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GREEN'S FUNCTIONS FOR A SCALAR FIELD IN A CLASS OF ROBERTSON–WALKER SPACE-TIMES

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Abstract. The retarded and advanced Green's functions for a massless nonconformally-coupled scalar field in a class of Robertson–Walker space-times are calculated analytically. The results are applied to the calculation of the Hadamard fundamental solutions in some special cases.

Key words: Green's functions, scalar field, Robertson–Walker space-times, Hadamard elementary solution, Huygens principle.

1. INTRODUCTION

There are many problems in both classical and quantum physics that are connected with Green's functions for linear second-order hyperbolic differential equations. When considering an external relativistic gravitational field, one is led to equations on a curved space-time.

As is known [1–3], the Huygens principle does not hold, as a rule, in the case of a curved space-time. This means that the Green's function has support inside the light cone due to the scatter of waves off the background curvatures. As a result, a single pulse of a source causes corresponding field pulses to appear in distant space regions followed by subsiding "tails". This is a classical analogue of particle creation in a varying gravitational field [4].

The Green's function approach is closely connected with the Hadamard fundamental solution which is an important tool for the regularization of the stress-energy tensor of a quantum field propagating in a curved space-time [4,5]. Exact fundamental solutions (Green's functions), however, have been found for some rare cases only, such as the Bianchi-type I metric and the de Sitter universe

[6,7]. For most cases only approximate forms of fundamental solutions have been calculated, mainly by the application of the Hadamard method.

Following the suggestions by Bunch and Davies [8], a scalar field propagating in a spatially flat Robertson–Walker (RW) universe is considered in this paper, with a power-law behaviour of the conformal scale factor for the metric. Such a metric has been discussed by Ford and Parker [9] in connection with infrared divergences, and by Bunch and Davies in connection with regularization of the stress-energy tensor of a scalar quantum field propagating in the RW background space-times.

In Section 2, the formalism for calculating the Green's function is presented. A similar technique was developed by Friedlander [1]. In Section 3, this technique is applied to find the Green's functions for a massless, nonconformally-coupled scalar field in RW background space-times. In Section 4, the properties of the Green's functions are discussed. Some special cases of the Hadamard fundamental solutions are evaluated explicitly.

2. FORMALISM FOR THE CALCULATION OF THE FUNDAMENTAL SOLUTIONS

This paper deals with the fundamental solutions G for a massless scalar field which satisfies the covariant wave equation

$$\hat{L}G(x,y) := \left(\Box + \left(\frac{1}{6}+\xi\right)R(x)\right)G(x,y) = \delta(x,y), \quad (1)$$

where ξ is a constant. The d'Alembertian has the form

$$\Box = g^{ik}\nabla_i \nabla_k, \quad (2)$$

where g^{ik} is a metric tensor of the pseudo-Riemannian space M with the signature $(+\,-\,-\,-)$ and ∇_i denotes a covariant derivative. The scalar curvature R has the same sign as in [1] and $\delta(x,y)$ is the Dirac delta distribution in M. Differentiations of two-point functions always refer to the first argument and small Latin indices run from 0 to 3.

A general theory of covariant Green's functions is presented in [1,10]. According to these, the retarded and advanced Green's functions $G^{\pm}$ in a causal domain $\Omega \subseteq M$ (see [1]) are given as

$$G^{\pm}(x,y) = \frac{1}{2\pi}\left(W(x,y)\delta^{\pm}(\sigma(x,y)) + V(x,y)\theta^{\pm}(\sigma(x,y))\right). \quad (3)$$

The world function $\sigma(x, y)$ equals the square of the geodesic distance between the points x and y. It is negative for space-like intervals and positive for time-like ones, satisfying

$$\nabla^i \sigma \, \nabla_i \sigma = 4\sigma . \tag{4}$$

The transport scalar $W(x,y)$ coincides with the scalarized Van Vleck determinant and satisfies the transport equation with an additional condition:

$$(2(\nabla^i \sigma)\nabla_i + (\Box\sigma - 8))W(x,y) = 0, \quad \forall x, y \notin \Omega, \quad W(y,y) = 1. \tag{5}$$

Delta distributions $\delta^{\pm}(\sigma(x,y))$ have supports on the future light cone $C^{+}(y)$ and on the past one $C^{-}(y)$, respectively. The Heaviside functions $\theta^{\pm}(\sigma)$ have supports in the closures $J^{\pm}(y) = C^{\pm}(y) \cup D^{\pm}(y)$, where $D^{\pm}(y)$ denote the interiors of the cones $C^{\pm}(y)$, respectively. In the region $D^{\pm}(y)$ the function $V(x, y)$ must satisfy the homogeneous differential equation

$$\hat{L}V(x,y) = 0 \tag{6}$$

with the characteristic initial condition

$$\hat{P}V(x,y) := (2(\nabla^i \sigma)\nabla_i + (\Box\sigma - 4))V(x,y) = -\hat{L}W(x,y), \quad \forall x \in C^{\pm}(y). \tag{7}$$

