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ON SMOOTHING PROBLEMS WITH WEIGHTS AND OBSTACLES

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Abstract. A system connecting weights and deviations of the solution from given values in multivariate smoothing problems is established. The system contains a symmetric regular matrix as free parameter. Certain properties of solvability are studied.

Key words: smoothing problem, multivariate natural spline.

1. INTRODUCTION

The minimization of a functional containing an integral part and weighted deviations of an unknown function from given data is a problem which possesses a unique solution in the form of a natural spline. Actual finding of this spline means the solution of a linear system. The minimization problem of the integral part of an unknown function satisfying certain obstacle conditions has also a natural spline as the solution but, to our mind, there is not yet satisfactory algorithm for finding this spline. For example, the convergence of coordinatewise descent and penalty methods is not known. An interesting method of adding–removing knots in the interpolation problem is described in [1] (pp. 68–69), but the presented proof of its finiteness is based on a false lemma (Lemma 10.3 in [1]). We see that for a problem with obstacles there is an equivalent problem with weights and below we describe the system connecting the weights and deviations in the corresponding problems.

In the one-variable case in [2] an attempt is made to solve the smoothing problem with obstacles by using another problem with weights. A system containing weights and deviations is established, but its efficiency is shown when there are obstacles only in one knot.

2. NOTATION AND PRELIMINARIES

For given integers r and n, with $2r > n \geq 1$, we denote by G the fundamental solution of the operator $\triangle^r$, where $\triangle$ is the n-dimensional Laplace operator. It is known that for n odd $G(X) = c_{nr}||X||^{2r-n}$ and for n even $G(X) = c_{nr}||X||^{2r-n}\ln||X||$ with some constants $c_{nr} > 0$ and $||X|| = (x_1^2 + \ldots + x_n^2)^{1/2}$. Let us denote by $L_2^{(r)}(I\!R^n)$ the space of functions defined on $I\!R^n$ having all partial (distributional) derivatives of order r in $L_2(I\!R^n)$. Put

$$Tf = \left\{ \sqrt{\frac{r!}{\alpha!}} D^\alpha f \;\middle|\; |\alpha| = r \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$, $\alpha! = \alpha_1! \ldots \alpha_n!$, and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Thus $T: L_2^{(r)}(I\!R^n) \to L_2(I\!R^n) \times \ldots \times L_2(I\!R^n)$ with the number of components in the product as far as there are different derivatives of order r. We also need the product

$$(Tf, Tg) = \sum_{|\alpha|=r} \frac{r!}{\alpha!} \int_{I\!R^n} D^\alpha f D^\alpha g \, dX, \quad f, g \in L_2^{(r)}(I\!R^n),$$

and the corresponding seminorm $||Tf|| = (Tf, Tf)^{1/2}$. Let $\mathcal{P}_{r-1}$ be the space of all polynomials of order $\leq r-1$ on $I\!R^n$ and $p = \dim \mathcal{P}_{r-1}$. We see that $\operatorname{Ker} T = \mathcal{P}_{r-1}$.

Given a finite number of points $X_i \in I\!R^n$, $i \in I$, a function

$$S(X) = P(X) + \sum_{i \in I} d_i G(X - X_i), \quad X \in I\!R^n, \tag{1}$$

with $P \in \mathcal{P}_{r-1}$ and

$$\sum_{i \in I} d_i Q(X_i) = 0 \;\; \forall Q \in \mathcal{P}_{r-1}, \tag{2}$$

is called the natural spline. It is known that any natural spline belongs to $L_2^{(r)}(I\!R^n)$. Furthermore, for all $f \in L_2^{(r)}(I\!R^n)$ and any natural spline S it holds

$$(TS, Tf) = (-1)^r \sum_{i \in I} d_i f(X_i). \tag{3}$$

In more detail about natural splines we refer the reader to [1], in the one-dimensional case to [3–5].

We suppose here and in the next section that the zero-valued interpolation problem with polynomials from $\mathcal{P}_{r-1}$ in the knots X_i, $i \in I$, possesses a unique solution, i.e., $P \in \mathcal{P}_{r-1}$, $P(X_i) = 0$, $i \in I$, implies that $P = 0$. Then for given data z_i, $i \in I$, there is a unique natural spline S satisfying $S(X_i) = z_i$, $i \in I$.

We use sometimes the notation $|I|$ for the number of elements in an index set I.

