-1},\, t_{n,i+2}]}$$

and by the inequalities (15)

$$||y - Q_n y||_{C[t_{n,i},\, t_{n,i+1}]} \le 4||y - T_{2,i}||_{C[t_{n,i-1},\, t_{n,i+2}]}, \quad i = 2, ..., \frac{n}{2} - 1. \qquad (20)$$

With the help of the inequality (2) we obtain for the error $||y - T_{2,i}||_{C[t_{n,i-1},\, t_{n,i+2}]}$:

$$\begin{aligned}
|y(t) - T_{2,i}(t)| &= \frac{1}{2}\Big|\int_{t_{n,i+2}}^{t} (t-s)^2 y'''(s)ds\Big| \le \frac{1}{2}\int_{t}^{t_{n,i+2}} (s-t)^2 |y'''(s)|ds \\
&\le \frac{1}{2}d_3 \int_{t}^{t_{n,i+2}} (s-t)^2 \left[(s-a)^{-\beta} + (b-s)^{-\beta}\right] ds \\
&\le \frac{1}{2}d_3 (t-a)^{-\beta} \int_{t}^{t_{n,i+2}} (s-t)^2 ds + \frac{1}{2}d_3 \int_{t}^{t_{n,i+2}} (s-t)^2 (b-s)^{-\beta} ds \\
&\le \frac{1}{6}d_3 (t_{n,i-1} - a)^{-\beta}(t_{n,i+2} - t_{n,i-1})^3 \\
&\quad + \frac{1}{6}d_3 (t_{n,i-1} - a)^{-\beta}(t_{n,i+2} - t_{n,i-1})^3 \\
&= \frac{1}{3}d_3 (t_{n,i-1} - a)^{-\beta}(t_{n,i+2} - t_{n,i-1})^3.
\end{aligned}$$

Using the formulas (4) it is easy to see that for $i = 0, ..., n/2 - 1$

$$t_{n,i-1} - a = (b-a)2^{r-1}(i-1)^r n^{-r}$$

and

$$t_{n,i+2} - t_{n,i-1} \le 3r(b-a)2^{r-1}(i+2)^{r-1} n^{-r}.$$

So we have

$$\begin{aligned}
|y(t) - T_{2,i}(t)| &\le \frac{1}{3}d_3 \frac{(t_{n,i+2} - t_{n,i-1})^3}{(t_{n,i-1} - a)^{\beta}} \le 9d_3 \frac{r^3 (b-a)^3 2^{3r-3}(i+2)^{3r-3} n^{-3r}}{(b-a)^{\beta} 2^{\beta r - \beta}(i-1)^{\beta r} n^{-\beta r}} \\
&= 9d_3 r^3 (b-a)^{3-\beta} 2^{(3-\beta)(r-1)} \frac{(i+2)^{\beta r}}{(i-1)^{\beta r}} (i+2)^{r(3-\beta)-3} n^{-r(3-\beta)} \\
&\le 9d_3 r^3 (b-a)^{3-\beta} 2^{(3-\beta)(r-1)} 4^{\beta r} (i+2)^{r(3-\beta)-3} n^{-r(3-\beta)}.
\end{aligned}$$

Let now $r \geq \frac{3}{3-\beta}$. Then

$$(i+2)^{r(3-\beta)-3} \leq n^{r(3-\beta)-3},\ i = 2, ...n/2-1,$$

and

$$|y(t) - T_{2,i}(t)| \leq 9 d_3 r^3 (b-a)^{3-\beta} 2^{r(\beta+3)+\beta-3} n^{-3}.$$

If $1 \leq r < \frac{3}{3-\beta}$, then

$$(i+2)^{r(3-\beta)-3} \leq 1,\ i = 2, ..., n/2-1,$$

and

$$|y(t) - T_{2,i}(t)| \leq 9 d_3 r^3 (b-a)^{3-\beta} 2^{r(\beta+3)+\beta-3} n^{-r(3-\beta)}.$$

Therefore

$$||y - T_{2,i}||_{C[t_{n,i-1},\, t_{n,i+2}]}$$
$$\leq 9 d_3 r^3 (b-a)^{3-\beta} 2^{r(\beta+3)+\beta-3} \begin{cases} n^{-3}, & r \geq \dfrac{3}{3-\beta}; \\ n^{-r(3-\beta)}, & 1 \leq r < \dfrac{3}{3-\beta}; \end{cases}$$

and by the inequalities (20) we get for $i = 2, ..., n/2-1$

$$||y - Q_n y||_{C[t_{n,i},\, t_{n,i+1}]}$$
$$\leq 9 d_3 r^3 (b-a)^{3-\beta} 2^{r(\beta+3)+\beta-1} \begin{cases} n^{-3}, & r \geq \dfrac{3}{3-\beta}; \\ n^{-r(3-\beta)}, & 1 \leq r < \dfrac{3}{3-\beta}. \end{cases}$$

