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2-SEMIPARALLEL SURFACES IN SPACE FORMS
2. The general case

Ülo LUMISTE

Institute of Pure Mathematics, University of Tartu, Vanemuise 46, 51014 Tartu, Estonia; lumiste@math.ut.ee

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Abstract. A surface in n-dimensional space form is said to be 2-semiparallel if $\bar{R} \circ \bar{\nabla} h = 0$. All such surfaces are classified (Theorem 6) and geometrically described. Also their existence is established, together with their arbitrariness in some subclasses (Propositions 4 and 5).

Key words: 2-semiparallel surfaces, locally Euclidean surfaces.

1. INTRODUCTION

In the previous paper [1] with the same basic title some particular classes of 2-semiparallel surfaces are studied in space forms. Now these surfaces are investigated in general.

The necessary definitions, primary results, apparatus, and references are given in [1].

2. GENERAL APPROACH TO 2-SEMIPARALLEL SURFACES

As shown in [1], a moving orthonormal frame $\{x, e_I\}$ in a space form $N^n(c)$, for which

$$dx = e_I\omega^I, \quad de_I = e_J\omega_I^J - xc\omega^I, \quad \omega_I^J + \omega_J^I = 0,$$

$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_I^J = \omega_I^K \wedge \omega_K^J + c\omega^J \wedge \omega^I$$

hold (here I, J, K run $\{1, ..., n\}$), can be adapted to a surface M^2 in $N^n(c)$ so that $\omega^\alpha = 0$, $\omega_i^\alpha = h_{ij}^\alpha \omega^j$ (here i, j run $\{1, 2\}$, but α runs $\{3, ..., n\}$), and $h_{11}^3 = \alpha + a$, $h_{12}^3 = h_{21}^3 = 0$, $h_{22}^3 = \alpha - a$, $h_{11}^4 = h_{22}^4 = \beta$, $h_{12}^4 = h_{21}^4 = b$, $h_{11}^5 = h_{22}^5 = \gamma$, $h_{12}^5 = h_{21}^5 = 0$, $h_{ij}^\rho = 0$ for $\rho = \{6, ..., n\}$ if $n \geq 6$. Moreover, $\bar{\nabla} h_{ij}^\alpha = h_{ijk}^\alpha \omega^k$, where $\bar{\nabla} h_{ij}^\alpha = dh_{ij}^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k + h_{ij}^\beta \omega_\beta^\alpha$ and $h_{ijk}^\alpha = h_{ikj}^\alpha$.

The only essential curvature 2-form of the connection ∇ on M^2 is $\Omega_1^2 = -K\omega^1 \wedge \omega^2$, where $K = c + \alpha^2 + \beta^2 + \gamma^2 - a^2 - b^2$ is the Gaussian curvature of M^2; the only essential curvature 2-form of the normal connection $\nabla^\perp$ of M^2 is $\Omega_3^4 = -2ab\omega^1 \wedge \omega^2$. All other curvature 2-forms Ω_j^i, Ω_β^α of the van der Waerden–Bortolotti connection $\bar{\nabla}$ are zero, except, perhaps, $\Omega_2^1 = -\Omega_1^2$ and $\Omega_4^3 = -\Omega_4^3$.

A surface M^2 in $N^n(c)$ is called 2-semiparallel if

$$h_{ljk}^\alpha \Omega_i^l + h_{ilk}^\alpha \Omega_j^l + h_{ijl}^\alpha \Omega_k^l - h_{ijk}^\beta \Omega_\beta^\alpha = 0,$$

or, in more detail,

$$3h_{112}^3 \Omega_1^2 + h_{111}^4 \Omega_3^4 = 0, \tag{1}$$

$$2h_{122}^3 \Omega_1^2 - h_{111}^3 \Omega_1^2 + h_{112}^4 \Omega_3^4 = 0, \tag{2}$$

$$-2h_{112}^3 \Omega_1^2 + h_{222}^3 \Omega_1^2 + h_{122}^4 \Omega_3^4 = 0, \tag{3}$$

$$-3h_{122}^3 \Omega_1^2 + h_{222}^4 \Omega_3^4 = 0, \tag{4}$$

$$-h_{111}^3 \Omega_3^4 + 3h_{112}^4 \Omega_1^2 = 0, \tag{5}$$

$$-h_{112}^3 \Omega_3^4 - h_{111}^4 \Omega_1^2 + 2h_{122}^4 \Omega_1^2 = 0, \tag{6}$$