3. SMOOTHING PROBLEMS WITH WEIGHTS

Let us consider the minimization problem

$$\inf_{f\in K_0}\left(||Tf||^2+\sum_{i\in I_1}\frac{|f(X_i)-z_i|^2}{w_i}\right),$$

where $K_0=\{f\in L_2^{(r)}(I\!R^n)\,\big|\,f(X_i)=z_i,\ i\in I_0\}$, $I_0\cup I_1=I$, and $I_0\cap I_1=\emptyset$, with given data $z_i,\ i\in I$, and weights $w_i>0,\ i\in I_1$. We call it the smoothing problem with weights.

Proposition 1. *There exists only one natural spline S satisfying*

$$\begin{aligned}(-1)^r d_i(S)w_i+S(X_i) &= z_i,\ i\in I_1,\\ S(X_i) &= z_i,\ i\in I_0,\end{aligned}\tag{4}$$

and this spline is the unique solution of the smoothing problem with weights.

Proof. If $|I|=m$, then there are $m+p$ coefficients to determine in the representation (1) of a natural spline. The conditions (4) with (2) give also $m+p$ linear equations. We show that the corresponding homogeneous system has only trivial solution. Suppose S_0 is a natural spline such that

$$\begin{aligned}(-1)^r d_i(S_0)w_i+S_0(X_i) &= 0,\ i\in I_1,\\ S_0(X_i) &= 0,\ i\in I_0.\end{aligned}$$

Then, from (3) with $S=f=S_0$, we obtain

$$\begin{aligned}0\le(TS_0,TS_0) &= (-1)^r\sum_{i\in I_1}d_i(S_0)S_0(X_i)\\ &= -(-1)^{2r}\sum_{i\in I_1}w_id_i^2(S_0),\end{aligned}$$

which gives $d_i(S_0)=0,\ i\in I_1$. We see that $S_0(X_i)=0,\ i\in I$, therefore $S_0=0$.

Let S be the natural spline satisfying (4). Each element in K_0 may be represented in the form $S+h$, where $h\in L_2^{(r)}(I\!R^n)$ and $h(X_i)=0,\ i\in I_0$. Denote by F the functional to minimize in the smoothing problem. Then, using also (3), we have

$$\begin{aligned}F(S+h) &= ||T(S+h)||^2+\sum_{i\in I_1}\frac{|S(X_i)+h(X_i)-z_i|^2}{w_i}\\ &= ||TS||^2+2(TS,Th)+||Th||^2\\ &\quad+\sum_{i\in I_1}\frac{|S(X_i)-z_i|^2}{w_i}+2\sum_{i\in I_1}\frac{(S(X_i)-z_i)h(X_i)}{w_i}+\sum_{i\in I_1}\frac{|h(X_i)|^2}{w_i}\end{aligned}$$

$$= F(S) + 2\sum_{i\in I_1}\left((-1)^r d_i(S) + \frac{S(X_i)-z_i}{w_i}\right)h(X_i)$$

$$+||Th||^2 + \sum_{i\in I_1}\frac{|h(X_i)|^2}{w_i}.$$

Hence, according to (4), we see that $F(S+h) \geq F(S)$. Moreover, $F(S+h) = F(S)$ yields $||Th|| = 0$, $h(X_i) = 0$, $i \in I_1$, and this with $h(X_i) = 0$, $i \in I_0$, gives $h = 0$, which completes the proof.

4. SMOOTHING PROBLEMS WITH OBSTACLES

Given $X_i,\ z_i,\ i \in I$, and $\varepsilon_i > 0,\ i \in I_1 \subset I$, let us pose the problem

$$\inf_{f\in K_\varepsilon} ||Tf||^2, \tag{5}$$

with $K_\varepsilon = \{f \in L_2^{(r)}(I\!R^2)\ \big|\ f(X_i) = z_i,\ i \in I_0,\ |f(X_i) - z_i| \leq \varepsilon_i,\ i \in I_1\}$, $I_0 = I\backslash I_1$. We consider also the problem

$$\inf_{f\in K_{\alpha\beta}} ||Tf||^2, \tag{6}$$

where $K_{\alpha\beta} = \{f \in L_2^{(r)}(I\!R^n)\ \big|\ f(X_i) = z_i, i \in I_0,\ \alpha_i \leq f(X_i) \leq \beta_i,\ i \in I_1\}$ and $\alpha_i < \beta_i$, $i \in I_1$. Here $\alpha_i = -\infty$ or $\beta_i = \infty$ for some i is not excluded, in this case the corresponding condition in $K_{\alpha\beta}$ must be read as $f(X_i) \leq \beta_i$ or $f(X_i) \geq \alpha_i$. We call (5) and (6) the smoothing problems with obstacles.

