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OPTIMIZATION OF PLASTIC SPHERICAL SHELLS OF PIECEWISE CONSTANT THICKNESS

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Abstract. An optimization procedure is developed for spherical shells of piecewise constant thickness. The shells under consideration are simply supported at the edge and subjected to the uniformly distributed external pressure. Material of the shells obeys the Tresca yield condition. The problem solved herein consists of the maximization of the load carrying capacity under the condition that the material volume of the shell is fixed.

Key words: optimization, spherical shell, plasticity.

1. INTRODUCTION

Rigid-plastic thin-walled shells have been investigated by several authors [1–10]. Hodge [2] has solved the problems of limit analysis of spherical caps subjected to the uniformly distributed loading, whereas Mróz and Bing-Ye [7] studied the case of loading in the form of loads distributed along the edge of a central hole. Popov [9] solved the same problem in the case of combined loading. In these studies the yield surface corresponding to the Tresca yield condition is presented in the form of two hexagons on the planes of moments and membrane forces, and shells of constant thickness are treated. Sankaranarayanan [10] introduced a generalized square yield condition.

Jones and Ich [3] proposed a new approximation of the yield surface which consists of two diamonds on the planes of bending moments and membrane forces.

In the works mentioned above shells of constant thickness are considered and the load carrying capacity is established under different assumptions. In [5,6] shallow spherical shells of piecewise constant thickness are studied.

In the present paper spherical caps of piecewise constant thickness are considered in the case of the material obeying the Tresca yield condition. The aim of the paper is to maximize the load carrying capacity under the given weight of the shell. Instead of the exact value of the limit load, the lower bound is used in the case of spherical caps with a finite central angle.

2. PROBLEM FORMULATION

Let us consider a spherical cap of the radius A subjected to the uniformly distributed external pressure of intensity P (Fig. 1). The external edge of the shell is simply supported at $\varphi = \beta$.

The thickness of the shell is assumed to be piecewise constant, e.g.

$$h = \begin{cases} h_0, & \varphi \in (0, \alpha), \\ h_1, & \varphi \in (\alpha, \beta), \end{cases} \tag{1}$$

where h_0, h_1, and α are treated as previously unknown constant parameters. However, β is considered to be a given constant. We are looking for the design of the cap for which load carrying capacity attains the maximum value for a fixed weight of the shell.

Calculating the volume of a body which is located between two spherical surfaces with radii $A - \frac{h}{2}$ and $A + \frac{h}{2}$, respectively, one can evaluate the weight (mass) of the cap as

$$V = (1 - \cos\alpha)\left(3A^2 h_0 + \frac{h_0^3}{4}\right) + (\cos\alpha - \cos\beta)\left(3A^2 h_1 + \frac{h_1^3}{4}\right). \tag{2}$$

Here $V = 3M/2\pi\varrho$, where M is the mass of the shell and ϱ is material density.

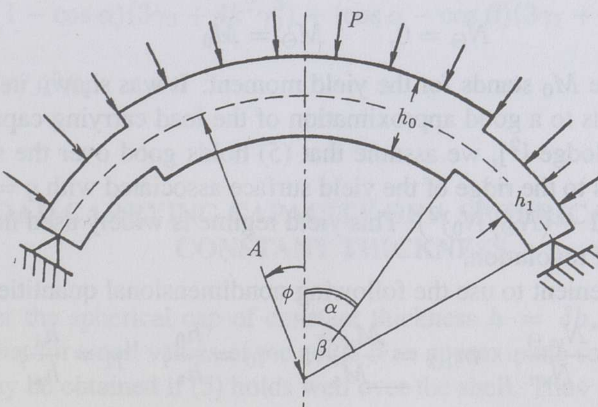

Fig. 1. Shell geometry.

3. GOVERNING EQUATIONS AND BASIC ASSUMPTIONS

For small strains and displacements, the equilibrium equations of a shell element have the form [2]

$$\begin{aligned}
&(N_\varphi \sin\varphi)' - N_\Theta \cos\varphi = S \sin\varphi, \\
&(N_\varphi + N_\Theta + PA)\sin\varphi = -(S\sin\varphi)', \\
&(M_\varphi \sin\varphi)' - M_\Theta \cos\varphi = AS \sin\varphi.
\end{aligned} \tag{3}$$

In (3), N_φ and N_Θ stand for the membrane forces, M_φ and M_Θ are the moments, and S stands for the shear force. Here and henceforth primes denote differentiation with respect to the angle φ.