The Hadamard fundamental solution $H(x, y)$ is a biscalar of the form

$$H(x,y) = \frac{1}{2\pi}\left(\frac{1}{\sigma}W(x,y) + V(x,y)\ln|\sigma| + w(x,y)\right), \tag{8}$$

where w is a biscalar free of singularities, satisfying the equation

$$\hat{L}w = -\frac{1}{\sigma}[\hat{P}V(x,y) + \hat{L}W(x,y)] \tag{9}$$

and the boundary condition

$$w(x,y) = 0, \quad \forall x \in C^{\pm}(y). \tag{10}$$

Next we consider a conformally flat background space-time M with the metric

$$dS^2 = \Omega^2(x) \cdot d\tilde{S}^2 = \Omega^2(x)\eta_{ik} dx^i dx^k , \tag{11}$$

where $\eta_{ik} = \mathrm{diag}(1, -1, -1, -1)$. If $\tilde{G}(x,y)$ is a fundamental solution of

$$\hat{\tilde{L}}\tilde{G}(x,y) := (\tilde{\nabla}^i\tilde{\nabla}_i + \Omega^{-2}(x)\cdot\xi\, R(x))\tilde{G}(x,y) = \tilde{\delta}(x,y), \tag{12}$$

then

$$G(x,y) = \Omega^{-1}(x)\Omega^{-1}(y)\tilde{G}(x,y) \tag{13}$$

is a fundamental solution of (1). Hereafter "~" labels the quantities referring to the conformal space-time $\tilde{M}$. It should be pointed out that the Hadamard fundamental solutions H and $\tilde{H}$ are also related to each other by (13).

It follows from (3), (5), (6), and (7) that

$$G^{\pm}(x,y) = \frac{1}{2\pi\,\Omega(x)\Omega(y)}\{\tilde{\delta}^{\pm}(\tilde{\sigma}(x,y)) + \tilde{V}(x,y)\tilde{\theta}^{\pm}(\tilde{\sigma}(x,y))\}, \tag{14}$$

where $\tilde{V}(x,y)$ is the solution of the corresponding Cauchy problem

$$\hat{\tilde{L}}\tilde{V}(x,y) = 0, \qquad x \in D^{\pm}(y), \tag{15}$$

$$\hat{\tilde{P}}\tilde{V} := [2(\tilde{\nabla}^i\tilde{\sigma})\tilde{\nabla}_i + 4]\tilde{V}(x,y) = -\xi\, R(x)\Omega^{-2}(x), \quad x \in C^{\pm}(y). \tag{16}$$

The world function $\tilde{\sigma}(x,y)$ takes the form

$$\tilde{\sigma}(x,y) = \eta_{ik}(x^i - y^i)(x^k - y^k) = (x^0 - y^0)^2 - r^2. \tag{17}$$

3. FUNDAMENTAL SOLUTIONS IN THE ROBERTSON–WALKER SPACE-TIMES

Remarkably, there is an example permitting an exact solution of (1):

$$\Omega^2(x) = \beta^2(x^0)^{2\gamma}, \tag{18}$$

where β and γ are constants. This power-law behaviour presents a solution of the Einstein's equation with a vanishing cosmological constant containing matter with the state equation

$$p = \alpha\rho, \quad \alpha = (2-\gamma)/3\gamma, \quad \gamma \neq 0, \tag{19}$$

where ρ is energy density and p is pressure. Thus, $\gamma = -1$ corresponds to the de Sitter space-time, $\gamma = 1$ to a radiation-, and $\gamma = 2$ to a dust-dominated expansion of the universe. The Minkowski space-time corresponds to $\gamma = 0$, which is excluded for (19).

The power-law behaviour of Ω leads to

$$R=\frac{6}{\beta^2}\gamma(\gamma-1)(x^0)^{-2(\gamma+1)} \tag{20}$$

and

$$\hat{\tilde{L}}=\tilde{\nabla}^i\tilde{\nabla}_i+\frac{K}{(x^0)^2} \tag{21}$$

with

$$K=6\xi\gamma(\gamma-1). \tag{22}$$

The Cauchy problem (15), (16) can be solved by applying the following ansatz:

$$\tilde{V}(x,y)=\frac{1}{x^0y^0}f(z), \qquad z:=\frac{\tilde{\sigma}}{x^0y^0}. \tag{23}$$

It follows from (15), (16), and (21) that

$$z(z+4)\frac{d^2}{dz^2}f+4(z+2)\frac{d}{dz}f+(k+2)f=0 \tag{24}$$

together with the initial condition $f(0)=-K/4$.