It is known ([1], pp. 64–67) that the problem (6) has a solution which is a natural spline of the form (1). We assume here and in the sequel the uniqueness of the solution. For this it is sufficient that the zero-valued interpolation problem in $\mathcal{P}_{r-1}$ with knots in I_0 has only trivial solution but, in general, such a condition is too restrictive.

Clearly, the problem (5) is a special case of (6), we have only to take $\alpha_i = z_i - \varepsilon_i$ and $\beta_i = z_i + \varepsilon_i$, $i \in I_1$. Conversely, if S is the solution of (6), set, for instance, $z_i = S(X_i)$, $i \in I_0$, $z_i = (\alpha_i + \beta_i)/2$, $\varepsilon_i = (\beta_i - \alpha_i)/2$, $i \in I_1$ (if, for instance, $\beta_i = \infty$, let $\varepsilon_i > 0$ be such that $S(X_i) \leq \alpha_i + 2\varepsilon_i$, then put $z_i = \alpha_i + \varepsilon_i$), and we see that there is a problem (5) having the same (unique) solution as (6). Thus, both stated smoothing problems with obstacles are equivalent to each other.

Given a problem (5), it is natural to ask whether there is a smoothing problem with weights having the same solution. Of course, we do not allow the change of z_i, I_0, and I_1 as given data. Our purpose is to determine suitable weights w_i, $i \in I_1$, using ε_i, $i \in I_1$. We obtain an answer to the question from an internal characterization of the solution of the problem (6) ([1], p. 66): a natural spline S of the form (1) such that $S \in K_{\alpha\beta}$ is the solution of the problem (6) if and only if in the knots of I_1 the spline S satisfies the conditions

$$\begin{aligned} d_i &= 0, && \text{if} \quad \alpha_i < S(X_i) < \beta_i, \\ (-1)^r d_i &\geq 0, && \text{if} \quad S(X_i) = \alpha_i, \\ (-1)^r d_i &\leq 0, && \text{if} \quad S(X_i) = \beta_i. \end{aligned} \tag{7}$$

It may happen that, for the solution S of (5), there is an index $i \in I_1$ such that $S(X_i) \neq z_i$ and $|S(X_i) - z_i| < \varepsilon_i$, therefore $d_i = 0$. We see that in this case the equalities (4) cannot be satisfied for any weights.

5. CONNECTION BETWEEN DEVIATIONS AND WEIGHTS IN SMOOTHING PROBLEMS

We saw in the preceding section that the smoothing problems with weights and obstacles having the same given data z_i, I_0, and I_1 are not, in general, equivalent.

Let us consider the problem (5) possessing a unique solution S such that its coefficients in the representation (1) satisfy $d_i \neq 0$, $i \in I_1$. Likewise we assume that in the initial problem the knots in I_1, corresponding to the coefficients $d_i = 0$, $i \in I_1$, are already removed. It is clear that after this removal of knots the new and initial problems have the same solution. In this case we know that $S(X_i) \neq z_i$, $i \in I_1$. Thus, taking

$$w_i = \frac{z_i - S(X_i)}{(-1)^r d_i}, \quad i \in I_1,$$

we see that (4) holds and S is the solution of an equivalent smoothing problem with weights. However, the question of how to actually find the weights remains open because S and d_i are unknown.

The uniqueness of the solution of (5) or (6) always implies that the zero-valued interpolation problem in $\mathcal{P}_{r-1}$ with knots in I has only trivial solution. Indeed, supposing the uniqueness of the solution (denote it by S), if there is a polynomial $P \neq 0$, $P(X_i) = 0$, $i \in I$, we get that $S + P \in K_\varepsilon$ or $S + P \in K_{\alpha\beta}$ and $||T(S + P)|| = ||TS||$, which contradicts to the uniqueness of the solution of (5) or (6).