The strain rates may be presented as [2]

$$\begin{aligned}
&\dot{\varepsilon}_\varphi = \frac{1}{A}(\dot{U}' - \dot{W}), \qquad \dot{\varepsilon}_\Theta = \frac{1}{A}(\dot{U}\cot\varphi - \dot{W}), \\
&\dot{K}_\varphi = -\frac{1}{A^2}(\dot{U} + \dot{W}')', \qquad \dot{K}_\Theta = -\frac{1}{A^2}\cot\varphi(\dot{U} + \dot{W}'),
\end{aligned} \tag{4}$$

where $\dot{U}$ and $\dot{W}$ denote the displacement rates in the meridional and normal directions, respectively.

The material of the shell is assumed to be rigid-plastic obeying the Tresca yield condition. The effects of elastic strains, strain hardening, and geometrical nonlinearity will be neglected in the present paper.

Yield surfaces in the space of generalized stresses N_φ, N_Θ, M_φ, M_Θ were derived for shells of a Tresca material by Hodge [2]. Exact yield surfaces are quite complicated in both cases associated with solid and sandwich shell walls, respectively. In applications different simplifications are introduced and exact solutions of complicated shell problems are quite rare. Hodge [2] suggested "one and two moment limited interaction" surfaces for materials which obey the Tresca condition on the plane of principal stresses. Moreover, in the case of small values of the angle β he assumed that for sandwich shells

$$N_\Theta = 0, \qquad M_\Theta = M_0 \tag{5}$$

over the shell. Here M_0 stands for the yield moment. It was shown in [2] that the assumption (5) leads to a good approximation of the load carrying capacity of the shell. Following Hodge [2], we assume that (5) holds good over the shell. Note that (5) corresponds to the ridge of the yield surface associated with $q = r$ (see [2]) and $M_\Theta/M_0 = \pm(1 - (N_\Theta/N_0)^2)$. This yield regime is widely used in the plastic analysis of shells of revolution.

It appears convenient to use the following nondimensional quantities:

$$\begin{aligned}
&n_{1,2} = \frac{N_{\varphi,\Theta}}{N_*}, \quad m_{1,2} = \frac{M_{\varphi,\Theta}}{M_*}, \quad \gamma_0 = \frac{h_0}{h_*}, \quad \gamma_1 = \frac{h_1}{h_*}, \\
&k = \frac{h_*}{4A}, \quad p = \frac{PA}{N_*}, \quad s = \frac{S}{N_*}, \quad w = \frac{W}{A}, \quad u = \frac{U}{A},
\end{aligned} \tag{6}$$

where $M_* = \sigma_0 h_*^2/4$, $N_* = \sigma_0 h_*$, and σ_0 is the yield stress. Here h_* stands for the thickness of the reference shell of constant thickness.

Making use of the nondimensional variables (6), we may present the equilibrium equations (3) as

$$\begin{aligned}(n_1 \sin\varphi)' - n_2 \cos\varphi &= s \sin\varphi,\\ (n_1 + n_2 + p)\sin\varphi &= -(s \sin\varphi)',\\ k[(m_1 \sin\varphi)' - m_2 \cos\varphi] &= s \sin\varphi.\end{aligned} \tag{7}$$

The strain rates (4) may be put into the form

$$\begin{aligned}\dot{\varepsilon}_\varphi &= \dot{u}' - \dot{w}, \quad \dot{\varepsilon}_\Theta = \dot{u}\cot\varphi - \dot{w},\\ \dot{k}_\varphi &= -k(\dot{u} + \dot{w}')', \quad \dot{k}_\Theta = -k\cot\varphi(\dot{u} + \dot{w}'),\end{aligned} \tag{8}$$

where

$$\dot{k}_\varphi = \frac{M_*}{AN_*}\dot{K}_\varphi, \quad \dot{k}_\Theta = \frac{M_*}{AN_*}\dot{K}_\Theta. \tag{9}$$

Boundary conditions for a simply supported spherical cap are

$$\begin{aligned}m_1(0) &= m_2(0), \quad m_1(\beta) = 0,\\ n_1(0) &= n_2(0).\end{aligned} \tag{10}$$

It is evident that in the case of the stepped shell the material of the cap is used maximally if the moment M_φ attains its maximum value at $\varphi = \alpha$. Thus, if $h_1 < h_0$, one has

$$m_1(\alpha) = \gamma_1^2. \tag{11}$$

Material volume of the shell (2) may be presented as

$$v = (1 - \cos\alpha)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos\alpha - \cos\beta)(3\gamma_1 + 4k^2\gamma_1^3), \tag{12}$$

where $v = V/A^2 h_*$.

