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ON STATISTICAL CONVERGENCE

Binod Chandra TRIPATHY

Mathematical Science Division, Institute of Advanced Study in Science and Technology, Khanapara, Guwahati-781 022, Assam, India

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Abstract. In this article the results of Dedekind and Abel for term product of series are extended to statistically convergent series. An extension of Leibniz's test is given and Tauberian theorems are proved.

Key words: statistical convergence, Tauberian theorem, alternating series, density, partial sums.

1. DEFINITION AND BACKGROUND

In order to extend the notion of convergence, statistical convergence of sequences was introduced by Fast [1] and Schoenberg [2]. Later on it was studied and linked with summability by Fridy [3], Tripathy [4], Connor [5], Rath and Tripathy [6], Sălăt [7], Maddox [8], Kolk [9] and many others. Statistical convergence of series was introduced by Tripathy [10]. In the present article mainly the term product of series and Tauberian theorems are discussed. The basic idea depends on the density of a certain subset A of N, the set of natural numbers. A subset A of N is said to have the density $\delta(A)$ if there exists $\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$, where $A(n) = \{k \le n : k \in A\}$ and $|A|$ denotes the cardinality of the set A. Clearly finite subsets of N have zero natural density and $\delta(A^c) = \delta(N-A) = 1-\delta(A)$, whenever either side exists. Throughout this paper A^c denotes the complement of A in N. For a series $\sum a_k$ we write $s_n = a_1 + a_2 + \ldots + a_n$, $n \in N$.

Definition 1. *A series* $\sum a_k$ *is said to be statistically convergent to* s*, written as* stat-lim $s_n = s$, *if the sequence of its partial sums* (s_n) *converges statistically to* s*, that is, for every* $\varepsilon > 0$, $\delta(\{n \in N : |s_n - s| \ge \varepsilon\}) = 0$.

Definition 2. *Let $p = (p_k)$ be a bounded sequence of non-negative numbers with* $\inf p_k > 0$ *and let X be a normed linear space. A sequence $x = (x_k)$, where $x_k \in X$, $k \in N$, is called strongly p-Cesàro summable if there exists $L \in X$ such that*

$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} \|x_k - L\|^{p_k} = 0 .$$

The space of all strongly p-Cesàro summable X-valued sequences is denoted by $w_{(p)}(X)$.

Throughout the paper C denotes the set of all complex numbers; ℓ_∞, $\bar{c}$, $\bar{c}_0$ denote the spaces of all *bounded*, *statistically convergent*, and *statistically null* sequences. The forward difference operator Δ is defined as $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$. Sums without limit mean that the summation is from $k = 1$ to ∞. For two statistically convergent series $\sum a_k$ and $\sum b_k$, the term product $\sum a_k b_k$ may or may not be statistically convergent. This is clear from the following example, taking $b_k = a_k$ for all $k \in N$.

Example. Let $\sum a_n$ be defined as $a_n = (-1)^n$ for $n = k^2$ and $(n-1) = k^2$, $k \in N$, and $a_n = n^{-2}$ otherwise.

The following lemmas will be used in establishing the results.

Lemma 1. *If $\sum a_k$ is statistically convergent, then* stat-lim $a_n = 0$, *but not conversely.*

Lemma 2. (Abel's summation formula). *Let $1 \le m \le n$ and $s_0 = 0$. Then*

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} s_k \Delta b_k + s_n b_n - s_{m-1} b_m .$$

Lemma 3. (Theorem 3, Fridy [3]). *If $x = (x_k)$ is a sequence such that* stat-lim $x_k = L$ *and $\{k \Delta x_k\} \in \ell_\infty$, then* $\lim x_k = L$.

Lemma 4. (Theorem 5, Fridy [3]). *Let (k_i) be an increasing sequence of positive integers such that* $\liminf \dfrac{k_{i+1}}{k_i} > 1$ *and let (x_k) be a corresponding gap sequence:* $\Delta x_k = 0$ *if $k \ne k_i$ for $i \in N$; if* stat-lim $x_k = L$, then $\lim x_k = L$.