On the other hand, if the zero-valued interpolation problem in $\mathcal{P}_{r-1}$ with knots in I has only trivial solution and the problem (5) or (6) has a solution S with $d_i(S) \neq 0$, $i \in I_1$, then there is no other solution of (5) or (6). Let us prove it. Suppose $f \in K_\varepsilon$ or $f \in K_{\alpha\beta}$ is a solution of (5) or (6). Then

$$\begin{aligned}\|Tf\|^2 - \|TS\|^2 &\\ &= \|Tf - TS\|^2 + 2(TS, Tf - TS)\\ &= \|T(f-S)\|^2 + 2\sum_{i\in I_1}(-1)^r d_i(S)(f(X_i) - S(X_i)).\end{aligned}$$

The straightforward control, using (7), shows that the sum term here is ≥ 0 which really gives $f - S = P \in \mathcal{P}_{r-1}$, $P(X_i) = 0$, $i \in I_1$. Taking also $P(X_i) = 0$, $i \in I_0$ (resulting from $f, S \in K_\varepsilon$ or $f, S \in K_{\alpha,\beta}$) into account, we conclude that $P = 0$. Thus, the solution of (5) or (6) is unique.

Choose a basis in $\mathcal{P}_{r-1}$, say, X^{β_j}, $j \in J$, $|J| = p$. Then a natural spline S may be presented as

$$S(X) = \sum_{j\in J} c_j X^{\beta_j} + \sum_{j\in I} d_j G(X - X_j).$$

Denoting the vectors $c = (c_j)_{j\in J}$, $d = (d_j)_{j\in I}$, $g = (S(X_i))_{i\in I}$ and matrices $G = (G(X_i - X_j))_{i,j\in I}$, $V = (X_i^{\beta_j})_{i\in I, j\in J}$, we get

$$g = Vc + Gd \tag{8}$$

and (2) has an equivalent form $d \in \mathrm{Ker} V^\top$. This means also that $\mathrm{Ker} V^\top$ do not depend on the choice of the basis in $\mathcal{P}_{r-1}$. Let us verify that $\dim \mathrm{Im}\, V = p$. Considering $V\colon \mathbb{R}^p \to \mathbb{R}^m$ and $Ve_j = (X_i^{\beta_j})_{i\in I}$, $j \in J$, it is sufficient to show that Ve_j are linearly independent. But the equalities $\sum_{j\in J} \xi_j X_i^{\beta_j} = 0$, $i \in I$, imply $\xi_j = 0$, $j \in J$. Now, as $\mathrm{Ker}\, V^\top = (\mathrm{Im}\, V)^\perp$, we have $\dim \mathrm{Ker}\, V^\top = m - p$.

Suppose S is the solution of (5) with $d_i \neq 0$, $i \in I_1$. We know that S is also the solution of a smoothing problem with some weights w_i, $i \in I_1$. We need the matrix $W = (w_{ij})_{i,j\in I}$, where $w_{ii} = w_i$ for $i \in I_1$, $w_{ii} = 1$ for $i \in I_0$, and $w_{ij} = 0$ for $i \neq j$. Then the conditions (4) may be written as

$$(-1)^r W d = \tilde{\varepsilon}, \tag{9}$$

with $\tilde{\varepsilon}_i = z_i - S(X_i)$ for $i \in I_1$ and $\tilde{\varepsilon}_i = (-1)^r d_i$ for $i \in I_0$. Denote also $\varepsilon = (z_i - g_i)_{i\in I}$, $z = (z_i)_{i\in I}$, $\chi\colon \mathbb{R}^m \to \mathbb{R}^m$ the projection such that $(\chi d)_i = d_i$, $i \in I_0$, and $(\chi d)_i = 0$, $i \in I_1$. Then $\tilde{\varepsilon} = z - g + (-1)^r \chi d = \varepsilon + (-1)^r \chi d = (I - \chi)\varepsilon + (-1)^r \chi d$. From (8) and (9) in the form

$$(-1)^r W d = z - g + (-1)^r \chi d$$

we obtain

$$(-1)^r W d + G d + V c = z + (-1)^r \chi d. \tag{10}$$

Take an arbitrary symmetric regular $m \times m$ matrix A, set $D = A^{-1}V$, and $\Pi = I - D(D^\top D)^{-1}D^\top$. Then $\Pi^2 = \Pi$. Let us show that Π is an orthogonal projection with $\mathrm{Im}\,\Pi = A\mathrm{Ker}V^\top$. Each $x \in A\mathrm{Ker}V^\top$ is representable as $x = Ay$, $y \in \mathrm{Ker}V^\top$, and

$$\Pi x = x - D(D^\top D)^{-1}D^\top Ay = x - D(D^\top D)^{-1}V^\top A^{-1}Ay = x.$$

