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TWO TAUBERIAN REMAINDER THEOREMS FOR THE CESÀRO METHOD OF SUMMABILITY

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Abstract. The Tauberian remainder theorems for the Cesàro method of summability are studied using two different methods of the proof. Theorem 1 is proved by applying the method of summability with a given rapidity. Theorem 2 is proved using several consequences of one basic Wiener's Tauberian remainder theorem proved by A. Beurling. Both theorems are connected with the hypothesis of G. Kangro (see Tammeraid, I. *Metody algebry i analiza*, III. 1988, 113–114).

Key words: Tauberian remainder theorems.

Let $\lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow$. The real sequence $x = \{\xi_n\}$, converging to ξ, is said (see [1] or [2]) to be λ-bounded if the sequence $\{\lambda_n(\xi_n - \xi)\}$ is bounded. Let us denote by m^λ the set of all λ-bounded sequences $x = \{\xi_n\}$ and by m_0^λ the subset of m^λ with $\xi = 0$. We say that the sequence x is λ-bounded by the method A if $Ax \in m^\lambda$. We define the set (A, m_0^λ) by

$$x \in (A, m_0^\lambda) \Leftrightarrow Ax \in m_0^\lambda,$$

where the summability method is determined by the matrix $A = (a_{nk})$. We say that the method A preserves λ-boundedness if $Am^\lambda \subset m^\lambda$. Kangro [1] proved that a regular method A with $\sum_{k=0}^\infty a_{nk} = 1$ preserves λ-boundedness if and only if

$$\lambda_n \sum_k \frac{|a_{nk}|}{\lambda_k} = O(1). \tag{1}$$

The Woronoi–Nörlund matrix method of summability (WN, p_n) is defined by the sequence of numbers $\{p_n\}$ in the sequence-to-sequence form by the lower triangular matrix (a_{nk}) with $a_{nk} = p_{n-k}/P_n \ (k \le n)$ and $P_n = \sum_{k=0}^{n} p_k \ne 0$. In the case $p_n = A_n^{\alpha-1}$ with $A_n^{\alpha} = \binom{n+\alpha}{n}$ the method $\left(WN, A_n^{\alpha-1}\right)$ is the Cesàro method C^{α}.

The Riesz matrix method of weighted means $P = (R, p_n)$ is defined by the sequence of numbers $\{p_n\}$, while $a_{nk} = p_k/P_n \ (k \le n)$ with $P_n = \sum_{k=0}^{n} p_k \ne 0$.

Kangro [3] proved several Tauberian remainder theorems for the Riesz method $P = (R, p_n)$. We present a consequence of them as Lemma 1.

Lemma 1. *If the regular Riesz method $P = (R, p_n)$, where $p_n > 0$, preserves λ-boundedness and the conditions*

$$1 \le \lambda_n/\tau_n \uparrow, \quad \lambda_n\tau_n \uparrow, \quad \mu_n = \sqrt{\lambda_n\tau_n}, \tag{2}$$

$$p_n\sqrt{\lambda_n} = O(P_{n-1}\sqrt{\tau_n}) \quad (n \to +\infty), \tag{3}$$

$$x = \{\xi_n\} \in ((R, p_n), m^{\lambda}),$$

$$\tau_n P_n \Delta\xi_n = O_L(p_n)$$

are fulfilled, then $x \in m^{\mu}$.

Kangro (see [4]) posed a hypothesis that an analogical assertion is valid for the Cesàro method C^{α} $(\alpha > 0)$. This hypothesis is not verified up to now.

Hypothesis. *Let the sequences $\lambda = \{\lambda_n\}$, $\tau = \{\tau_n\}$, and $\mu = \{\mu_n\}$ satisfy the conditions* (2). *Let the Cesàro method C^{α} $(\alpha > 0)$ preserve λ-boundedness. If a sequence $x = \{\xi_n\}$ is λ-bounded by the method C^{α} and the left-handed Tauberian condition*

$$\tau_n\,(n+1)\,\Delta\xi_n = O_L(1)$$

is satisfied, then $x \in m^{\mu}$.

Theorems 1 (see also [5–8]) and 2 (see also [9,10]) are closely related to the Hypothesis, but unfortunately do not verify its trueness or falsehood. The assumptions and assertions of Theorems 1 and 2 are essentially conditioned by the method used for the proof.

