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SUBSYSTEM FORMATION IN THE TWO-LEVEL CONTROL SCHEME

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1. INTRODUCTION

It is well known that the multilevel control methods are implemented in two general steps: decomposition of the original problem into subproblems and coordination of the local solutions until they constitute the solution of the original problem. The coordination process is essentially iterative. The convergence of coordinative iterations is, as a rule, only assumed provided the solution of original (centralized) problem exists [1–3].

Despite the (assumed) fact of convergence the speed may depend on the strength of interconnections between subproblems. If there is some freedom in decomposing the n-dimensional original problem into n_i-dimensional subproblems, then the structure with the best convergence is clearly preferable.

In this paper a measure which enables us to estimate the convergence of the "state and costate coordination" method [2] is proposed. The measure is a scalar, the Euclidean norm of a matrix, characterizing the strength of interconnections between subproblems.

2. THE PROBLEM ASSOCIATED WITH THE SELECTED TWO-LEVEL METHOD

Consider the n-dimensional linear discrete-time object

$$x_{k+1} = Ax_k + Bu_k, \ \ x_0 \text{ – given,}$$

which must be controlled by the sequence of unconstrained u_k, so that the weighted sum

$$J = \frac{1}{2}\sum_{k=1}^{N-1}\left(x_k' Q x_k + u_k' R u_k\right)$$

takes the minimal value. Here matrices B, R, and Q are diagonal, $R > 0$, $Q \geq 0$, k denotes the time step.

The paper aims at decomposion of this centralized problem into two subproblems of the dimensions n_1 and n_2, so that $n_1 + n_2 = n$, and solving the subproblems by the "state and costate coordination" method, using the classical scheme [2].

Applying the discrete maximum principle, we get the system of two-point boundary value problems:

$$x_{i,k+1} = A_i x_{i,k} - G_{i,k} p_{i,k+1} + z_{i,k},$$

$$x_{i,0} \text{ – given,}$$

$$p_{i,k} = A_i' p_{i,k+1} + Q_i x_{i,k} + y_{i,k},$$

$$y_{i,N} = 0,$$

where

$$G_{i,k} = B_i R_i^{-1} B_i',$$

$$z_{i,k} = A_{ij} x_{j,k},$$

$$y_{i,k} = A_{ji}' p_{j,k+1}.$$

Here and henceforth the indices are used in the following sense:
$i, j = 1,\ 2;\ i \neq j;\ k = 0,\ 1,\ 2, \ldots,\ N-1$; N is the given number of time steps.

If the quantities $z_{i,k}$ and $y_{i,k}$ are fixed, then we have to do with two independent subproblems. Solving the given two-point boundary value problem backward in time, we can finally write:

$$u_{i,k} = -R_i^{-1} B_i' F_{i,k}\left[S_{i,k+1}\left(A_i x_{i,k} + z_{i,k}\right) + g_{i,k+1}\right],$$

where

$$F_{i,k} = \left(I + S_{i,k+1}G_{i,k}\right)^{-1}.$$

The vector $g_{i,k}$ is computed backward in time after every updating of $z_{i,k}$ and $y_{i,k}$:

$$g_{i,k} = A_i' F_{i,k}\left(S_{i,k+1}z_{i,k} + g_{i,k+1}\right) + y_{i,k},$$

$$g_{i,N} = 0.$$

Matrices $S_{i,k}$ are control independent and are computed once, also backward in time:

$$S_{i,k} = A_i' F_{i,k} S_{i,k+1} A_i + Q_i,$$

$$S_{i,N} = 0.$$

At the first level of this two-level scheme two independent subproblems with fixed coordination parameters are solved and the corresponding state and costate trajectories are obtained. The task of the second level is to calculate new values of coordination parameters for the use at the first level. The simplest way to correct the fixed quantities is direct updating:

$$z_{i,k}^{l+1} = A_{ij} x_{j,k}^{l},$$

$$y_{i,k}^{l+1} = A_{ji}' p_{j,k+1}^{l}.$$

Here l denotes the number of coordination iterations.

We must now try to give a quantitative measure which estimates the convergence of x_i^l to its optimal value. One possible way is to express the subproblems in the static form and define the (contraction) operator between the closed-loop state trajectory iterations.

3. CONVERGENCE MEASURE

Suppose l iterations are accomplished and from this point controls keep their last values whereas only $z_{i,k}$ is updated. This special case characterizes the influence of the interconnection matrices A_{ij}.