This means that Π: $A\mathrm{Ker}V^\top \to A\mathrm{Ker}V^\top$ is an identity and $A\mathrm{Ker}V^\top \subset \mathrm{Im}\,\Pi$. On the other hand, if $x \in \mathrm{Im}\,\Pi$, then $x = \Pi y = y - D(D^\top D)^{-1}D^\top y$ for some $y \in I\!R^m$. Then, as $D^\top = V^\top A^{-1}$, we have

$$V^\top A^{-1}x = V^\top A^{-1}y - V^\top A^{-1}D(D^\top D)^{-1}D^\top y = V^\top A^{-1}y - D^\top y = 0.$$

Hence $A^{-1}x \in \mathrm{Ker}V^\top$ or $x \in A\mathrm{Ker}V^\top$. We have proved that $\mathrm{Im}\,\Pi \subset A\mathrm{Ker}V^\top$ and, consequently, $\mathrm{Im}\,\Pi = A\mathrm{Ker}V^\top$. In addition, for each $x, y \in I\!R^m$, we have $(\Pi x, y) = (x, \Pi y)$, which means that the projection Π is orthogonal.

Because $d \in \mathrm{Ker}V^\top$, we have $Ad \in A\mathrm{Ker}V^\top$ and $\Pi Ad = Ad$ or $A^{-1}\Pi Ad = d$. Let us show that $\Pi A^{-1}Vc = 0$ for all $c \in I\!R^p$. It is equivalent to $A^{-1}Vc \in (\mathrm{Im}\,\Pi)^\perp = (A\mathrm{Ker}V^\top)^\perp$ or $(A^{-1}Vc, Ax) = 0$ for all $x \in \mathrm{Ker}V^\top$. But $(A^{-1}Vc, Ax) = (Vc, x) = (c, V^\top x) = 0$. Now, applying ΠA^{-1} to (10), we obtain

$$(-1)^r \Pi A^{-1}WA^{-1}Ad + \Pi A^{-1}GA^{-1}\Pi Ad = \Pi A^{-1}z + (-1)^r \Pi A^{-1}\chi d$$

or, taking (9) into account,

$$(\Pi A^{-1}WA^{-1} + (-1)^r \Pi A^{-1}GA^{-1}\Pi)AW^{-1}\tilde{\varepsilon} = \Pi A^{-1}(z + (-1)^r \chi d). \quad (11)$$

The system (11) connects the vector of weights $w = (w_i)_{i \in I_1}$ and the vector of deviations ε.

6. PROPERTIES OF THE SYSTEM CONNECTING WEIGHTS AND DEVIATIONS

Since $d \in \mathrm{Ker}V^\top$, from (9) we know that $W^{-1}\tilde{\varepsilon} \in \mathrm{Ker}V^\top$ and $AW^{-1}\tilde{\varepsilon} \in A\mathrm{Ker}V^\top$. Concerning the system (11), the following is important.

Proposition 2. *The operator*

$$\Pi A^{-1}WA^{-1} + (-1)^r \Pi A^{-1}GA^{-1}\Pi \colon A\mathrm{Ker}V^\top \to A\mathrm{Ker}V^\top$$

is invertible.

Proof. First, suppose $d = (d_i) \in \mathrm{Ker}V^\top$, $d \neq 0$. For $S(X) = \sum_{j \in I} d_j G(X - X_j)$, according to (3), we have

$$\begin{aligned}||TS||^2 &= (-1)^r \sum_{i\in I} d_i S(X_i) = (-1)^r \sum_{i,j\in I} d_i d_j G(X_i - X_j) \\ &= (-1)^r (Gd, d).\end{aligned}$$

Hence $(-1)^r(Gd, d) > 0$ because $S \notin \mathcal{P}_{r-1}$.

Let $x \in A\mathrm{Ker}V^\top$. It means that $x = Ay$, $y \in \mathrm{Ker}V^\top$. Then

$$\begin{aligned}((\Pi A^{-1}WA^{-1} &+ (-1)^r \Pi A^{-1}GA^{-1}\Pi)x, x) \\ &= (Wy, y) + (-1)^r(Gy, y) \\ &\geq \gamma||y||^2 \geq \gamma_0||x||^2\end{aligned}$$

for some $\gamma, \gamma_0 > 0$. This completes the proof.