Theorem 1. *Let $\alpha \in \mathbf{N}$ and*

$$\lambda_n \uparrow, \ \ \tau_n \uparrow, \ \ \lambda_n/\tau_n \uparrow, \ \mu_n = \lambda_n^{2^{-\alpha}}\, \tau_n^{1-2^{-\alpha}}.$$

If the Cesàro method C^1 preserves λ-boundedness, then from the C^{α}-boundedness of the sequence $x = \{\xi_n\}$ and left-handed Tauberian condition

$$\tau_n (n+1) \Delta \xi_n = O_L (1) \tag{4}$$

follows the μ *-boundedness of* x.

Proof. According to (1), the method C^1 preserves λ-boundedness, iff

$$\frac{\lambda_n}{n+1} \sum_{\nu=0}^{n} \frac{1}{\lambda_\nu} = O(1). \tag{5}$$

As the condition $\lambda_n / \tau_n \uparrow$ implies

$$\frac{\tau_n}{\tau_\nu} \leq \frac{\lambda_n}{\lambda_\nu} \quad (\nu \leq n), \tag{6}$$

then from the connections (5) and (6) we get

$$\frac{\tau_n}{n+1} \sum_{\nu=0}^{n} \frac{1}{\tau_\nu} = \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{\tau_n}{\tau_\nu} \leq \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{\lambda_n}{\lambda_\nu} = O(1),$$

that means

$$\frac{\tau_n}{n+1} \sum_{\nu=0}^{n} \frac{1}{\tau_\nu} = O(1). \tag{7}$$

Therefore the method C^1 preserves also τ-boundedness. For arbitrary $n \in \mathbf{N}$ the quantity

$$\varphi_n (\beta) = \frac{\tau_n}{A_n^\beta} \sum_{\nu=0}^{n} \frac{A_\nu^{\beta-1}}{\tau_\nu} \quad (\beta \in \mathbf{N})$$

is a decreasing function (see [6]) of the variable β. As by the relation (7) we have $\varphi_n(1) = O(1)$, then $\varphi_n(\beta) = O(1)$. It means that the Riesz method $P = (R, A_n^{\beta-1})$ preserves both τ-boundedness and λ-boundedness. Let

$$\sigma_n^\beta = \sum_{\nu=0}^{n} \frac{A_{n-\nu}^{\beta-1}}{A_n^\beta} \xi_\nu.$$

As $\varphi_n(\beta) = O(1)$, the relation

$$(n+1) \tau_n \Delta \sigma_n^{\beta+1} = O_L(1)$$

follows from the relation

$$(n+1) \tau_n \Delta \sigma_n^\beta = O_L(1).$$

Indeed, as at $\beta \in \mathbf{N} \cup \{0\}$ we have (see [6])

$$\sigma_n^{\beta+1} = \sum_{\nu=0}^{n} \frac{A_\nu^\beta}{A_n^{\beta+1}} \sigma_\nu^\beta$$

and

$$\Delta\sigma_n^{\beta+1} = \frac{\beta+1}{(n+\beta+2)\, A_n^{\beta+1}} \sum_{\nu=0}^{n} A_\nu^{\beta+1} \Delta\sigma_\nu^\beta,$$

then

$$\begin{aligned}(n+1)\,\tau_n \Delta\sigma_n^{\beta+1} &= \frac{(\beta+1)\,(n+1)\,\tau_n}{(n+\beta+2)\, A_n^{\beta+1}} \sum_{\nu=0}^{n} A_\nu^{\beta+1} \Delta\sigma_\nu^\beta \\ &= O_L(1)\, \frac{\tau_n}{A_n^{\beta+1}} \sum_{\nu=0}^{n} \frac{A_\nu^{\beta+1}}{(\nu+1)\,\tau_\nu} \\ &= O_L(1)\, \frac{\tau_n}{A_n^{\beta+1}} \sum_{\nu=0}^{n} \frac{A_\nu^{\beta}}{\tau_\nu} = O_L(1).\end{aligned}$$