Because of the symmetry argument the estimates obtained in parts I and II also hold on the other half of the interval $[a, b]$, i.e.

$$||y - Q_n y||_{C[t_{n,n-2},\, t_{n,n}]}$$
$$\leq d_3 \left[\frac{2^{\beta-2}(b-a)^{3-\beta} 6^{r(3-\beta)}}{3-\beta} + \frac{(b-a)^{3-\beta} 6^{3r} 2^{\beta(r+1)}}{12} \right] n^{-r(3-\beta)}$$

and

$$||y - Q_n y||_{C[t_{n,i},\, t_{n,i+1}]}$$
$$\leq 9 d_3 r^3 (b-a)^{3-\beta} 2^{r(\beta+3)+\beta-1} \quad \begin{cases} n^{-3}, & r \geq \dfrac{3}{3-\beta}; \\ n^{-r(3-\beta)}, & 1 \leq r < \dfrac{3}{3-\beta}; \end{cases}$$
$$i = \frac{n}{2}, ..., n-3.$$

Using now the inequalities (17), we get the estimate (16), which concludes the proof.

4. COLLOCATION METHOD

We seek the approximation y_n of the solution y of Eq. (1) in the linear space $S_{2,1}(\Delta_n^r)$ such that Eq. (1) is satisfied at the interpolation points $x_i,\ i = 0, ..., n+1$ (see the formulas (6)):

$$y_n(x_i) = \int_a^b g(x_i, s)\kappa(x_i - s)y_n(s)ds + f(x_i), \quad i = 0, ..., n + 1. \tag{21}$$

Considering that we can present the function y_n in the form

$$y_n(t) = \sum_{i=0}^{n+1} c_i\varphi_i(t), \qquad t \in [a, b],$$

where $\{\varphi_0, ..., \varphi_{n+1}\}$ is a basis in the space $S_{2,1}(\Delta_n^r)$ and $c_i,\ i = 0, ..., n + 1$, are constants to be determined, the collocation conditions (21) lead to a linear system with respect to the unknowns c_i:

$$\sum_{j=0}^{n+1} \left[\varphi_j(x_i) - \int_a^b g(x_i, s)\kappa(x_i - s)\varphi_j(s)ds \right] c_j = f(x_i), \quad i = 0, ..., n+1. \tag{22}$$

Solving this system we obtain the values of the parameters $c_i,\ i = 0, ..., n + 1$, and the analytical form of the approximation y_n.

The next two results (which formulate the convergence and the rate of convergence of the method (21)) we state without proofs. Using Remark 2 they can be proved analogously with Theorems 3 and 4, respectively, in [5].

Theorem 2. *Let the assumptions* (A1) *be fulfilled and let* $f \in C[a, b]$. *Assume also that the homogeneous equation*

$$y(t) = \int_a^b g(t, s)\kappa(t - s)y(s)ds, \qquad t \in [a, b],$$

has only the trivial solution $y = 0$ *and the interpolation points* (6) *with the partition* {(3),(4)} *of the interval* $[a, b]$ *are used.*

Then Eq. (1) *has a unique solution* $y \in C[a, b]$ *and there exists* $n_0 \in \mathbf{N}$ *such that for* $n \geq n_0$ *the collocation equations* (21) *determine a unique approximation* $y_n \in S_{2,1}(\Delta_n^r)$ *to* y *with*

$$||y_n - y||_{C[a,b]} \to 0, \qquad n \to \infty.$$

Theorem 3. *Let the conditions of Theorem* 2 *hold and let* $f \in C^{3,\beta}[a,b]$ $(\beta \in (0,3))$. *Then there exists* $n_0 \in \mathbf{N}$ *such that for* $n \geq n_0$

$$||y_n - y||_{C[a,b]} \leq c \begin{cases} n^{-3}, & r \geq \dfrac{3}{3-\beta}; \\ n^{-r(3-\beta)}, & 1 \leq r < \dfrac{3}{3-\beta}; \end{cases}$$

where y *is the solution of Eq.* (1), y_n *is the approximation to* y *defined by the collocation equations* (21) *and* c *is a positive constant not depending on* n.

5. NUMERICAL EXAMPLES

Let us consider the weakly singular integral equation

$$y(t) = \int_0^1 |t-s|^{-1/2} y(s)ds + f(t), \qquad t \in [0,1],$$

where

$$f(t) = \sqrt{t} - \frac{\pi}{2}t - \frac{(1+\sqrt{1-t})^2}{4} + \frac{t^2}{4(1+\sqrt{1-t})^2} - t\ln(1+\sqrt{1-t}) + \frac{t\ln t}{2}$$

and the exact solution $y(t) = \sqrt{t}$. If we take $g(t,s) = 1$, $\kappa(\tau) = |\tau|^{-1/2}$, $a = 0,\ b = 1$, the assumptions (A1) and (A2) are fulfilled with $\beta = \frac{5}{2}$.