Define the vectors

$$X_i = \begin{bmatrix} x_{i,1} \\ \cdot \\ \cdot \\ \cdot \\ x_{i,N} \end{bmatrix}, \quad U_i = \begin{bmatrix} B_i u_{i,1} \\ \cdot \\ \cdot \\ \cdot \\ B_i u_{i,N} \end{bmatrix}, \quad W_i = \begin{bmatrix} A_i x_{i,0} \\ A_i^2 x_{i,0} \\ \cdot \\ \cdot \\ A_i^N x_{i,0} \end{bmatrix} + \begin{bmatrix} I\, A_{ij} x_{j,0} \\ A_i\, A_{ij} x_{j,0} \\ \cdot \\ \cdot \\ A_i^{N-1}\, A_{ij} x_{j,0} \end{bmatrix}$$

and matrices

$$K_i = \begin{bmatrix} I & 0 & \cdot & \cdot & 0 \\ A_i & I & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_i^{N-1} & A_i^{N-2} & \cdot & A_i & I \end{bmatrix}, \quad L_{ij} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ A_{ij} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & A_{ij} & 0 \end{bmatrix}.$$

The subsystem state equation

$$x_{i,k} = A_i x_{i,k-1} + B_i u_{i,k} + A_{ij} x_{j,k-1}$$

can now be rewritten in the static form:

$$X_i = K_i U_i + P_{ij} X_j + W_i,$$

where $P_{ij} = K_i L_{ij}$.

We consider X_i^l as a function of X_j^{l-1} and get similar expressions for $X_i^l, X_j^{l-1}, X_i^{l-2}, X_j^{l-3}$. Now we can expose the dependence between even state trajectory iterations. The constant terms cancel:

$$X_i^l - X_i^{l-2} = P_{ij} P_{ji} \left(X_i^{l-2} - X_i^{l-4} \right).$$

The matrix $P_{ij} P_{ji}$, composed of the elements of the state matrix A, has the key role in the last expression. The measure of closeness of state trajectories may be characterized by a scalar, the Euclidean norm

$$\rho_1 = \left\| P_{ij} P_{ji} \right\|.$$

In a general case we must expose the dependence between the state trajectories X_i^l and X_j^{l-1} from the original optimization problem given in the static form: minimize

$$J^* = \frac{1}{2}\sum_{i=1}^{2}\left(X_i' \, Q_i^* X_i + U_i' R_i^* U_i\right)$$

with respect to U_i subject to state equations. Here $Q_i^* = I_N \otimes Q_i$, $R_i^* = I_N \otimes R_i$, $\otimes$ indicates direct multiplication of matrices.

Following the classical scheme of static constrained optimization, we get the necessary conditions for optimality of the corresponding Lagrangian:

$$X_i' Q_i^* - \lambda_i' + \lambda_j' P_{ji} = 0,$$

$$U_i' R_i^* + \lambda_i' K_i = 0,$$

$$K_i U_i - X_i + P_{ij} X_j + W_i = 0.$$

Next the expression for the costate vector from the second equation is substituted into the first, costate equation. The result is U_i as a function of U_j and state X_i, provided K_j^{-1} exists:

$$U_i = R_i^{-1*} K_i P_{ji} K_j^{-1} R_j^* U_j - R_i^{-1*} K_i Q_i^* X_i .$$

From the state equation for X_j

$$U_j = K_j^{-1}\left(X_j - P_{ji} X_i - W_j\right).$$

Substitution of U_i and U_j into the state equation for X_i gives the desired result:

$$X_i = \Phi_{ij} X_j + W_i^* ,$$

where

$$\Phi_{ij} = \left[I + K_i^* Q_i^* + K_{ij}^* P_{ij}\right]^{-1}\left(K_{ij}^* + P_{ij}\right)$$

and

$$K_{ij}^* = K_i^* P_{ij} K_j^{-1*},$$

$$K_i^* = K_i R_i^{-1*} K_i,$$

$$W_i^* \text{ – constant terms.}$$

Following the steps of the previous special case for ρ_1, we get:

$$X_i^l - X_i^{l-2} = \Phi_{ij} \Phi_{ji}\left(X_i^{l-2} - X_i^{l-4}\right) \qquad \text{and } \rho_2 = \left\|\Phi_{ij}\Phi_{ji}\right\| .$$

4. EXAMPLE

Let us decompose a 4×4 problem into two 2×2 subproblems and compare the three possible structures.

For $Q = I$, $R = I$, $N = 8$, and

$$A = 0.1 \begin{pmatrix} 4 & 1 & 2 & 2 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

the result is

$$\rho_1 = 0.71, \quad \rho_2 = 0.41 \text{ for } (1,2-3,4),$$
$$\rho_1 = 0.74, \quad \rho_2 = 0.43 \text{ for } (1,3-2,4),$$
$$\rho_1 = 0.58, \quad \rho_2 = 0.23 \text{ for } (1,4-2,3).$$

The subsystem with the first and fourth elements of the state vector and the subsystem with the second and third elements give a clearly preferable decomposition.

5. CONCLUDING REMARKS

A measure, the Euclidean norm of a matrix, characterizing the strength of interconnections between the decomposed parts of a controlled object and between subproblems of the linear-quadrative optimal control, is derived. Both open-loop and closed-loop measures indicate the same decomposition. This fact points out the dominant role of interconnection matrices and the possibility for the use of ρ_1 as a preliminary measure of partitioning. Also, the ratio ρ_1/ρ_2 could serve as a kind of measure for the influence of optimal controls.

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