$$-h_{122}^3 \Omega_3^4 - 2h_{112}^4 \Omega_1^2 + h_{222}^4 \Omega_1^2 = 0, \tag{7}$$

$$-h_{222}^3 \Omega_3^4 - 3h_{122}^4 \Omega_1^2 = 0, \tag{8}$$

$$3h_{112}^\xi \Omega_1^2 = (2h_{122}^\xi - h_{111}^\xi)\Omega_1^2 = (-2h_{112}^\xi + h_{222}^\xi)\Omega_1^2 = 3h_{122}^\xi \Omega_1^2 = 0, \tag{9}$$

where

$$\xi \in \{5, ..., n\}.$$

It is seen that every surface M^2 with flat connection $\bar{\nabla}$ in $N^n(c)$, which is characterized by $\Omega_1^2 = \Omega_3^4 = 0$, is 2-semiparallel. Also, every parallel surface M^2 in $N^n(c)$, characterized by $h_{ijk}^\alpha = 0$, is 2-semiparallel. These are so-called trivial cases (cf. Proposition 4 in [1]).

The system of equations (1)–(8) is a linear homogeneous system for h^3_{ijk} and h^4_{ijk}, which has the determinant

$$D = \begin{vmatrix} 0 & 3\Omega_1^2 & 0 & 0 & \Omega_3^4 & 0 & 0 & 0 \\ -\Omega_1^2 & 0 & 2\Omega_1^2 & 0 & 0 & \Omega_3^4 & 0 & 0 \\ 0 & -2\Omega_1^2 & 0 & \Omega_1^2 & 0 & 0 & \Omega_3^4 & 0 \\ 0 & 0 & -3\Omega_1^2 & 0 & 0 & 0 & 0 & \Omega_3^4 \\ -\Omega_3^4 & 0 & 0 & 0 & 0 & 3\Omega_1^2 & 0 & 0 \\ 0 & -\Omega_3^4 & 0 & 0 & -\Omega_1^2 & 0 & 2\Omega_1^2 & 0 \\ 0 & 0 & -\Omega_3^4 & 0 & 0 & -2\Omega_1^2 & 0 & \Omega_1^2 \\ 0 & 0 & 0 & -\Omega_3^4 & 0 & 0 & -3\Omega_1^2 & 0 \end{vmatrix},$$

equal to $D = [9(\Omega_1^2)^2 - (\Omega_3^4)^2]^2[(\Omega_1^2)^2 - (\Omega_3^4)^2]^2$.

If this determinant is nonzero, then $h^3_{ijk} = h^4_{ijk} = 0$. If, moreover, $\Omega_1^2 \neq 0$, then from (9) also $h^\xi_{ijk} = 0$, and this gives a trivial case, a parallel surface.

It is seen that for a nontrivial case there are only three following possibilities:

1) $(\Omega_3^4)^2 = (\Omega_1^2)^2 \neq 0$, or
2) $(\Omega_3^4)^2 = (3\Omega_1^2)^2 \neq 0$, or
3) $\Omega_1^2 = 0$, $\Omega_3^4 \neq 0$.

3. NONEXISTENCE OF NONTRIVIAL 2-SEMIPARALLEL SURFACES WITH $D = 0$

In [1] two principal cases are distinguished: (I) $a > b \geq 0$ and (II) $a = b \geq 0$. Here $a = b = 0$ leads to a totally umbilical (in particular, geodesic) surface, which is a parallel one (see [2]) and thus belongs to a trivial case. Therefore (II) in a nontrivial case is (II) $a = b > 0$.

Let us consider first possibility 1). Then $\Omega_3^4 = \varepsilon\Omega_1^2 \neq 0$, where $\varepsilon = 1$ or $\varepsilon = -1$; thus $ab \neq 0$. Now from (1)–(8), if we denote $h^3_{112} = A$, $h^4_{112} = B$, it follows that

$$h^4_{111} = -3\varepsilon A, \quad h^4_{122} = -\varepsilon A, \quad h^4_{222} = 3B, \quad h^3_{122} = \varepsilon B.$$

Here the formulae (14) and (22) of [1] lead to

$$2a\omega_1^2 - b\omega_3^4 = A\omega^1 + \varepsilon B\omega^2, \quad 2b\omega_1^2 - a\omega_3^4 = \varepsilon A\omega^1 + B\omega^2,$$

therefore $2a\omega_1^2 - b\omega_4^3 = \varepsilon(2b\omega_1^2 - a\omega_3^4)$ and so

$$2(a - \varepsilon b)\omega_1^2 = -\varepsilon(a - \varepsilon b)\omega_3^4. \tag{10}$$