Consequently, the relation

$$(n+1)\,\tau_n \Delta\sigma_n^\beta = O_L(1) \quad (\beta \in \mathbf{N} \cup \{0\})$$

follows from the condition (4). As by the assumption the Cesàro method C^1 preserves λ-boundedness, then (see [1]) $\lambda_n = o(n)$. That is why at $\tau_n \uparrow$ the additional condition (3), the specific one in the Kangro's Tauberian theorems with the one-sided Tauberian conditions,

$$\lambda_n = O((n+1)^2\, \tau_n),$$

is fulfilled. Let us apply Lemma 1, selecting $p_n = A_n^{\alpha-1}$. We get

$$C^{\alpha-1} x \in m^{\mu^{(1)}},$$

while $\mu_n^{(1)} = \sqrt{\lambda_n \tau_n}$. As the next step, selecting $p_n = A_n^{\alpha-2}$, we prove that

$$C^{\alpha-2} x \in m^{\mu^{(2)}}$$

with $\mu_n^{(2)} = \sqrt{\sqrt{\lambda_n \tau_n}\, \tau_n} = \lambda_n^{2^{-2}}\, \tau_n^{1-2^{-2}}$. Step by step we get the assertion of Theorem 1.

Lemma 2 is used for the proof of Theorem 2. Lemma 2 follows from Corollary 1 and Remark 1 of the article [10] (see also Beurling's basic Tauberian remainder theorem [11]).

Lemma 2. *Let $\alpha \in \mathbf{N}$ and a normal matrix method D satisfy the condition*

$$(PD^{-1})m_0^{\lambda} \subset m_0^{\lambda}, \tag{8}$$

while P is a matrix of the Woronoi–Nörlund method $(WN,\ (n+1)^{\alpha} - n^{\alpha})$. Let

$$\lambda = \{\lambda_n\},\ \mu = \{\mu_n\},\ \lambda_n = (n+1)^s,\ \mu_n = (n+1)^{s/(1+\alpha)},\ 0 < s < 1/(2\pi). \tag{9}$$

If $x = \{\xi_n\}$ is a bounded sequence satisfying the left-handed Tauberian condition

$$\mu_n(\xi_m - \xi_n) = O_L(1) \quad (n_0 < n < m < n\exp(1/\mu_n),\ \ n \to \infty) \tag{10}$$

and $x \in (D, m_0^{\lambda})$, then $x \in m_0^{\mu}$.

Theorem 2. *Let $\alpha \in \mathbf{N}$. Let the sequences $\lambda = \{\lambda_n\}$ and $\mu = \{\mu_n\}$ be determined by* (9). *If $x = \{\xi_n\}$ is a bounded sequence satisfying the left-handed Tauberian condition* (10) *and $x \in (C^{\alpha}, m_0^{\lambda})$, then $x \in m_0^{\mu}$.*

Proof. We intend to apply Lemma 2 with $D = C^{\alpha}$. If we denote by

$$\varsigma_n^{\alpha} = (n+1)^{-\alpha} \sum_{k=0}^{n} \left((n+1-k)^{\alpha} - (n-k)^{\alpha}\right)\xi_k$$

and

$$\sigma_n^{\alpha} = \sum_{k=0}^{n} (A_{n-k}^{\alpha-1}\xi_k)/A_n^{\alpha},$$

then (see [12])

$$\xi_n = \sum_{k=0}^{n} A_{n-k}^{-\alpha-1} A_k^{\alpha} \sigma_k^{\alpha}$$

and

$$\begin{aligned}
\varsigma_n^{\alpha} &= \sum_{k=0}^{n} \frac{((n+1-k)^{\alpha} - (n-k)^{\alpha})}{(n+1)^{\alpha}} \sum_{\nu=0}^{k} A_{k-\nu}^{-\alpha-1} A_{\nu}^{\alpha} \sigma_{\nu}^{\alpha} \\
&= \sum_{\nu=0}^{n} A_{\nu}^{\alpha}\sigma_{\nu}^{\alpha} \sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-1} \frac{((n+1-k)^{\alpha} - (n-k)^{\alpha})}{(n+1)^{\alpha}} \\
&= \sum_{\nu=0}^{n} A_{\nu}^{\alpha}\sigma_{\nu}^{\alpha} \sum_{i=0}^{n-\nu} A_{i}^{-\alpha-1} \frac{((n+1-\nu-i)^{\alpha} - (n-\nu-i)^{\alpha})}{(n+1)^{\alpha}} \\
&= \sum_{\nu=0}^{n} \frac{A_{\nu}^{\alpha}\sigma_{\nu}^{\alpha}}{(n+1)^{\alpha}} \sum_{i=0}^{n-\nu+1} A_{i}^{-\alpha-2}(n-\nu+1-i)^{\alpha}.
\end{aligned}$$