There is an important special case of the choice of A, namely, take $A = W^{1/2}$. Then the system (11) may be written in the form

$$(I + Q)W^{-1/2}\tilde{\varepsilon} = \Pi W^{-1/2}(z + (-1)^r\chi d),$$

with $Q = (-1)^r \Pi W^{-1/2} G W^{-1/2} \Pi$. Here the operator $I + Q$ is invertible in the whole space $\mathbb{R}^m$.

Theoretically, the system (11) may be used to determine the vector of deviations ε for given W or to determine the weights w_i for given ε. However, in general, both cases have accompanying unknowns χd.

Proposition 3. *Under the condition*

$$\mathrm{Ker}V^\top \bot \chi \mathrm{Ker}V^\top \tag{12}$$

the system (11) *determines uniquely the deviations by the weights and the weights by the deviations.*

Proof. Let us denote $B = \Pi A^{-1}WA^{-1} + (-1)^r \Pi A^{-1}GA^{-1}\Pi$. First, suppose that W is given. Consider the corresponding homogeneous system

$$BAW^{-1}((I - \chi)\varepsilon + (-1)^r \chi d) = (-1)^r \Pi A^{-1}\chi d.$$

It gives (here B^{-1} means the inverse of B in $A\mathrm{Ker}\, V^\top$)

$$(I - \chi)\varepsilon + (-1)^r\chi d = (-1)^r W A^{-1} B^{-1} \Pi A^{-1} \chi d,$$

from which, applying $I - \chi$ and χ, we get

$$\begin{aligned}\varepsilon &= (-1)^r (I - \chi) W A^{-1} B^{-1} \Pi A^{-1} \chi d, \\ \chi d &= \chi W A^{-1} B^{-1} \Pi A^{-1} \chi d.\end{aligned}$$

Since $W\chi d = \chi d$, we obtain

$$\begin{aligned}(\chi d, \chi d) &= (\chi W A^{-1} B^{-1} \Pi A^{-1} \chi d, \chi d) \\ &= (B^{-1} \Pi A^{-1} \chi d, A^{-1} \chi d) \\ &= (Ay, A^{-1} \chi d) = (y, \chi d) = 0,\end{aligned}$$

where $y \in \mathrm{Ker} V^\top$ because $\Pi A^{-1} \chi d \in A\mathrm{Ker} V^\top$ and $B^{-1} \Pi A^{-1} \chi d \in A\mathrm{Ker} V^\top$. Thus $\chi d = 0$ which gives also $\varepsilon = 0$. Consequently, the system (11) determines uniquely the vector of deviations ε.

Conversely, suppose that ε and the corresponding d are given or, we may say that $\tilde{\varepsilon}$ is given. Let w be determined from (11), i.e., the triple w, ε, and d satisfies (11). Assume that the triple w', ε, and d also satisfies (11). Then let us solve the smoothing problem with weights w', as a result, we get some ε' and d'. Thus, the triple w', ε', and d' satisfies (11) too. In the first part of the proof we showed that for given weights (we mean now w') there is a unique corresponding vector of deviations. Consequently, $\varepsilon' = \varepsilon$. The proof is complete.

Let us remark that the condition (12) is satisfied if, for instance, $I_0 = \emptyset$, then $\chi = 0$. This case is quite typical in practice when the data z are not known exactly.

7. DIAGONALIZATION OF THE MAIN MATRIX

Denote here $Q = (-1)^r \Pi A^{-1} G A^{-1} \Pi$. The system (11) may be written

$$\begin{aligned}QAW^{-1}\tilde{\varepsilon} &= -\Pi A^{-1}((I-\chi)\varepsilon + (-1)^r \chi d) \\ &\quad + \Pi A^{-1}(z + (-1)^r \chi d) \\ &= \Pi A^{-1}(z - (I-\chi)\varepsilon) = \Pi A^{-1}(z - (I-\chi)\tilde{\varepsilon}).\end{aligned}$$