Here for case (I), and also for (II) with $\varepsilon = -1$, one obtains $a - \varepsilon b \neq 0$ and so $2\omega_1^2 = -\varepsilon\omega_3^4$. This gives by exterior differentiation

$$2\Omega_1^2 = -\varepsilon\Big(\Omega_3^4 - \sum_\xi \omega_3^\xi \wedge \omega_4^\xi\Big),$$

but from (9) $h_{ijk}^\xi = 0$, and therefore the formula (10) of [1] leads by $\alpha = \xi$ to $-b\omega_4^\xi = 0$, thus $\omega_4^\xi = 0$ (because here $0 \neq \Omega_3^4 = -2ab\omega^1 \wedge \omega^2$). The result is a contradiction $4(\Omega_1^2)^2 = (\Omega_3^4)^2$ to the supposition $(\Omega_3^4)^2 = (\Omega_1^2)^2 \neq 0$ of the considered possibility 1).

In the same way it can be shown that in case (I) and in case (II) with $\varepsilon = -1$ possibility 2) also leads to a contradiction. Indeed, then $\Omega_3^4 = 3\varepsilon\Omega_1^2$ and from (1)–(8)

$$h_{111}^4 = -\varepsilon A, \quad h_{122}^4 = \varepsilon A, \quad h_{222}^4 = -B, \quad h_{122}^3 = -\varepsilon B,$$

so

$$2a\omega_1^2 - b\omega_3^4 = A\omega^1 - \varepsilon B\omega^2, \quad 2b\omega_1^2 - a\omega_3^4 = \varepsilon A\omega^1 - B\omega^2.$$

Therefore $2a\omega_1^2 - b\omega_3^4 = \varepsilon(2b\omega_1^2 - a\omega_3^4)$, as above, but this gives the same $4(\Omega_1^2)^2 = (\Omega_3^4)^2$, which contradicts $(\Omega_3^4)^2 = 9(\Omega_1^2)^2 \neq 0$.

The analysis is more complicated in case (II) with $\varepsilon = 1$. Then, due to $a = b$, (10) turns into an identity and does not give any connection between ω_1^2 and ω_3^4. Here, as before, $h_{ijk}^\xi = 0$, $\xi \in \{5, ..., a\}$, hence (9)–(11) of [1] give by $\alpha = \rho \in \{6, ..., n\}$, due to $h_{ij}^\rho = 0$ and $a = b > 0$, that

$$-(\alpha + a)\omega_3^\rho - \beta\omega_4^\rho - \gamma\omega_5^\rho = 0,$$
$$-a\omega_4^\rho = 0,$$
$$-(\alpha - a)\omega_3^\rho - \beta\omega_4^\rho - \gamma\omega_5^\rho = 0.$$

Thus $\omega_3^\rho = \omega_4^\rho = \gamma\omega_5^\rho = 0$. In addition, (19)–(21) of [1] give analogously $\omega_3^5 = \omega_4^5 = d\gamma = 0$, but the system (13)–(18) of [1] reduces, for possibility 1), to

$$da = B\omega^1 - A\omega^2,$$
$$a(2\omega_1^2 - \omega_3^4) = A\omega^1 + B\omega^2,$$
$$d\alpha = \beta\omega_3^4 + 2(B\omega^1 + A\omega^2),$$
$$d\beta = -\alpha\omega_3^4 + 2(-A\omega^1 + B\omega^2).$$

These equations by exterior differentiation lead to

$$(dB + A\omega_1^2) \wedge \omega^1 - (dA - B\omega_1^2) \wedge \omega^2 = 0,$$
$$[dA + B(\omega_1^2 - \omega_3^4)] \wedge \omega^1 + [dB - A(\omega_1^2 - \omega_3^4) + 2a^3\omega^1] \wedge \omega^2 = 0, \tag{11}$$
$$[dB - A(\omega_1^2 - \omega_3^4)] \wedge \omega^1 + [dA + B(\omega_1^2 - \omega_3^4) - a^2\beta\omega^1] \wedge \omega^2 = 0,$$
$$-[dA + B(\omega_1^2 - \omega_3^4)] \wedge \omega^1 + [dB - A(\omega_1^2 - \omega_3^4) + a^2\alpha\omega^1] \wedge \omega^2 = 0.$$