Equation (24) has an appropriate solution [11]

$$f(z)=(z(z+4))^{-1/2}P_\nu^1(1+z/2), \tag{25}$$

where the constant ν is a root of the equation

$$\nu(\nu+1)=-K\equiv 6\xi(1-\gamma)\gamma \tag{26}$$

and P_ν^1 is the Legendre function, being the same for both roots of (26). Hence, the two-point function $\tilde{V}(x,y)$ can be expressed in terms of a Legendre function, or, alternatively, a hypergeometric function F:

$$\begin{aligned}\tilde{V}(x,y)&=\frac{1}{\sqrt{\tilde{\sigma}(\tilde{\sigma}+4x^0y^0)}}P_\nu^1\left(1+\frac{\tilde{\sigma}}{2x^0y^0}\right)\\&=\frac{\nu(\nu+1)(\tilde{\sigma}+4x^0y^0)^{\nu-1}}{(4x^0y^0)^\nu}F\left(-\nu,(1-\nu);\,2;\,\frac{\tilde{\sigma}}{\tilde{\sigma}+4x^0y^0}\right).\end{aligned} \tag{27}$$

It can be found by means of (14), (18), and (27) that

$$G^{\pm}(x,t) =$$

$$\frac{1}{2\pi\beta^2 (x^0 y^0)^\gamma} \left\{ \tilde{\delta}^{\pm}(\tilde{\sigma}) + [\tilde{\sigma}(\tilde{\sigma} + 4x^0 y^0)]^{-1/2} P_\nu^1 \left(1 + \frac{\tilde{\sigma}}{2x^0 y^0} \right) \tilde{\theta}^{\pm}(\tilde{\sigma}) \right\}. \quad (28)$$

Curiously, the solution (28) is very similar to that of a massive conformally-coupled scalar field propagating in the de Sitter universe ($\gamma = -1$). The retarded and advanced Green's functions $G_m^{\pm}$ of a massive conformally-coupled scalar field satisfy the equation

$$\left(\Box + \frac{1}{6} R(x) + m^2 \right) G_m^{\pm}(x,y) = \delta(x,y), \quad (29)$$

where m^2 is a constant. It can be easily seen that for the de Sitter space-time $R(x) = 12/\beta_s^{\,2} =$ const. Hence, $\mathbf{G} + (1/6) \cdot R + m^2 = \mathbf{G} + (1/6) + \xi^*)R$ with

$$\xi^* = \frac{m^2 \beta_s^2}{12}. \quad (30)$$

Now $G_m^{\pm}$ can be obtained by substituting $\gamma = -1$ and (30) to Eqs. (26) and (28). Note, however, that in this case ν has quite a different meaning:

$$\nu(\nu+1) = -m^2 \beta_s^2. \quad (31)$$

We note that the regular part $w = (1/\beta^2 (x^0 y^0)^\gamma)\tilde{w}$ of the Hadamard fundamental solution (8) can be found using (28) and the integral representation of the solution of the nonhomogeneous characteristic initial value problem, given in [1] (see Theorem 5.4.3 therein).

4. PROPERTIES OF THE GREEN'S FUNCTIONS

Some properties of $V(x, y)$ should be pointed out as consequences of (26), (27), and (28).

1. The Huygens principle is valid ($V(x, y) = 0$) iff $\nu = 0$. Following (26), one can distinguish two special cases: If $\xi = 0$, then $\nu = 0$ for each γ; if $\gamma = 0$, 1, then $\nu = 0$ for each ξ. Note that a space-time with $\gamma = 1$ is a radiation-dominated Friedmann universe. Actually, this is also an immediate consequence of (15) and (16), as $\hat{L}$ is conformally equivalent to the d'Alembertian of a flat space-time.

2. In the case of $\nu = n$, $n = 1, 2, \ldots$, the function $V(x, y)$ has a rather simple form, being a polynomial of the order $n - 1$ with respect to $\tilde{\sigma}$. Such ν occurs if

$\xi \notin (0, (2/3)n(n+1))$ and $\gamma = 1/2\left(1 \pm \sqrt{1 - 2n(n+1)/3\xi}\right)$. At minimal coupling ($\xi = -1/6$) one gets $\gamma = -n, (n+1)$.