Consider the decomposition $Q = P\Lambda P^\top$, where $Qp_i = \lambda_i p_i$, $i \in I$, $P = (p_i)_{i \in I}$, $P^\top = P^{-1}$, and Λ is the diagonal matrix with eigenvalues λ_i on the diagonal. In the proof of Proposition 2 we have actually shown that Q is positively definite on $A\mathrm{Ker} V^\top$ and we know also that $\dim A\mathrm{Ker} V^\top = m - p$. Since Π is a projection with $\mathrm{Im}\,\Pi = A\mathrm{Ker} V^\top$, the matrix Q has $m - p$ eigenvalues $\lambda_i > 0$ and p eigenvalues $\lambda_i = 0$. As $P^\top = P^{-1}$, we get

$$\Lambda P^\top A W^{-1} \tilde{\varepsilon} = P^\top \Pi A^{-1}(z - (I - \chi)\tilde{\varepsilon}). \tag{13}$$

For the definiteness suppose that in Section 5 we have introduced the index set $J \subset I$ in such a way that $\lambda_i = 0$, $i \in J$, and $\lambda_i > 0$, $i \in I \backslash J$. Now, let us complement the $p \times m$ matrix $V^\top = (X_i^{\beta_j})_{j \in J,\, i \in I}$ to $m \times m$ matrix $\bar{V}^\top$ by adding zero-rows. Introducing $C = \bar{V}^\top A^{-1} P$, taking $V^\top W^{-1}\tilde{\varepsilon} = 0$ or $\bar{V}^\top A^{-1} P P^\top A W^{-1}\tilde{\varepsilon} = C P^\top A W^{-1}\tilde{\varepsilon} = 0$ into account, and unifying the last equality with the system (13), we get

$$(C + \Lambda) P^\top A W^{-1}\tilde{\varepsilon} = P^\top \Pi A^{-1}(z - (I - \chi)\tilde{\varepsilon}). \tag{14}$$

Let us remark that if we mean $V^\top = (v_i)_{i \in J}$, then in the matrix $C + \Lambda$ the rows are $(v_i, A^{-1}p_j)_{j \in I}$ if $i \in J$ (i.e., $\lambda_i = 0$) or only $\lambda_i > 0$ on the diagonal if $i \in I \backslash J$.

Solving the system (11) relative to $\tilde{\varepsilon}$, we should not forget that $\tilde{\varepsilon}$ belongs to the subspace $W \mathrm{Ker} V^\top$. At the same time, in (14) there is no such concern about $\tilde{\varepsilon}$, a priori it may be an arbitrary vector in $I\!R^m$, since (14) yields $C P^\top A W^{-1}\tilde{\varepsilon} = 0$.

The matrix $C + \Lambda$ is regular if and only if the matrix $(v_i, A^{-1}p_j)_{i,j \in J}$, i.e., $(A^{-1}v_i, p_j)_{i,j \in J}$ is regular. The last matrix, being the Gram matrix of linear independent systems of vectors $A^{-1}v_i$, $i \in J$, and p_i, $i \in J$, is regular if and only if the linear spans of $A^{-1}v_i$, $i \in J$, and p_i, $i \in J$, coincide. We show that the last condition holds in any case. Since $\Pi D = 0$ or $\Pi A^{-1} V = 0$, we get $Q A^{-1} V = 0$. This gives $\Lambda P^\top A^{-1} V = 0$ or $\lambda_i (p_i, A^{-1} v_j) = 0$, $i \in I$, $j \in J$. As $\lambda_i > 0$ for $i \in I \backslash J$, we obtain $(p_i, A^{-1}v_j) = 0$, $i \in I \backslash J$, $j \in J$. Because p_i, $i \in I$, is an orthogonal basis in $I\!R^m$, we conclude that $A^{-1}v_j$ for all $j \in J$ belongs to the linear span of p_i, $i \in J$.

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KAALUDEGA JA TÕKETEGA SILUMISÜLESANNETEST

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On käsitletud mitme muutuja funktsioonide tõketega silumisülesande lahendamist kaaludega silumisülesandele taandamise teel ning kirjeldatud ekvivalentsete ülesannete olemasolu. On leitud süsteem, mis seob tõketega ülesande lahendi lähteandmetest kõrvalekaldeid ekvivalentse ülesande kaaludega. Süsteemis esineb vaba parameetrina suvaline sümmeetriline regulaarmaatriks. On esitatud tingimused süsteemi üheseks lahenduvuseks, kus tundmatuteks võivad olla nii kaalud kui ka kõrvalekalded. Kui viia süsteemi maatriks diagonaalkujule, siis õnnestub süsteem teisendada nii, et selle omadused on tunduvalt paremad kui üldjuhul.