By Cartan's lemma

$$dB + A\omega_1^2 = P\omega^1 + Q\omega^2, \quad -(dA - B\omega_1^2) = Q\omega^1 + R\omega^2, \tag{12}$$

and now substitution into the last three exterior equations gives, due to the expression of $a(2\omega_1^2 - \omega_3^4)$, that

$$P = a^{-1}A^2 + \tfrac{1}{2}a^2(2a - \alpha),$$
$$Q = a^{-1}AB - \tfrac{1}{2}a^2\beta,$$
$$R = a^{-1}B^2 + \tfrac{1}{2}a^2(2a + \alpha).$$

Substituting into (12) and differentiating then exteriorly, one obtains

$$A\beta - B(\alpha + 8a) = 0, \tag{13}$$

$$A(\alpha - 8a) + B\beta = 0. \tag{14}$$

If here $\alpha^2 + \beta^2 \neq 64a^2$, then $A = B = 0$, but if $\alpha^2 + \beta^2 = 64a^2$, then differentiation gives

$$\alpha B - \beta A = 32aB, \quad \alpha A + \beta B = -32aA$$

which, together with (13) and (14), lead to $40aB = 40aA = 0$, thus again to $A = B = 0$. As a result, (11) reduces to a contradiction $a^3\omega^1 \wedge \omega^2 = 0$!

For possibility 2), when $\Omega_3^4 = 3\Omega_1^2 \neq 0$ and thus

$$8a^2 = 3(c + \alpha^2 + \beta^2 + \gamma^2), \tag{15}$$

the system (13)–(18) of [1] reduces to

$$da = B\omega^1 + A\omega^2,$$

$$a(2\omega_1^2 - \omega_3^4) = A\omega^1 + B\omega^2,$$
$$d\alpha = \beta\omega_3^4, \quad d\beta = -\alpha\omega_3^4,$$

and, as before, $\omega_3^\xi = \omega_4^\xi = \gamma\omega_5^\xi = d\gamma = 0$. The equations with $d\alpha, d\beta$ lead by exterior differentiation to $2a^2\beta = 2a^2\alpha = 0$, thus to $\alpha = \beta = 0$. From (15) it follows that $8a^2 = 3(c + \gamma^2) = \text{const}$, thus $A = B = 0$. But instead of (11) now

$$[dA + B(\omega_1^2 - \omega_3^4)] \wedge \omega^1 + [dB + A(3\omega_1^2 - \omega_3^4) - \tfrac{2}{3}a^3\omega^1] \wedge \omega^2 = 0,$$

and this again leads to a contradiction.

As a result, both possibilities 1) and 2) give only contradictions, and so there holds

Proposition 1. *A surface M^2 in $N^n(c)$ can be a nontrivial 2-semiparallel one only if possibility 3) above is realized, i.e. only if $\Omega_1^2 = 0$, $\Omega_3^4 \neq 0$, or, more generally, only if M^2 is locally Euclidean and has non-flat normal connection.*

4. LOCALLY EUCLIDEAN 2-SEMIPARALLEL SURFACES WITH NON-FLAT NORMAL CONNECTION

It remains to investigate more thoroughly the case of the last proposition. Taking into account the expressions above for Ω_1^2 and Ω_3^4, this case is characterized by

$$c + \alpha^2 + \beta^2 + \gamma^2 - a^2 - b^2 = 0, \quad ab \neq 0.$$

Now from (1)–(8) it follows that $h_{ijk}^3 = h_{ijk}^4 = 0$, but this shows that span$\{h_{ijk}\}$ reduces to span$\{h_{ijk}^\xi e_\xi\}$ and therefore is orthogonal to $I_x M^2$ at each point $x \in M^2$ (see Remark 1 and Sec. 3 in [1]). Since the conditions (9) leave h_{ijk}^ξ free, due to $\Omega_1^2 = 0$, this orthogonality, conversely, implies $h_{ijk}^3 = h_{ijk}^4 = 0$ and thus the 2-semiparallelity of the considered M^2. The result can be formulated as

Proposition 2. *A locally Euclidean surface M^2 with non-flat normal connection in space form $N^n(c)$ is a nontrivial 2-semiparallel surface if and only if its* span$\{h_{ijk}\}$ *is orthogonal to the plane of normal curvature indicatrix at each point $x \in M^2$, i.e. orthogonal to $I_x M^2$.*