4. LOAD CARRYING CAPACITY OF A SPHERICAL CAP OF CONSTANT THICKNESS

Consider the spherical cap of constant thickness $h = \delta h_*$. It was shown by Hodge [2] that for small values of the angle β an approximate solution of the posed problem may be obtained if (5) holds well over the shell. Thus

$$n_2 = 0, \quad m_2 = \delta^2. \tag{13}$$

Integrating the set (7) and taking (13) into account, and satisfying (10), one eventually obtains

$$s = -\frac{p}{2}\varphi,$$

$$n_1 = \frac{p}{2}(\varphi \cot \varphi - 1), \tag{14}$$

$$m_1 = \delta^2 - \frac{p}{2k}(1 - \varphi \cot \varphi).$$

Substituting $m_1(\beta) = 0$ in (14) gives

$$p = \frac{2k\delta^2}{1 - \beta \cot \beta}. \tag{15}$$

The value of the load intensity (15) is a lower bound of the load carrying capacity, since (15) corresponds to the statically admissible stress distribution (14). For the solution (15) to be the exact solution, it is necessary that it meets the kinematic requirements. Making use of (8), (9) and the associated flow law, one can state that the solution is kinematically admissible for small values of the angle β. Thus, for small values of β, (15) presents the exact limit load. In the case of greater values of the angle β the current solution gives the lower bound for the limit load.

5. STEPPED SPHERICAL CAP

Consider now the simply supported spherical shell of piecewise constant thickness (1), whereas nondimensional thicknesses are γ_0 and γ_1. In this case, according to (5) and (6), $n_2 = 0$ and

$$m_2 = \begin{cases} \gamma_0^2, & \varphi \in [0, \alpha], \\ \gamma_1^2, & \varphi \in [\alpha, \beta]. \end{cases} \tag{16}$$

Substituting (16) in (7) and integrating under the boundary conditions (10), one easily finds

$$s = -\frac{p}{2}\varphi,$$

$$n_1 = \frac{p}{2}(\varphi \cot \varphi - 1) \tag{17}$$

for $\varphi \in [0, \beta]$ and

$$m_1 = \gamma_0^2 - \frac{p}{2k}(1 - \varphi \cot \varphi) \tag{18}$$

for $\varphi \in [0, \alpha]$. Similarly, for $\varphi \in [\alpha, \beta]$, one obtains

$$m_1 = \gamma_1^2 - \frac{p}{2k}(1 - \varphi \cot \varphi) + \frac{\sin \alpha}{\sin \varphi}(\gamma_0^2 - \gamma_1^2), \tag{19}$$

where the continuity requirement for m_1 at $\varphi = \alpha$ is taken into account. Satisfying the boundary condition $m_1(\beta) = 0$ in (19) leads to the lower bound of the load carrying capacity of the shell of piecewise constant thickness

$$p = \frac{2k}{1 - \beta \cot \beta} \left[\gamma_1^2 + \frac{\sin \alpha}{\sin \beta} (\gamma_0^2 - \gamma_1^2) \right]. \tag{20}$$

In order to solve the optimization problem, one has to maximize the load carrying capacity under the condition that the material volume of the shell (12) is given. Instead of the exact load carrying capacity, the lower bound (20) is used in the present paper. It is reasonable to assume that the shell material is maximally stressed if the condition (11) is satisfied. Thus, according to (11) and (18),

$$\gamma_0^2 - \gamma_1^2 - \frac{1 - \alpha \cot \alpha}{1 - \beta \cot \beta} \left[\gamma_1^2 + \frac{\sin \alpha}{\sin \beta} (\gamma_0^2 - \gamma_1^2) \right] = 0. \tag{21}$$