For example, if $\nu = 1$, then

$$V(x,y) = \frac{1}{2\beta^2}(x^0 y^0)^{-(\gamma+1)} \quad \text{and} \quad w = 0. \tag{32}$$

This is possible if $\xi \notin (0, 4/3)$ and $\gamma = 1/2\left(1 \pm \sqrt{1 - 4/(3\xi)}\right)$. At the minimal coupling the case considered corresponds either to the Friedmann dust-dominated universe or to the de Sitter one. If $\nu = 2$, then

$$V(x,y) = \frac{3}{2\beta^2}(x^0 y^0)^{-(\gamma+1)}\left(1 + \frac{\tilde{\sigma}}{x^0 y^0}\right). \tag{33}$$

The regular part of the Hadamard fundamental solution takes the form

$$w = \frac{3}{4\beta^2 (x^0 y^0)^{\gamma+1}}\left\{\tilde{\sigma} - \frac{1}{r}\left[(r - y^0)(x^{0\,2} - (r - y^0)^2)\ln\left|\frac{x^0 + y^0 - r}{2y^0}\right|\right.\right.$$
$$\left.\left. + (r + y^0)(x^{0\,2} - (r + y^0)^2)\ln\left|\frac{x^0 + y^0 + r}{2y^0}\right|\right]\right\}. \tag{34}$$

3. In the de Sitter universe, which is conformal to the RW one, the equation of the massless scalar field (1) corresponds to that of a massive field (29). Mass there is real ($m^2 \geq 0$) only if $\xi < 0$ and $\gamma \in (0, 1)$, or if $\xi > 0$ and $\gamma \notin (0, 1)$. In all other cases the mass is imaginary ($m^2 < 0$).

4. In the case of $\gamma \neq 0, -1, 1$ the following co-ordinate transformation can be performed:

$$x^0 = \left(\frac{|\nu|}{m\beta}\right)^{\frac{1}{1+\gamma}} \exp\left(-\frac{m}{|\nu|}\bar{x}^0\right),$$
$$x^\alpha = \frac{1}{\beta}\left(\frac{m\beta}{|\nu|}\right)^{\frac{\gamma}{1+\gamma}} \bar{x}^\alpha, \quad \alpha = 1, 2, 3, \tag{35}$$

where $m > 0$ is a constant. It turns out that in the limit $|\xi| \to \infty$ the wave equation $\hat{L}\Phi = 0$ transforms to the Klein–Gordon equation of the Minkowski space

$$\left(\left(\frac{\partial}{\partial \bar{x}^0}\right)^2 - \left(\frac{\partial}{\partial \bar{x}^1}\right)^2 - \left(\frac{\partial}{\partial \bar{x}^2}\right)^2 - \left(\frac{\partial}{\partial \bar{x}^3}\right)^2 + \varepsilon m^2\right)\bar{\Phi} = 0, \qquad (36)$$

where $\varepsilon = \pm 1$. It can be verified that in case of $\xi \to -\infty$ and $\gamma \in (0, 1)$, or, alternatively, if $\xi \to \infty$ and $\gamma \notin (0, 1)$, then the following identities hold:

$$\lim_{|\xi| \to \infty} V = -\frac{m}{2\sqrt{\bar{\sigma}}} J_1\left(m\sqrt{\bar{\sigma}}\right),$$
$$\bar{\sigma} = (\bar{x}^0 - \bar{y}^0)^2 - \sum_{\alpha=1}^{3} (\bar{x}^\alpha - \bar{y}^\alpha)^2, \qquad (37)$$

where $J_1(x)$ denotes the first-order Bessel function. This result corresponds to a real mass ($\varepsilon = 1$) in the Klein–Gordon equation (36).

If either $\xi \to -\infty$ and $\gamma \notin (0, 1)$ or $\xi \to \infty$ and $\gamma \in (0, 1)$, then

$$\lim_{|\xi| \to \infty} V = \frac{m}{2\sqrt{\bar{\sigma}}} I_1\left(m\sqrt{\bar{\sigma}}\right), \qquad (38)$$

where $I_1(x)$ is the first-order modified Bessel function and (38) corresponds to a field with imaginary mass ($\varepsilon = -1$).

It should be pointed out that an analogous limit procedure can also be carried out in the de Sitter space-time ($\gamma = -1$).

We have seen that, by the application of the above-mentioned co-ordinate transformation and the unrestricted growth of the coupling constant ξ, an initially massless scalar field in the RW universe can obtain mass in the corresponding asymptotic Minkowski space. This is a conclusion that might be an object of further interpretations.

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SKALAARVÄLJA GREENI FUNKTSIOONID TEATUD KLASSI ROBERTSONI–WALKERI AEGRUUMIDE KORRAL

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Spetsiaalse klassi Robertsoni–Walkeri aegruumide foonil on leitud mittekonformse seosega massita skalaarvälja retardeeritud ja avansseeritud Greeni funktsioonid. Erijuhtudel on tulemusi rakendatud Hadamardi elementaarlahendite saamiseks.