To study further this kind of 2-semiparallel surfaces, let us turn to the formulae (13)–(18) and (22) of [1]. They give that

$$da = db = 2a\omega_1^2 - b\omega_3^4 = 2b\omega_1^2 - a\omega_3^4 = 0, \tag{16}$$

$$d\alpha = \beta\omega_3^4 + \gamma\omega_3^5, \quad d\beta = -\alpha\omega_3^4 + \gamma\omega_4^5, \tag{17}$$

but the formulae (19)–(21) and (9)–(11) of [1] by $\alpha = \rho \in \{6, ..., n\}$ yield

$$d\gamma + \alpha\omega_3^5 + \beta\omega_4^5 = \tfrac{1}{2}(p^5 + r^5)\omega^1 + \tfrac{1}{2}(q^5 + s^5)\omega^2, \tag{18}$$

$$b\omega_4^\xi = q^\xi\omega^1 + r^\xi\omega^2, \tag{19}$$

$$a\omega_3^\xi = \tfrac{1}{2}(p^\xi - r^\xi)\omega^1 + \tfrac{1}{2}(q^\xi - s^\xi)\omega^2, \tag{20}$$

$$\alpha\omega_3^\rho + \beta\omega_4^\rho + \gamma\omega_5^\rho = \tfrac{1}{2}(p^\rho + r^\rho)\omega^1 + \tfrac{1}{2}(q^\rho + s^\rho)\omega^2, \tag{21}$$

where, recall, $\xi \in \{5, ..., n\}$ and the denotations $p^\xi = h_{111}^\xi$, $q^\xi = h_{112}^\xi$, $r^\xi = h_{122}^\xi$, $s^\xi = h_{222}^\xi$ are used.

Equations (16) give by exterior differentiation a relation $d\omega_3^4 = 0$ which is equivalent to

$$4ab + \mathbf{p}\cdot\mathbf{r} + \mathbf{q}\cdot\mathbf{s} - \mathbf{q}^2 - \mathbf{r}^2 = 0, \tag{22}$$

where $\mathbf{p}\cdot\mathbf{r} = \sum_\xi p^\xi r^\xi$, etc.

Equations (17) lead by exterior differentiation to

$$[-\beta\omega_4^5 + \tfrac{1}{2}(p^5 + r^5)\omega^1 + \tfrac{1}{2}(q^5 + s^5)\omega^2] \wedge \omega_3^5 + \gamma\omega_3^\rho \wedge \omega_\rho^5 = 0, \tag{23}$$

$$[-\alpha\omega_3^5 + \tfrac{1}{2}(p^5 + r^5)\omega^1 + \tfrac{1}{2}(q^5 + s^5)\omega^2] \wedge \omega_4^5 + \gamma\omega_4^\rho \wedge \omega_\rho^5 = 0, \tag{24}$$

where $\omega_3^5, \omega_3^\rho, \omega_4^5, \omega_4^\rho$ can be expressed from (19) and (20), and then (21) implies

$$-2ab\gamma\omega_5^\rho = \{b[(\alpha - a)p^\rho - (\alpha + a)r^\rho] + 2a\beta q^\rho\}\,\omega^1$$
$$+ \{b[(\alpha - a)q^\rho - (\alpha + a)s^\rho] + 2a\beta r^\rho\}\,\omega^2.$$

Substitution into (23) and (24) gives some relations between $a, b, \alpha, \beta, p^\xi, q^\xi, r^\xi$, and s^ξ. Some of them are rather complicated and make it difficult to study the locally Euclidean 2-semiparallel surfaces M^2 with non-flat $\nabla^\perp$ in general. But one of their properties can be shown immediately: from (16) it follows that a and b are some nonzero constants so that the following statement holds.

Proposition 3. *The surface M^2 of Proposition 2 has the property that its normal curvature ellipses are congruent at two arbitrary points.*

5. EXISTENCE AND GEOMETRICAL PROPERTIES OF 2-SEMIPARALLEL SURFACES

The difficulty with the considered class of 2-semiparallel surfaces makes the problem of their existence topical.

Let this problem be considered at first for trivially 2-semiparallel surfaces (see Proposition 4 of [1]), i.e. for parallel surfaces and surfaces with flat $\bar{\nabla}$. For the former ones the problem is solved by Theorem 2 of [1]. Note that non-flat $\bar{\nabla}$ among parallel surfaces have only totally umbilical surfaces and Veronese surfaces.