As $A_i^{-\alpha-2} = 0 \quad (i > \alpha+1)$ and $A_i^{-\alpha-2} = (-1)^i A_i^{\alpha+1-i} \quad (0 \le i \le \alpha+1)$ (see [12]), we have

$$\varsigma_n^\alpha = \sum_{\nu=0}^{n} \frac{A_\nu^\alpha \sigma_\nu^\alpha}{(n+1)^\alpha} \sum_{i=0}^{\min(n+1-\nu,\alpha+1)} (-1)^i A_i^{\alpha+1-i} (n+1-\nu-i)^\alpha \cdot$$

Using a generating function

$$(x-1)^{\alpha+1} x^{n-\nu-\alpha} = \sum_{i=0}^{\alpha+1} (-1)^i A_i^{\alpha+1-i} x^{n+1-\nu-i},$$

we get an equality

$$\sum_{i=0}^{\alpha+1} (-1)^i A_i^{\alpha+1-i} (n+1-\nu-i)^\alpha = 0. \tag{11}$$

Accordingly, we obtain

$$\varsigma_n^\alpha = \sum_{\nu=0}^{n} \frac{A_\nu^\alpha \sigma_\nu^\alpha}{(n+1)^\alpha} \sum_{i=0}^{\min(n+1-\nu,\alpha)} (-1)^i A_i^{\alpha+1-i} (n+1-\nu-i)^\alpha .$$

Through the mathematical induction for every $\alpha \in \mathbf{N}$ it is easy to prove that

$$\sum_{i=n+2-\nu}^{\alpha+1} (-1)^i A_i^{\alpha+1-i} (n+1-\nu-i)^\alpha < 0 \quad (1 \le n+1-\nu \le \alpha). \tag{12}$$

From (11) and (12) we conclude

$$\sum_{i=0}^{n+1-\nu} (-1)^i A_i^{\alpha+1-i} (n+1-\nu-i)^\alpha > 0 \quad (1 \le n+1-\nu \le \alpha).$$

It means that the matrix (a_{nk}) of the method

$$(WN, (n+1)^\alpha - n^\alpha) (C^\alpha)^{-1}$$

is a lower α-diagonal matrix with positive elements on these diagonals. Thus this method is a regular method (see [12]) with $\sum_{k=0}^{\infty} a_{nk} = 1$ and the condition (1) is fulfilled. We consequently get that this method preserves λ-boundedness when the condition (8) is satisfied. Therefore the normal matrix method $D = C^\alpha$ satisfies the conditions of Lemma 2 and the trueness of Theorem 2 follows from this lemma.

Remark 1. The condition

$$(n+1)\Delta\xi_n = O_L(1) \qquad (n \to \infty)$$

implies (10).

Remark 2. The analogous theorems to Theorems 1 and 2 are also valid in the cases of right-handed or two-sided Tauberian conditions. In the case of two-sided Tauberian conditions the assertions of the analogous theorems stay valid for sequences belonging to an arbitrary Banach space (see [1,13,14]). Using [15], new Tauberian remainder theorems can be obtained from Theorems 1 and 2.

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KAKS JÄÄKLIIKMEGA TAUBERI TEOREEMI CESÀRO MENETLUSE JAOKS

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On uuritud jääkliikmega Tauberi teoreeme Cesàro menetluse korral. Tõestamiseks on kasutatud kaht meetodit. Teoreem 1 on tõestatud meetodil, mis põhineb kiirusega summeeruvuse omadustel. Teoreemi 2 tõestamiseks on kasutatud mõningaid järeldusi A. Beurlingi tõestatud Wieneri tüüpi jääkliikmega Tauberi teoreemist. Mõlemad teoreemid on seotud G. Kangro hüpoteesiga.