Assume that the quantity v in (12) is equal to the nondimensional volume associated with the uniform thickness $\gamma = 1$. This conjecture leads to the relation

$$(1-\cos \alpha)(3\gamma_0+4k^2\gamma_0^3)+(\cos \alpha-\cos \beta)(3\gamma_1+4k^2\gamma_1^3)-(1-\cos \beta)(3+4k^2) = 0. \tag{22}$$

In order to maximize (20) under constraints (21) and (22), let us introduce the augmented functional

$$\begin{aligned} J_* = {} & \frac{2k}{\sin \beta - \beta \cos \beta} [\gamma_1^2 \sin \beta + \sin \alpha (\gamma_0^2 - \gamma_1^2)] \\ & +\lambda_1[(1 - \cos \alpha)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos \alpha - \cos \beta)(3\gamma_1 + 4k^2\gamma_1^3) \\ & -(1 - \cos \beta)(3 + 4k^2)] + \lambda_2\{\gamma_0^2 - \gamma_1^2 \\ & -\frac{1 - \alpha \cot \alpha}{\sin \beta - \beta \cos \beta} [\gamma_1^2 \sin \beta + \sin \alpha (\gamma_0^2 - \gamma_1^2)]\}. \end{aligned} \tag{23}$$

Necessary conditions of the minimum of (23)

$$\frac{\partial J_*}{\partial \alpha} = 0, \quad \frac{\partial J_*}{\partial \gamma_0} = 0, \quad \frac{\partial J_*}{\partial \gamma_1} = 0$$

may be presented as

$$\frac{2k\cos\alpha(\gamma_0^2-\gamma_1^2)}{\sin\beta-\beta\cos\beta}+\lambda_1[\sin\alpha(3\gamma_0+4k^2\gamma_0^3)-\sin\alpha(3\gamma_1+4k^2\gamma_1^3)]$$
$$+\frac{\lambda_2}{\sin\beta-\beta\cos\beta}\left[\left(\cot\alpha-\frac{\alpha}{\sin^2\alpha}\right)(\gamma_1^2\sin\beta-\sin\alpha(\gamma_0^2-\gamma_1^2))\right.$$
$$\left.-(1-\alpha\cot\alpha)\cdot\cos\alpha(\gamma_0^2-\gamma_1^2)\right]=0,$$

$$\frac{4k\gamma_0\sin\alpha}{\sin\beta-\beta\cos\beta}+\lambda_1(1-\cos\alpha)(3+12k^2\gamma_0^2)$$
$$+2\lambda_2\left[\gamma_0-\frac{1-\alpha\cot\alpha}{\sin\beta-\beta\cos\beta}\cdot\gamma_0\sin\alpha\right]=0, \tag{24}$$

$$\frac{4k}{\sin\beta-\beta\cos\beta}(\gamma_1\sin\beta-\gamma_1\sin\alpha)+\lambda_1(\cos\alpha-\cos\beta)(3+12k^2\gamma_1^2)$$
$$-2\lambda_2\left[\gamma_1+\frac{1-\alpha\cot\alpha}{\sin\beta-\beta\cos\beta}(\gamma_1\sin\beta-\gamma_1\sin\alpha)\right]=0.$$

The set of algebraic equations (24) must be solved together with (21), (22) with respect to $\alpha,\ \gamma_0,\ \gamma_1,\ \lambda_1,\ \lambda_2$. This has been done numerically with the aid of the Newton method.

6. DISCUSSION

The results of calculations are presented in Figs. 2 and 3, and in Tables 1 and 2 for several values of the angle β. Table 1 corresponds to the case $k = 0.005$, whereas Table 2 is associated with $k = 0.001$. The quantity e in Tables 1 and 2 can be considered as the economy coefficient defined as

$$e = \frac{p}{p_0}.$$

Here p stands for the lower bound of the load carrying capacity of the stepped shell, whereas p_0 is the limit load of the reference shell of constant thickness. In the latter case $\gamma_0 = \gamma_1 = 1$.

The calculations carried out show that the lower bound of the load carrying capacity of the shell can be increased by more than 22% (in the case $\beta = \pi/2$). For smaller values of β, the economy coefficient attains smaller values. However, the limit load can be increased by more than 15% anyway.

Numerical analysis reveals somewhat unexpectedly that the optimal values of $\alpha,\ \gamma_0,$ and γ_1 only weakly depend on the geometrical parameter k. For instance, if $k = 0.005$ and $\beta = 0.8$, then $\alpha = 0.64107$, $\gamma_0 = 1.1355$, and $\gamma_1 = 0.7431$. However, if $k = 0.001$, we have $\alpha = 0.6411$, $\gamma_0 = 1.1355$, and $\gamma_1 = 0.7432$.