Let now the surfaces M^2 with flat $\bar{\nabla}$, i.e. with $b = 0$ and $K = 0$, in Euclidean space E^n be considered. Then $c = 0$ and $K \equiv \alpha^2 + \beta^2 + \gamma^2 - a^2 = 0$, thus $h_{11} = (\alpha + a)e_3 + \beta e_4 + \gamma e_5$, $h_{22} = (\alpha - a)e_3 + \beta e_4 + \gamma e_5$ are orthogonal; moreover, $h_{12} = 0$. Supposing first that h_{11} and h_{22} are both nonzero, a new

orthonormal frame part $\{\tilde{e}_3, \tilde{e}_4\}$ in two-dimensional span$\{h_{ij}\}$ can be taken so that $h_{11} = k\tilde{e}_3$, $h_{22} = l\tilde{e}_4$, $kl \neq 0$ (cf. [2]). Then

$$\tilde{\omega}_1^3 = k\omega^1, \quad \tilde{\omega}_2^4 = l\omega^2, \quad \tilde{\omega}_2^3 = 0, \quad \tilde{\omega}_1^4 = 0, \quad \omega_1^\xi = 0, \quad \omega_2^\xi = 0.$$

Exterior differentiation leads to

$$\begin{gathered} d\ln k \wedge \omega^1 + \omega_1^2 \wedge \omega^2 = 0, \\ -\omega_1^2 \wedge \omega^1 + d\ln l \wedge \omega^2 = 0, \\ -k\omega_1^2 \wedge \omega^1 + l\tilde{\omega}_3^4 \wedge \omega^2 = 0, \\ k\tilde{\omega}_3^4 \wedge \omega^1 - l\omega_1^2 \wedge \omega^2 = 0, \\ \omega_3^\xi \wedge \omega^1 = 0, \quad \omega_4^\xi \wedge \omega^2 = 0, \end{gathered}$$

and then, due to Cartan's lemma,

$$d\ln k = p\omega^1 + q\omega^2, \quad \omega_1^2 = q\omega^1 + r\omega^2, \quad d\ln l = -r\omega^1 + s\omega^2,$$

$$\tilde{\omega}_3^4 = -kl^{-1}r\omega^1 - k^{-1}lq\omega^2, \quad \omega_3^\xi = P^\xi\omega^1, \quad \omega_4^\xi = Q^\xi\omega^2.$$

Here the rank of polar system is $s_1 = 4 + 2(n-4)$, and the number N of new essential coefficients p, q, r, s, P^ξ, Q^ξ is also $4 + 2(n-4)$. Hence Cartan's test condition is satisfied (see [3,4]) and the considered surface exists with arbitrariness of $2n - 4$ real holomorphic functions of one real argument.

If $h_{11} \neq 0$, $h_{22} = 0$, then $s_1 = N = 3 + 2(n-4)$, and the considered surface exists in this case with arbitrariness of $2n - 5$ real holomorphic functions of one real argument.

The result is the same if instead of E^n is $N^n(c)$ with $c \neq 0$. Then the investigation must be done in Euclidean E^{n+1} (or in Minkowski E_1^{n+1}), which contains $N^n(c)$ as a hypersphere of radius $(\sqrt{c})^{-1}$ [or $i(\sqrt{|c|})^{-1}$, respectively].

Let us turn now to the problem of the existence of nontrivial 2-semiparallel surfaces M^2 in $N^n(c)$, i.e. of surfaces of Propositions 1–3. As noted above, this problem is rather complicated in general.

Below, the investigation of this problem will be restricted to the subclass of these surfaces in $N^6(c)$, which satisfy the complementary condition that the mean curvature vector H at each point $x \in M^2$ is orthogonal to the plane $I_x M^2$ of the normal curvature ellipse. This subclass is characterized by $\alpha = \beta = 0$, and contains all minimal surfaces (i.e. with $H = 0$), for which, in addition, also $\gamma = 0$.

So let further $\alpha = \beta = 0$. Then $\gamma^2 = a^2 + b^2 - c = \text{const} \geq 0$ (see Proposition 3). Let first $\gamma \neq 0$, so that $a^2 + b^2 > 0$. From (17) now $\omega_3^5 = \omega_4^5 = 0$, thus due to (19) and (20) $p^5 = q^5 = r^5 = s^5 = 0$, and (18) turns into an identity.

Moreover, since now $n = 6$, it is suitable to denote p^6, q^6, etc. by p, q, etc. Then the relations (19)–(22) reduce to

$$b\omega_4^6 = q\omega^1 + r\omega^2, \tag{19'}$$

$$a\omega_3^6 = \tfrac{1}{2}(p-r)\omega^1 + \tfrac{1}{2}(q-s)\omega^2, \tag{20'}$$

$$\gamma\omega_5^6 = \tfrac{1}{2}(p+r)\omega^1 + \tfrac{1}{2}(q+s)\omega^2, \tag{21'}$$

$$4ab + pr + qs - q^2 - r^2 = 0, \tag{22'}$$

but (23) and (24) give

$$ps - qr = 0, \quad pr - qs - q^2 + r^2 = 0. \tag{23'}$$

Exterior differentiation yields from (19′)–(21′), due to (16),

$$[dq + (p-2r)\omega_1^2] \wedge \omega^1 + [dr + (2q-s)\omega_1^2] \wedge \omega^2 = 0,$$

$$[d(p-r) + (s-5q)\omega_1^2] \wedge \omega^1 + [d(q-s) + (p-5r)\omega_1^2] \wedge \omega^2 = 0,$$

$$[d(p+r) - (q+s)\omega_1^2] \wedge \omega^1 + [d(q+s) + (p+r)\omega_1^2] \wedge \omega^2 = 0.$$