Lemma 5. (Theorem, Tripathy [4]). *Let $p = (p_k)$ be a bounded sequence of non-negative real numbers such that* $\inf p_k > 0$. *Then*

$$w_{(p)}(X) \cap \ell_\infty(X) = \bar{c}(X) \cap \ell_\infty(X).$$

2. MAIN RESULTS

The first proposition follows immediately from Lemma 1.

Proposition 1. *If $\sum a_k$ converges statistically, then $\sum \Delta a_k$ converges statistically to a_1.*

Theorem 1. *Let $a_k, b_k \in C$ and let*

(2.1) (s_n) *be bounded,*

(2.2) $\sum |\Delta b_k|$ *be convergent,*

(2.3) $(b_n) \in \bar{c}_0$.

Then $\sum a_k b_k$ converges statistically to $\sum s_k \Delta b_k$.

Proof. The statement follows from the equality

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} s_k \Delta b_k + s_n b_n \tag{2.4}$$

which is true by Lemma 2.

Theorem 2. *Let $a_k, b_k \in C$ with* (2.2) *holding and*

(2.5) $(s_n) \in w_{(p)}(C) \cap \ell_\infty(C)$,

whenever $0 < \inf_k p_k \le p_k \le \sup_k p_k < \infty$. Then $\sum a_k b_k$ converges statistically.

Proof. We have for all $n \in N$

$$b_n = b_1 - \sum_{k=1}^{n-1} \Delta b_k. \tag{2.6}$$

By (2.2) and (2.5) the sequence of numbers $\sum_{k=1}^{n-1} s_k \Delta b_k$ is absolutely convergent. Since (s_n) is statistically convergent by Lemma 5 and (b_n) is convergent by (2.6) and (2.2), $(s_n b_n)$ is statistically convergent. Thus, by (2.4) $\sum a_k b_k$ is statistically convergent as the set of all statistically convergent sequences is a linear space (see Schoenberg [2]).

Theorem 3. *If* $\sum a_k$ *is statistically convergent and* $(na_n) \in \ell_\infty$, *then* $\sum a_k$ *is convergent.*

Proof. The proof follows from Lemma 3.

The following result is an extension of Leibniz's test for the convergence of series.

Theorem 4. *Let* $\sum a_k$ *be an arbitrary term series of reals such that*

(2.7) *for* $M = \{k_n : k_1 < k_2 < k_3 \ldots\} \subset N, a_{k_n} = (-1)^n b_n, n \in N,$

(2.8) (b_n) *is non-increasing, non-negative and* $\lim b_n = 0$,

(2.9) $\sum_{k \in M^c} a_k$ *is statistically convergent.*

Then $\sum a_k$ *is statistically convergent.*

Proof. The proof follows immediately from the decomposition

$$\sum_k a_k = \sum_{k \in M} a_k + \sum_{k \in M^c} a_k.$$

From the above theorems we get the next two propositions.

Proposition 2. *Let* $\sum a_k$ *be a series satisfying the conditions* (2.7), (2.8), (2.9), *and* $(na_n) \in \ell_\infty$ *for* $n \in M^c$. *Then* $\sum a_k$ *is convergent.*

Proposition 3. *Let* (k_i) *be an increasing sequence of positive integers such that* $\liminf \frac{k_{i+1}}{k_i} > 1$ *and let* $a_k = 0$ *if* $k \neq k_i$ *for* $i \in N$. *If* stat-lim $s_n = L$, *then* $\lim s_n = L$.

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STATISTILISEST KOONDUVUSEST

Binod Chandra TRIPATHY

Artiklis on Dedekindi ja Abeli tulemused korrutisridade kohta üle kantud statistilise koonduvuse juhule, on esitatud Leibnizi koonduvustunnuse üks üldistus ning tõestatud mõned Tauberi teoreemid.