Proc. Estonian Acad. Sci. Phys. Math., 1997, **46**, 4, 235–240
https://doi.org/10.3176/phys.math.1997.4.01

AN EMBEDDING THEOREM FOR FINITE GROUPS

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Received 5 May 1997

Abstract. It is shown that for each finite group G there exists a finite group H such that the direct product $G \times H$ is determined by its endomorphism semigroup in the class of all groups.

Key words: group, semigroup, endomorphism.

1. INTRODUCTION

Let G be a fixed group and End(G) the semigroup of all endomorphisms of G. If for an arbitrary group H the isomorphism of semigroups End(G) and End(H) implies the isomorphism of groups G and H, then we say that the group G is determined by its semigroup of endomorphisms (in the class of all groups). There are many groups that are determined by their endomorphism semigroups, for example, finite Abelian groups ([1], Theorem 4.2), nontorsion divisible Abelian groups ([2], Theorem), Sylow subgroups of finite symmetric groups ([3], Corollary 1). On the other hand, there exist also groups that are not determined by their endomorphism semigroups: the alternating group A_4 [4], some semidirect products of two finite cyclic groups [5]. In this connection let us set a problem: For a given group G, find a group K such that $G \subset K$ and K is determined by its endomorphism semigroup. For example, as the finite symmetric groups are determined by their endomorphism semigroups [6], this problem has an affirmative solution for all finite groups.

In [7], the mentioned problem is specified (problem 3.49): Is it possible to find a group H for a given group G such that the direct product $G \times H$ is determined by its endomorphism semigroup? For Abelian groups this question has an affirmative answer ([8,9]). In the present paper this problem is solved for finite groups. Using the ideas of [1] and [10], we will prove the following theorem.

Theorem. *Let G be a finite group. Then there exists a finite group H such that the direct product $G \times H$ is determined by its endomorphism semigroup in the class of all groups.*

2. PRELIMINARIES

Let G be a group. If A is a subgroup of the centre $Z(G)$ of G, then G is a central extension of A by the group $B \cong G/A$. Let $\{g_\alpha \mid \alpha \in B\}$ be the set of representatives of elements of the factor group G/A and $M = \{m_{\alpha,\beta} \mid \alpha, \beta \in B\}$ (shortly $M = \{m_{\alpha,\beta}\}$) the corresponding system of factors, i.e., $g_\alpha g_\beta = g_{\alpha\beta} m_{\alpha,\beta}$. Then

$$G = \{\, g_\alpha a \,|\, a \in A,\ \alpha \in B \,\}$$

and the product in G is given by

$$(g_\alpha a)\,(g_\beta b) = g_{\alpha\beta}(m_{\alpha,\beta} ab) \tag{1}$$

([11], pp. 315–323). Note that by (1) the group G is well defined if there are given a group B, a commutative group A, and the factor system M on A (see [11] for conditions that must hold for factor systems). In these notions the following lemmas hold.

Lemma 1 ([10], Lemma 2). *Let $\psi \in \mathrm{End}(A)$ be such that the factor systems $M = \{\, m_{\alpha,\beta} \,\}$ and $\psi(M) = \{\, \psi(m_{\alpha,\beta}) \,\}$ are equivalent, i.e., $\psi(m_{\alpha,\beta}) = c_{\alpha\beta}^{-1} m_{\alpha,\beta} c_\alpha c_\beta$ for some $c_\alpha \in A$, $\alpha \in B$. Then ψ can be extended to an endomorphism φ of G by setting*

$$\varphi(g) = \varphi(g_\alpha a) = g_\alpha c_\alpha\, \psi(a), \ g = g_\alpha a \in G.$$

Furthermore, if $\psi \in \mathrm{Aut}(A)$, then the corresponding φ is an automorphism of G.

Lemma 2 ([12], p. 248). *Assume that B is of the order n and $M = \{\, m_{\alpha,\beta} \,\}$ is a factor system on A. Then the factor systems $\{\, m_{\alpha,\beta}^n \,\}$ and $\{\, 1_{\alpha,\beta} \,\}$, where $1_{\alpha,\beta}$ is the unit element of A for each α, $\beta \in B$, are equivalent.*

Lemma 3. *Let B be a finite group of the order n and m a natural number, $n \leq m$. Then there exists only a finite number of nonisomorphic finite groups G such that $\mathrm{End}(G) = m$ and the group of all inner automorphisms of G is isomorphic to B.*

Proof. Choose B, n, and m as stated in the formulation of Lemma 3. Assume that G is a finite group such that

$$|\mathrm{End}(G)| = m \tag{2}$$

and the group of all inner automorphisms of G is isomorphic to B, i.e., $B \cong G/A$, where $A = Z(G)$. Then G is a central extension of A by B. Let $M = \{ m_{\alpha,\beta} \}$ be the corresponding factor system.