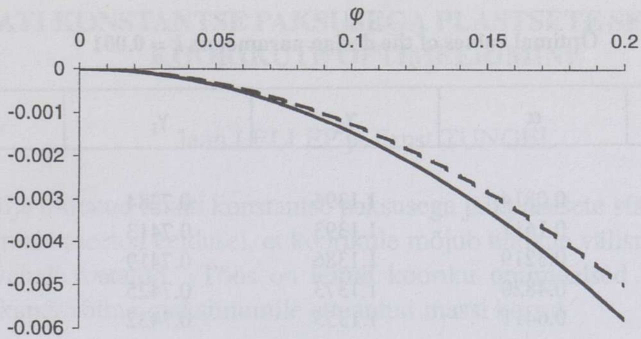

Fig. 2. Membrane force.

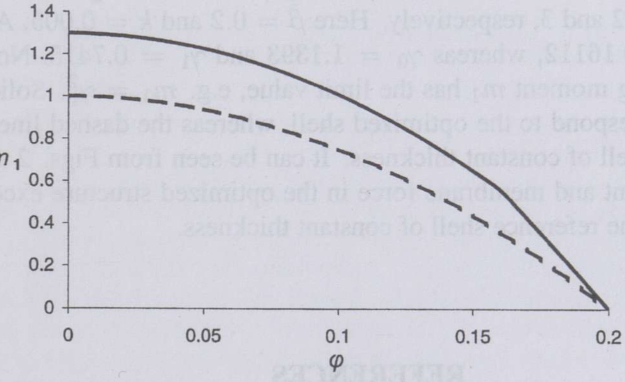

Fig. 3. Bending moment.

Table 1

Optimal values of the design parameters $k = 0.005$

β	α	γ_0	γ_1	e
0.1	0.08056	1.1395	0.7417	1.15345
0.15	0.12086	1.1394	0.7415	1.1537
0.2	0.16112	1.1393	0.7415	1.1540
0.3	0.24156	1.1390	0.7417	1.1550
0.4	0.32188	1.1386	0.7419	1.1564
0.5	0.40201	1.1380	0.7422	1.1583
0.6	0.48195	1.1373	0.7425	1.1606
0.8	0.64107	1.1355	0.7431	1.1668
1.0	0.7991	1.1330	0.7437	1.1754
1.2	0.9559	1.1298	0.7440	1.1871
1.4	1.1116	1.1257	0.7437	1.2028
$\pi/2$	1.2442	1.1215	0.7428	1.2205

Table 2

Optimal values of the design parameters $k = 0.001$

β	α	γ_0	γ_1	e
0.1	0.0814	1.1396	0.7284	1.1533
0.2	0.1611	1.1393	0.7413	1.1541
0.4	0.3219	1.1386	0.7419	1.1565
0.6	0.4820	1.1373	0.7425	1.1606
0.8	0.6411	1.1355	0.7432	1.1660

The distributions of the membrane force n_1 and the bending moment m_1 are presented in Figs. 2 and 3, respectively. Here $\beta = 0.2$ and $k = 0.005$. According to Table 1, $\alpha = 0.16112$, whereas $\gamma_0 = 1.1393$ and $\gamma_1 = 0.7415$. Note that at $\varphi = \alpha$ the bending moment m_1 has the limit value, e.g. $m_1 = \gamma_1^2$. Solid lines in Figs. 2 and 3 correspond to the optimized shell, whereas the dashed lines are due to the reference shell of constant thickness. It can be seen from Figs. 2 and 3 that the bending moment and membrane force in the optimized structure exceed those corresponding to the reference shell of constant thickness.

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TÜKATI KONSTANTSE PAKSUSEGA PLASTSETE SFÄÄRILISTE KOORIKUTE OPTIMEERIMINE

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On välja töötatud tükati konstantse paksusega jäikplastsete sfääriliste koorikute optimeerimise meetod eeldusel, et koorikule mõjub ühtlane välisrõhk ning kooriku serv on vabalt toetatud. Töös on leitud kooriku optimaalsed parameetrid, mis vastavad kandevõime maksimumile etteantud massi korral.