Therefore by Cartan's lemma

$$dp = 3q\omega_1^2 + P\omega^1 + Q\omega^2, \tag{25}$$

$$dq = (2r-p)\omega_1^2 + Q\omega^1 + R\omega^2, \tag{26}$$

$$dr = (s-2q)\omega_1^2 + R\omega^1 + S\omega^2, \tag{27}$$

$$ds = -3r\omega_1^2 + S\omega^1 + T\omega^2. \tag{28}$$

Differentiating in (22′) and (23′), using (25)–(28), (16), and the same (23′), one can find that the terms with ω_1^2 cancel, but the coefficients of ω^1 and ω^2 lead to

$$rP + (s-2q)Q + (p-2r)R + qS = 0, \tag{29}$$

$$rQ + (s-2q)R + (p-2r)S + qT = 0, \tag{30}$$

$$-sP + rQ + qR - pS = 0, \tag{31}$$

$$-sQ + rR + qS - pT = 0, \tag{32}$$

$$rP - (s+2q)Q + (p+2r)R - qS = 0, \tag{33}$$

$$rQ - (s+2q)R + (p+2r)S - qT = 0. \tag{34}$$

This is a linear homogeneous system for P, Q, R, S, and T, whose determinant turns to be nonzero, in general. Therefore $P = Q = R = S = T = 0$, and so (25)–(28) reduce to

$$dp = 3q\omega_1^2, \quad dq = (2r-p)\omega_1^2, \quad dr = (s-2q)\omega_1^2, \quad ds = -3r\omega_1^2.$$

The last differential system is, due to $d\omega_1^2 = \Omega_1^2 = 0$, completely integrable.

This system coincides with $\bar{\nabla} h_{ijk}^6 = 0$; recall that $h_{ijk}^3 = h_{ijk}^4 = h_{ijk}^5 = 0$. But $\bar{\nabla} h_{ijk}^3$, $\bar{\nabla} h_{ijk}^4$, $\bar{\nabla} h_{ijk}^5$ need not be zero, in general.

The result can be formulated as

Proposition 4. *In $N^6(c)$ there exist, with arbitrariness of some constants, the 2-semiparallel locally Euclidean surfaces M^2 with non-flat $\nabla^\perp$, whose mean curvature vector H at each point $x \in M^2$ is orthogonal to the plane of the normal curvature ellipse. These ellipses at two arbitrary points of such M^2 are congruent, and the length of H (i.e. the distance of x from the plane of the ellipse) is constant. Such an M^2 is neither parallel, nor 2-parallel, nor semiparallel.*

Finally let us investigate the minimal 2-semiparallel surfaces M^2 in $N^n(c)$ with $\Omega_1^2 = 0$, $\Omega_3^4 \neq 0$, i.e. let in addition to $\alpha = \beta = 0$ also $\gamma = 0$. Then $K = 0$ implies $c = a^2 + b^2$, so that these surfaces can exist only in elliptic spaces. Now Eqs. (17) are satisfied, but from (18) and (21) it follows that $r^\xi = -p^\xi$, $s^\xi = -q^\xi$. So the only essential equations among (17)–(21) are

$$a\omega_3^\xi = p^\xi\omega^1 + q^\xi\omega^2, \qquad b\omega_4^\xi = q^\xi\omega^1 - p^\xi\omega^2.$$