Let $n = 4, 6, 8, ...$, and let $0 = t_{n,0} < t_{n,1} < ... < t_{n,n} = 1$ be a partition of the interval $[0,1]$ determined by the formulas

$$\begin{cases} t_{n,i} = \dfrac{1}{2}\left(\dfrac{2i}{n}\right)^r, & i = 0, ..., \dfrac{n}{2}; \\ t_{n,n/2+i} = 1 - t_{n,n/2-i}, & i = 1, ..., \dfrac{n}{2}; \end{cases}$$

where $r \in [1,\infty)$ is a fixed real number not depending on n. We choose the interpolation points as follows:

$$x_0 = 0, \qquad x_i = \frac{t_{n,i-1} + t_{n,i}}{2}, \quad i = 1, ..., n, \qquad x_{n+1} = 1.$$

To estimate the norm $||y - y_n||_{C[0,1]}$, we give another partition of the interval $[0,1]$ with the knots $\tau_{ij},\ i = 0, ..., n-1,\ j = 0, ..., 10$, namely

$$\tau_{ij} = t_{n,i} + j\left(\frac{t_{n,i+1} - t_{n,i}}{10}\right).$$

In the role of the basis functions we take the B-splines of the second degree $B_{2,i},\ i = 0, ..., n+1$, defined by the expressions {(13),(14)}.

Table 1. Numerical results

n	$r=1$		$r=3$		$r=4$		$r=6$	
	ε_n	ρ_n	ε_n	ρ_n	ε_n	ρ_n	ε_n	ρ_n
4	0.0681131		0.0141109		0.0078840		0.0109106	
8	0.0433667	1.571	0.0046903	3.008	0.0017569	4.487	0.0018694	5.836
16	0.0289248	1.499	0.0016290	2.879	0.0004123	4.261	0.0001951	9.582
32	0.0197214	1.467	0.0005733	2.841	0.0001009	4.085	0.0115118	0.017
64	0.0136114	1.449	0.0002024	2.832	0.0000258	3.911	2.0648866	0.006
128	0.0094672	1.438	0.0000715	2.830	0.0019489	0.013	2.9964757	0.689

Under these assumptions we solved the system (22) (the integrals $\int_a^b g(x_i,s)\kappa(x_i-s)B_{2,j}(s)ds = \int_{t_{n,j-2}}^{t_{n,j+1}} g(x_i,s)\kappa(x_i-s)B_{2,j}(s)ds,\ \ i,j = 0,...,n+1$, were computed exactly). In Table 1 we present the errors

$$\varepsilon_n = \max_{\substack{0\le i\le n-1\\ 0\le j\le 10}} |y(\tau_{ij}) - y_n(\tau_{ij})|$$

and the ratios $\rho_n = \varepsilon_{n/2}/\varepsilon_n$ characterizing the rate of convergence of the method (22) for four values of the parameter r: $r=1,\ r=3,\ r=4$, and $r=6$. It follows from Theorem 3 that if $\beta = \frac{5}{2}$, then in case $r=1$ the ratio ρ_n must be approximately $2^{1/2} \approx 1.414$; in case $r=3$ the ratio ρ_n must be $2^{3/2} \approx 2.828$; in case $r=4$ the ratio ρ_n must be $2^2=4$; and in case $r=6$ the ratio ρ_n must be $2^3=8$. From the table we can see that numerical results are quite in accord with theoretical estimations except the case $r=6$, where the loss of accuracy appears: the error for n is bigger than the error for $n/2$ if $n=32,\ n=64$, and $n=128$. This is due to the roundoff errors in numerical evaluation of exact formulas of coefficients of the system (22).

All calculations were made with double precision.

ACKNOWLEDGEMENT

This work was supported by the Estonian Science Foundation (research grant No. 4410).

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Ruutsplain-kollokatsioonimeetod nõrgalt singulaarsete integraalvõrrandite lahendamiseks astmeliste võrkude korral

Rene Pallav

On vaadeldud splain-kollokatsioonimeetodit teist liiki nõrgalt singulaarsete Fredholmi integraalvõrrandite numbriliseks lahendamiseks siledate ruutsplainide korral. Nende võrrandite lahendamisel on kasutatud astmelisi (erikujulisi) võrke, mille korral on tuletatud meetodi koonduvuskiiruse hinnang maksimumnormis.