Let us fix now an arbitrary natural number k and consider the endomorphism $\psi(k)$ of A given by

$$(\psi(k))(c) = c^{kn+1}, \ c \in A.$$

In view of Lemma 2, the factor systems $M = \{ m_{\alpha,\beta} \}$ and

$$(\psi(k))(M) = \{ (\psi(k))(m_{\alpha,\beta}) \} = \{ m_{\alpha,\beta}^{kn+1} \}$$

are equivalent. Therefore, by Lemma 1, the endomorphism $\psi(k)$ of A can be extended to the endomorphism $\varphi(k)$ of G. By (2) there exist k and l such that $0 \le l < k \le m$ and $\varphi(k) = \varphi(l)$, i.e., $c^{kn+1} = c^{ln+1}$ and $c^{(k-l)n} = 1$ for each $c \in A$. Suppose r is the smallest positive integer with the property $c^{rn} = 1$ for each $c \in A$. Therefore $A = Z(G)$ is bounded and

$$r \le m. \tag{3}$$

Denote

$$H = \langle g_\alpha, \, m_{\alpha,\beta} \mid \alpha, \beta \in B \rangle.$$

Then $H \triangleleft G$, $G/H = AH/H$ is commutative and

$$|H| \le |B| \, (rn)^{|M|} = n \, (rn)^{n^2}. \tag{4}$$

If $H \ne G$, then the factor group G/H splits up:

$$G/H = \oplus_{i \in I} \langle a_i H \rangle,$$

where $a_i \in A \setminus H$. There exists a nonzero homomorphism

$$\tau_i : \langle a_i H \rangle \longrightarrow \langle a_i \rangle$$

for each $i \in I$. Each τ_i generates an endomorphism $\xi_i = \tau_i \pi_i \epsilon$ of G, where $\epsilon : G \longrightarrow G/H$ is the canonical homomorphism and $\pi_i : G/H \longrightarrow \langle a_i H \rangle$ is the projection. It follows from here and Eq. (2) that $|I| \le m$, $|G/H| \le (rn)^{|I|}$. Thus

$$|G/H| \le (rn)^m. \tag{5}$$

The inequality (5) holds in the case $G = H$ too. Therefore it holds always.

By the inequalities (3)–(5),

$$|G| = |G/H| \, |H| \le (rn)^m \, n \, (rn)^{n^2}$$

$$= n \, (rn)^{m+n^2} \le n \, (mn)^{m+n^2}. \tag{6}$$

As m and n are fixed, there exists only a finite number of nonisomorphic finite groups G satisfying the inequality (6). Lemma 3 is proved.

Lemma 4. *Let G be a finite group. Then there exists only a finite number of nonisomorphic groups H such that* $\mathrm{End}(G) \cong \mathrm{End}(H)$.

Proof. Let G be a finite group. Then $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ are also finite. Suppose that $B_1, \ldots, B_k$ is the list of all subgroups of $\mathrm{Aut}(G)$. Assume now that H is a group such that $\mathrm{End}(H) \cong \mathrm{End}(G)$. By [10], Theorem 2, the group H is finite. Since $\mathrm{Aut}(H) \cong \mathrm{Aut}(G)$, the group of inner automorphisms of H is isomorphic to B_i for some $i \in \{1, 2, \ldots, k\}$. Hence, $H/Z(H) \cong B_i$. In view of Lemma 3, there exists a finite number of nonisomorphic groups H such that $\mathrm{End}(H) \cong \mathrm{End}(G)$ and $H/Z(H) \cong B_i$. As i takes k different values, the statement of the lemma is evidently true. The lemma is proved.

3. PROOF OF THE THEOREM

Let us start now with the proof of the theorem. Assume that G is a finite group. By Lemma 4, there exists a finite number of nonisomorphic groups K such that $\mathrm{End}(G) \cong \mathrm{End}(K)$. Suppose that all those groups are $G = G_0, G_1, \ldots, G_n$. Denote the direct product of $G_0, G_1, \ldots, G_n$ by Q:

$$Q = G_0 \times G_1 \times \ldots \times G_n = G \times G_1 \times \ldots \times G_n. \tag{7}$$

Our aim is to show that the group Q is determined by its endomorphism semigroup.