By exterior differentiation they give, due to (16), that

$$(dp^\xi - 3q^\xi\omega_1^2) \wedge \omega^1 + (dq^\xi + 3p^\xi\omega_1^2) \wedge \omega^2 = 0,$$

$$(dq^\xi + 3p^\xi\omega_1^2) \wedge \omega^1 - (dp^\xi - 3q^\xi\omega_1^2) \wedge \omega^2 = 0,$$

and from here by Cartan's lemma

$$dp^\xi - 3q^\xi\omega_1^2 = P^\xi\omega^1 + Q^\xi\omega^2,$$

$$dq^\xi + 3p^\xi\omega_1^2 = Q^\xi\omega^1 - P^\xi\omega^2.$$

Let further $n = 5$, so that ξ takes only one value 5; let us denote p^5, q^5, etc. by p, q, etc. The relation (22), which reduces to $2ab - p^2 - q^2 = 0$, gives by differentiation $pP + qQ = 0$, $pQ - qP = 0$. Here $p^2 + q^2 = 0$ leads to a contradiction $ab = 0$. Therefore $P = Q = 0$, and the investigation finishes with a completely integrable system $dp = 3q\omega_1^2$, $dq = -3p\omega_1^2$.

The result is

Proposition 5. *In the elliptic space $N^5(c)$, $c > 0$, there exist, with arbitrariness of some constants, the 2-semiparallel minimal locally Euclidean surfaces M^2 with non-flat $\nabla^\perp$, whose normal curvature ellipses at two arbitrary points are congruent. Such an M^2 is neither parallel, nor 2-parallel, nor semiparallel.*

6. MAIN THEOREM AND CONCLUDING REMARKS

The main results of our previous paper [1] and of the present paper can be summarized by the following classification theorem.

Theorem 6. *A surface M^2 in a space form $N^n(c)$ is 2-semiparallel if and only if it belongs to one of the following three mutually exclusive classes consisting of:*

(i) *surfaces with flat van der Waerden–Bortolotti connection $\bar{\nabla}$,*

(ii) *parallel surfaces with non-flat $\bar{\nabla}$, i.e. totally umbilical surfaces and Veronese surfaces,*

(iii) *locally Euclidean surfaces (i.e. with flat ∇, or equivalently with vanishing Gaussian curvature), whose normal connection $\nabla^{\perp}$ is non-flat and* span$\{h_{ijk}\}$ *at an arbitrary point $x \in M$ is orthogonal to the plane of normal curvature ellipse at this point x; these ellipses at two arbitrary points of such an M^2 are congruent.*

All these classes are non-empty; especially for class (iii) this is in some low dimension n stated by Propositions 4 and 5. Classes (i) and (ii) can be considered as trivial (see Proposition 4 of [1]).

Remark 1. Propositions 4 and 5 do not exclude the case when $a = b$, which is the case of pointwise isotropic surfaces. Thus these propositions establish that the surfaces of type (ii*) in Theorem 5 of [1] do exist, and so the problem which was left open in part 1 is now partly solved.

Remark 2. The minimal locally Euclidean surfaces M^2 with non-flat $\nabla^{\perp}$ of Proposition 5 were found earlier in [5]. This paper deals with the minimal surfaces in space forms, whose normal curvature ellipses are similar at two arbitrary points of the surface. The special case when these ellipses are circles (and thus the surface is pointwise isotropic) was indicated already in [6].

Remark 3. In [7] Mirzoyan proved that a submanifold M^m in $N^n(c)$ is an envelope of order s for some family of m-dimensional s-parallel submanifolds if and only if it is l-semiparallel for the values $l = s - 1$ and $l = s$ if $s \geq 2$, and for the value $l = s$ if $s = 1$ (here 1-parallel and 1-semiparallel mean simply parallel and semiparallel, respectively).

According to Theorem 3 of [1], every 2-parallel surface M^2 in $N^n(c)$ has flat $\bar{\nabla}$, and this flatness of $\bar{\nabla}$ holds also for the envelope of order 2 (in the sense of [7]) of their family. Propositions 4 and 5 above show now that the condition of l-semiparallelity for both values $l = s - 1$ and $l = s \geq 2$ in Mirzoyan's theorem is essential, at least by $l = 2$, and cannot be weakened to $l = s$ only.

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2-SEMIPARALLEELSED PINNAD RUUMIVORMIDES

2. Üldjuht

Ülo LUMISTE

On antud üldiste 2-semiparalleelsete pindade klassifikatsioon n-mõõtmelistes ruumivormides ning näidatud, et need on kas (i) kõverusvaba seostusega $\bar{\nabla}$ pinnad või (ii) paralleelsed pinnad või (iii) lokaalselt eukleidilised pinnad mittekõverusvaba seostusega $\nabla^{\perp}$, mille span$\{h_{ijk}\}$ on igas punktis ortogonaalne normaalkõverusellipsi tasandiga. On tõestatud, et tüüpi (i), (ii) ja (iii) pinnad tõepoolest eksisteerivad, ning selgitatud nende mõningaid geomeetrilisi omadusi.