Assume that R is a group and $\mathrm{End}(R) \cong \mathrm{End}(Q)$. We shall show that the groups R and Q are isomorphic. Fix an isomorphism

$$^*: \mathrm{End}(Q) \longrightarrow \mathrm{End}(R). \tag{8}$$

Let π_i be the projection of Q onto the subgroup G_i ($i = 0, 1, \ldots, n$). By [1], pp. 79, 85, and 86, the group R splits up

$$R = \mathrm{Im}\,\pi_0^* \times \ldots \times \mathrm{Im}\,\pi_n^* \tag{9}$$

and

$$\mathrm{End}(\mathrm{Im}\,\pi_i^*) \cong \mathrm{End}(\mathrm{Im}\,\pi_i) = \mathrm{End}(G_i) \cong \mathrm{End}(G)$$

for each $i \in \{0, 1, \ldots, n\}$. Moreover, π_i^* is the projection of R onto $\mathrm{Im}\,\pi_i^*$. By the choice of groups $G_0, \ldots, G_n$, each $\mathrm{Im}\,\pi_i^*$ is isomorphic to a group from the set $\{G_0, \ldots, G_n\}$.

Note that each $\xi \in \mathrm{Hom}(\mathrm{Im}\,\pi_i, \mathrm{Im}\,\pi_j) = \mathrm{Hom}(G_i, G_j)$ can be naturally extended to an endomorphism of Q by setting

$$\xi(gh) = \xi g,$$

where $g \in \operatorname{Im} \pi_i = G_i$ and $h \in \operatorname{Ker} \pi_i$. We shall identify ξ with its extension. Then

$$\operatorname{Hom}(\operatorname{Im} \pi_i, \operatorname{Im} \pi_j) = \{ \xi \in \operatorname{End}(Q) \mid \xi \pi_i = \xi = \pi_j \xi \}.$$

By the isomorphism (8),

$$(\operatorname{Hom}(\operatorname{Im} \pi_i, \operatorname{Im} \pi_j))^* = \{ \xi^* \in \operatorname{End}(R) \mid \xi^* \pi_i^* = \xi^* = \pi_j^* \xi^* \}$$
$$= \operatorname{Hom}(\operatorname{Im} \pi_i^*, \operatorname{Im} \pi_j^*). \quad (10)$$

As the groups $G_0 = \operatorname{Im} \pi_0, \dots, G_n = \operatorname{Im} \pi_n$ are nonisomorphic, for different i and j there exist no endomorphisms $\alpha, \beta \in \operatorname{End}(Q)$ such that $\alpha \in \operatorname{Hom}(\operatorname{Im} \pi_i, \operatorname{Im} \pi_j)$, $\beta \in \operatorname{Hom}(\operatorname{Im} \pi_j, \operatorname{Im} \pi_i)$, and $\beta\alpha = \pi_i$, $\alpha\beta = \pi_j$. In view of (10), the same holds also in $\operatorname{End}(R)$, i.e., for different i and j there are no endomorphisms $\alpha^* \in \operatorname{Hom}(\operatorname{Im} \pi_i^*, \operatorname{Im} \pi_j^*)$ and $\beta^* \in \operatorname{Hom}(\operatorname{Im} \pi_j^*, \operatorname{Im} \pi_i^*)$, so that $\beta^* \alpha^* = \pi_i^*$ and $\alpha^* \beta^* = \pi_j^*$. Therefore, the subgroups $\operatorname{Im} \pi_0^*, \dots, \operatorname{Im} \pi_n^*$ of R are nonisomorphic. Since each $\operatorname{Im} \pi_i^*$ is isomorphic to a group from the set $\{ G_0, \dots, G_n \}$, it follows from the decompositions (7) and (9) that the groups Q and R are isomorphic. Consequently, the group $Q = G \times H$, where $H = G_1 \times \dots \times G_n$, is determined by its endomorphism semigroup (if $Q = G_0 = G$, then $H = \langle 1 \rangle$). The theorem is proved.

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LÕPLIKE RÜHMADE SISESTUSTEOREEM

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On tõestatud, et iga lõpliku rühma G jaoks leidub selline lõplik rühm H, mille puhul nende otsekorrutis $G \times H$ on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.